# Perturbations beyond Schauder

Cristiana De Filippis (University of Parma)
cristiana.defilippis@unipr.it

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# Uniform vs nonuniform ellipticity

Consider the linear equation

$$-\operatorname{div}\,(a(x)Du) = 0\,.$$

(assume that $a(\cdot)$ is symmetric).
This is called uniformly elliptic if

$$\sup_x \frac{\text{highest eigenvalue of } a(x)}{\text{lowest eigenvalue of } a(x)} \leq c$$

It is called nonuniformly elliptic otherwise, and typically serves to prove regularity of $u$.

In the nonlinear case

$$-\text{div}\, A(Du) = 0$$

one looks at the differentiated (linearized) equation

$$-\text{div}\,(a(x)DD_s u) = 0, \qquad a(x) = \partial_z A(Du(x))$$

to get gradient estimates. Therefore, if ellipticity is described by

$$g_1(|z|)\mathbb{I}_d \leq \partial_z A(z) \leq g_2(|z|)\mathbb{I}_d,$$

then uniform ellipticity occurs when

$$\sup_{|z|} \mathcal{R}(z) = \sup_{|z|} \frac{g_2(|z|)}{g_1(|z|)} < \infty \quad \leftsquigarrow \quad \left( \sup_{|z|>1} \frac{g_2(|z|)}{g_1(|z|)} < \infty \right).$$

# Examples of uniformly elliptic equations

- Laplace

$$-\Delta u = 0;$$

- $p$-Laplace (degenerate, but uniformly elliptic)

$$-\Delta_p u = -\mathrm{div}\,(|Du|^{p-2} Du) = 0, \qquad p > 1.$$

- In general

$$-\mathrm{div}\, A(Du) = 0$$

under the classical assumptions (Ladyzhenskaya & Uraltseva)

$$|z|^{p-2}\mathbb{I}_d \lesssim \partial_z A(z) \lesssim |z|^{p-2}\mathbb{I}_d.$$

*In such cases the ellipticity ratio $\mathcal{R}(z)$ stays uniformly bounded away from zero and infinity.*

# More uniformly elliptic operators

Uniformly elliptic equations are not necessarily of polynomial type. For instance

$$-\mathrm{div}\,(\tilde{a}(|Du|)Du) = f$$

is uniformly elliptic provided

$$-1 < i_a \leq \frac{\tilde{a}'(t)t}{\tilde{a}(t)} \leq s_a < \infty \quad \text{for every } t > 0.$$

Indeed it follows that

$$\mathcal{R}(z) \lesssim \frac{\max\{1, s_a + 1\}}{\min\{1, i_a + 1\}}\,.$$

*These operators have been recently the object of intensive investigation (see papers of Baroni, Beck & Mingione, Byun, Cianchi & Maz'ya, Diening, Lieberman, Stroffolini amongst the others).*

# The non-autonomous case is more delicate

For equations of the type

$$-\text{div}\, A(x, Du) = f$$

we describe ellipticity in the more flexible way

$$g_1(x, |z|)\mathbb{I}_d \le \partial_z A(x, z) \le g_2(x, |z|)\mathbb{I}_d,$$

and we consider two ratios:

- the classical **pointwise** ellipticity ratio

$$\mathcal{R}(x, z) := \frac{g_2(x, |z|)}{g_1(x, |z|)}$$

- and the new **nonlocal** one

$$\mathcal{R}(z, B) := \frac{\sup_{x\in B}\, g_2(x, |z|)}{\inf_{x\in B}\, g_1(x, |z|)}\,.$$

# The non-autonomous case is more delicate

Obviously

$$\mathcal{R}(x,z) \le \mathcal{R}(z,B) \qquad \forall\ x \in B\,.$$

Consider the equation

$$\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du\right) = 0 \qquad 0 \le a(x) \le L$$

which is Euler-Lagrange equation of the double phase functional

$$w \mapsto \int \left[|Dw|^p + a(x)|Dw|^q\right] dx \qquad 1 < p < q\,.$$

# Two notions of uniform ellipticity

Then

$$\mathcal{R}(x,z) \lesssim \frac{\max\{1, q-1\}}{\min\{1, p-1\}}$$

while

$$\mathcal{R}(z,B) \approx \|a\|_{L^\infty(B)}|z|^{q-p} \to \infty \qquad \text{as } |z| \to \infty$$

The equation is therefore **pointwise uniformly elliptic**, but the **nonlocal ellipticity ratio** blows up, cf. D. & Mingione (ARMA 2021) for a discussion.

Solutions to

$$-\Delta u = -\text{div}\,(Du) = 0$$

are smooth. Adding coefficients (ingredients)

$$-\text{div}\,(A(x)Du) = -(A^{ij}(x)D_j u)_{x_i} = 0\,.$$

preserves as much regularity as the degree of smoothness of coefficients allow.

This holds provided the matrix $A(\cdot)$ is bounded and elliptic

$$\nu \mathbb{I}_d \leq A(x) \leq L\mathbb{I}_d\,.$$

# Classical Schauder estimates

As $Du$ and $A(x)$ directly couple in the equation, we have

$$A(\cdot) \in C^{0,\alpha} \Longrightarrow Du \in C^{0,\alpha} \qquad 0 < \alpha < 1\,.$$

This kind of results where first obtained by **Hopf** (1929), **Caccioppoli** (1934) and **Schauder** (1934), in various forms, and are today known as Schauder estimates. The original proofs involve heavy potential theory.

Modern proofs have been given by **Campanato**, **Trudinger** and **Leon Simon**. All the proofs rely, in a way or in another, on **perturbation** methods. This means that regularity estimates for solutions to

$$-\mathrm{div}\,(A(x)Du) = 0$$

are obtained via comparison with solutions to equations with more regular coefficients, as for instance frozen equations

$$-\mathrm{div}\,(A(x_0)Dv) = 0\,.$$

# Nonlinear Schauder estimates

The nonlinear theory is a more recent story, dating back the beginning of the 80s.

A model example is given by the $p$-Laplacean equation with coefficients

$$-\mathrm{div}\,(c(x)|Du|^{p-2}Du) = 0\,, \qquad 0 < \nu \leq c(\cdot) \in C^{0,\alpha}\,.$$

The possibility of Schauder estimates in the nonlinear case relies on the fact that solutions $v$ to frozen equations

$$-\mathrm{div}\,(c(x_0)|Dv|^{p-2}Dv) = 0$$

still enjoy good regularity estimates (this is **Uraltseva-Uhlenbeck** theory).

*In this case we can say that Du is Hölder continuous for some exponent.*

# General uniformly elliptic equations

These estimates hold for general equations of the type

$$-\text{div}\, A(x, Du) = 0$$

under the classical assumptions (**Ladyzhenskaya & Uraltseva**)

$$\begin{cases} |z|^{p-2}\mathbb{I}_d \lesssim \partial_z A(x,z) \lesssim |z|^{p-2}\mathbb{I}_d \\ |A(x_1,z) - A(x_2,z)| \leq L|x_1 - x_2|^{\alpha}|z|^{p-1}\,. \end{cases}$$

This is due work of **Giaquinta & Giusti**, **Manfredi**, **Di Benedetto**, **Lewis**, **Tolksdorf**. For a general approach see also **Kuusi & Mingione** (JFA 2012). Hölder continuity can be relaxed up to Dini continuity, see also recent work of **Maz'ya & McOwen** (Arxiv).

# Freezing & Comparing

- Summarizing, both in the linear and in the nonlinear case, **perturbation** methods are crucial.
- Such methods work provided all the estimates involved are **homogeneous** and therefore can be matched and iterated.
- This happens in the uniformly elliptic case, but **not** in the nonuniformly elliptic one.
- Proving Schauder estimates in the general nonuniformly elliptic case has remained an **open problem**.

# Schauder estimates in the variational setting

When dealing with functionals of the type

$$w \mapsto \int_\Omega \mathrm{c}(x) F(Dw)\, dx$$

(think for instance of $F(Dw) \equiv |Dw|^p$) we can use the Euler-Lagrange equation

$$-\mathrm{div}\,(\mathrm{c}(x)\partial_z F(Du)) = 0\,.$$

In this case regularity follows from those known for equations.

# Non-differentiable functionals

What happens when dealing for instance with classical model functionals as

$$w \mapsto \int_\Omega [F(Dw) + \mathtt{h}(x, w)]\, dx$$

when $v \mapsto \mathtt{h}(\cdot, v)$ is not differentiable, but only Hölder? As $\mathtt{h}(\cdot)$ is not differentiable, the Euler-Lagrange equation

$$-\text{div}\, \partial_z F(Du) + \partial_u \mathtt{h}(x, u) = 0$$

does not exists.

These are treated in papers by **Frehse**, **Giaquinta & Giusti**, **Ivert**, **Manfredi**, from the beginning of the 80s, who still proved that $Du$ is locally Hölder continuous. They still use **perturbation techniques**, this time directly relying on minimality. Results also by **Kristensen & Mingione**.

# Non-differentiable functionals

These arguments can be carried through up to general functionals of the type

$$w \mapsto \int_\Omega c(x, w) F(Dw)\, dx$$

therefore falling outside the realm of traditional Schauder estimates.

In this case crucial use is made of the fact that **$u$ s a priori known to be Hölder continuous**, so that

$$x \mapsto c(x, u(x)) \quad \text{is Hölder continuous.}$$

This extra information comes directly from growth conditions

$$|z|^p \lesssim F(x, w) \lesssim |z|^p + 1$$

via standard **De Giorgi's theory**.

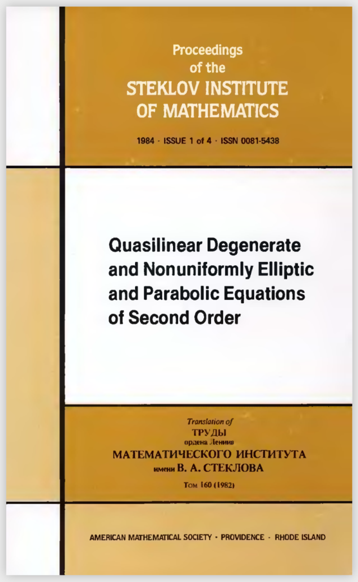
# Nonuniformly elliptic classics $\leq 70s$

- Ladyzhenskaya & Uraltseva (Book + CPAM 1970)
- Gilbarg (1963)
- Stampacchia (CPAM 1963)
- Hartman & Stampacchia (Acta Math. 1965)
- Ivočkina & Oskolkov (Zap. LOMI 1967)
- Oskolkov (Trudy Mat. Inst. Steklov 1967)
- Serrin (Philos. Trans. Roy. Soc. London Ser. A 1969)
- Ivanov (Proc. Steklov Inst. Math. 1970)
- Trudinger (Thesis, Bull. AMS 1967, ARMA 1971)
- Leon Simon (Indiana Univ. Math. J. 1976)

# Nonuniformly elliptic classics $\geq 80s$

- Trudinger (Invent. Math. 1981)
- N. N. Ural'tseva & A. B. Urdaletova (Vestnik Leningrad Univ. Math. 1984)
- Lieberman (Indiana Univ. Math. J. 1983)
- **Ivanov** (Proc. Steklov Inst. Math. Book 1984)
- **Marcellini** (ARMA 1987, JDE 1991, Ann. Pisa 1996)
- **Zhikov** (Papers from the '80s, Math. of USSR-Izvestia. 1995, Russian J. Math. Phys. 1997)

# Ivanov's book.



# A Ladyzhenskaya & Uraltseva classic

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XXIII, 677–703 (1970)

## Local Estimates for Gradients of Solutions of Non-Uniformly Elliptic and Parabolic Equations

O. A. LADYZHENSKAYA AND N. N. URAL'TSEVA
*Leningrad University*

Various classes of non-uniformly elliptic (and parabolic) equations of second order of the form

$$\text{(1.1)} \qquad \sum_{i,j=1}^{n} a_{ij}(x, u, u_x) u_{ij} = a(x, u, u_x), \qquad a_{ij}(x, u, u_x)\xi_i\xi_j > 0 \quad \text{for} \quad |\xi| = 1,$$

for all solutions $u(x)$ of which $\max_\Omega |u_x|$ can be estimated by $\max_\Omega |u|$ and $\max_{\partial\Omega} |u_x|$, were discussed in [1] (see also [2]).[1] The method used was introduced in [3]. In the same paper a method was suggested for obtaining local estimates of $|u_x|$, i.e., estimates of $\max_{\Omega'} |u_x|$ in terms of $\max_\Omega |u|$ and the distance $d(\Omega', \partial\Omega)$ of $\Omega' \subset \Omega$ from the boundary $\partial\Omega$. In a series of papers (concerning these see [4] and [5]) we have shown that this method is applicable to the whole class of uniformly elliptic and parabolic equations. In the present paper we investigate the possibility of applying it to non-uniformly elliptic and parabolic equations. It turns out that it is applicable, roughly speaking, to those classes of [1] for which the order of nonuniformity of the quadratic form $a_{ij}(x, u, u_x)\xi_i\xi_j$ is less than two. The first part of this paper is devoted to the proof of this assertion.

In the second part we analyze a different method of obtaining local estimates for $|u_x|$ which is applicable to elliptic equations of the form

$$\text{(1.2)} \qquad -\sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, u, u_x) + a(x, u, u_x) = 0,$$

and embraces such interesting cases as equations for the mean curvature of a

[1] We shall use the notation

$$u_{x_i} = u_i, \quad u_x = (u_1, \cdots, u_n), \quad u_{x_i x_j} = u_{ij},$$
$$u_x^2 = \sum_{i=1}^{n} u_i^2, \quad |u_x| = \sqrt{u_x^2}, \quad u_{xx}^2 = \sum_{i,j=1}^{n} u_{ij}^2, \quad |u_{xx}| = \sqrt{u_{xx}^2}.$$

**THE DIRICHLET PROBLEM FOR NONUNIFORMLY ELLIPTIC EQUATIONS**[1]

BY NEIL S. TRUDINGER

Communicated by F. John, January 23, 1967

**Introduction.** Let $\Omega$ be a bounded domain in $E^n$. The operator

$$Qu = a^{ij}(x, u, u_x)u_{x_i x_j} + a(x, u, u_x)$$

acting on functions $u(x) \in C^2(\Omega)$ is *elliptic* in $\Omega$ if the minimum eigenvalue $\lambda(x, u, p)$ of the matrix $[a^{ij}(x, u, p)]$ is positive in $\Omega \times E^{n+1}$. Here

$$u_x = (u_{x_1}, \cdots u_{x_n}), \qquad p = (p_1, \cdots p_n)$$

and repeated indices indicate summation from 1 to $n$. The functions $a^{ij}(x, u, p)$, $a(x, u, p)$ are defined in $\Omega \times E^{n+1}$. If furthermore for any $M>0$, the ratio of the maximum to minimum eigenvalues of $[a^{ij}(x, u, p)]$ is bounded in $\Omega \times (-M, M) \times E^n$, $Qu$ is called *uniformly elliptic*. A solution of the *Dirichlet problem* $Qu=0$, $u=\phi(x)$ on $\partial\Omega$ is a $C^0(\bar{\Omega}) \cap C^2(\Omega)$ function $u(x)$ satisfying $Qu=0$ in $\Omega$ and agreeing with $\phi(x)$ on $\partial\Omega$.

When $Qu$ is elliptic, but not necessarily uniformly elliptic, it is referred to as *nonuniformly elliptic*. In this case it is well known from two dimensional considerations, that in addition to smoothness of the boundary data $\partial\Omega$, $\phi(x)$ and growth restrictions on the coefficients of $Qu$, geometric conditions on $\partial\Omega$ may play a role in the solvability of the Dirichlet problem. A striking example of this in higher dimensions is the recent work of Jenkins and Serrin [**4**] on the minimal surface equation, mentioned below.

*On the Regularity of Generalized Solutions of Linear, Non-Uniformly Elliptic Equations*

NEIL S. TRUDINGER

*Communicated by* J. C. C. NITSCHE

**1. Introduction**

We consider in this paper the simplest form of a second order, linear, divergence structure equation in $n$ variables, namely

$$\frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial u}{\partial x_j}\right)=0 \tag{1.1}$$

where the coefficients $a^{ij}$, $1 \leq i, j \leq n$, are measurable functions on a domain $\Omega$ in Euclidean $n$ space $E^n$. Following the usual summation convention, repeated indices will indicate summation from 1 to $n$. We assume always that $n \geq 2$.

Equation (1.1) is *elliptic* in $\Omega$ if the coefficient matrix $\mathscr{A}(x)=[a^{ij}(x)]$ is positive almost everywhere in $\Omega$. Let $\lambda(x)$ denote the minimum eigenvalue of $\mathscr{A}(x)$ and set

$$\mu(x) = \sup_{1 \leq i, j \leq n} |a^{ij}(x)| \tag{1.2}$$

so that

$$\lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq n^2\mu(x)|\xi|^2 \tag{1.3}$$

for all $\xi \in E^n$, $x \in \Omega$. We will say that equation (1.1) is *uniformly elliptic* in $\Omega$ if the function $\gamma(x)=\mu(x)/\lambda(x)$ is essentially bounded in $\Omega$. If $\gamma$ is not necessarily bounded, then equation (1.1) is referred to as *non-uniformly elliptic*. We note here that uniformly elliptic equations for which $\lambda^{-1}$ is unbounded have sometimes been referred to as *degenerate elliptic* [9].

Uniformly elliptic equations of the form (1.1), with bounded $\lambda^{-1}$ and $\mu$, have been extensively studied in the literature, two of the major results being a *Hölder estimate* for generalized solutions, due to DEGIORGI [1] and NASH [11], and a *Harnack inequality*, due to MOSER [7]. The purpose of this paper is to extend these results to a class of non-uniformly elliptic equations. In order to accomplish this, our methods differ substantially from those previously proposed and hence may be considered as new proofs of the original results. Various features of our proofs do coincide, however, with techniques in MOSER's two papers [6], [7]. An essential difference is that in order to obtain the stronger results we need to extract more information from the equation.

# *Interior Gradient Bounds for Non-uniformly Elliptic Equations*

LEON SIMON

In [1] Bombieri, De Giorgi and Miranda were able to derive a local interior gradient bound for solutions of the minimal surface equation with $n$ independent variables, $n \geqq 2$, thus extending the result previously established by Finn [2] for the case $n = 2$. Their method was to use test function arguments together with a Sobolev inequality on the graph of the solution (Lemma 1 of [1]). A much simplified proof of their result was later given by Trudinger in [12].

Since the essential features of the test function arguments given in [1] generalized without much difficulty to many other non-uniformly elliptic equations, it was apparent that interior gradient bounds could be obtained for these other equations provided appropriate analogues for the Sobolev inequality of [1] could be established. Ladyzhenskaya and Ural'tseva obtained such inequalities ([4], Lemma 1) for a rather large class of equations, including the minimal surface equation as a special case. They were thus able to obtain gradient bounds for this class of equations.

In §2 of [5] a general Sobolev inequality was established on certain generalized submanifolds of Euclidean space. In the special case of nonparametric hypersurfaces in $\mathbf{R}^{n+1}$ of the form $x_{n+1} = u(x)$, where $u$ is a $C^2$ function defined on an open subset $\Omega \subset \mathbf{R}^n$, the inequality of [5] implies

$$(1) \qquad \left\{\int_\Omega h^{n/(n-1)} v\, dx\right\}^{(n-1)/n} \leqq c \int_\Omega \left[\left\{\sum_{i,j=1}^n g^{ij} h_{x_i} h_{x_j}\right\}^{1/2} + h\,|H|\right] v\, dx$$

for each non-negative $C^1$ function $h$ with compact support in $\Omega$, where

$$v = (1 + |Du|^2)^{1/2}$$

$$g^{ij} = \delta_{ij} - \nu_i \nu_j\,, \qquad \nu_i = u_{x_i}/v, \qquad i, j = 1, \cdots, n$$

$$H = \frac{1}{n} v^{-1} \sum_{i,j=1}^n g^{ij} u_{x_i x_j}\,,$$

and where $c$ is a constant depending only on $n$. (See the discussion in §2 below.) The quantity $H$ appearing in this inequality is in fact the mean curvature of the hypersurface $x_{n+1} = u(x)$ and in the special case when $H \equiv 0$ (*i.e.* when $u$

The variational setting

$$w \mapsto \int_\Omega F(Dw)\, dx$$

turns out to be the most appropriate for local estimates.

The Euler-Lagrange reads as

$$-\mathrm{div}\, \partial_z F(Du) = 0$$

Nonuniform ellipticity reads as

$$\lim_{|z|\to\infty} \mathcal{R}_{\partial_z F}(z) = \lim_{|z|\to\infty} \frac{\text{highest eigenvalue of } \partial_{zz} F(z)}{\text{lowest eigenvalue of } \partial_{zz} F(z)} = \infty\,.$$

# Polynomial Nonuniform Ellipticity

This happens, when, for $|z|$ is large,

$$\mathcal{R}_{\partial_z F}(z) \approx |z|^{\delta} \qquad \text{for some } \delta > 0$$

These are usually formulated prescribing

$$|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$$

so that

$$\mathcal{R}_{\partial_z F}(z) \lesssim |z|^{q-p}, \ \text{ for } |z| \text{ large}, \quad 1 < p < q\,.$$

These are called $(p, q)$-growth conditions in Marcellini's terminology. Conditions of the type

$$\frac{q}{p} < 1 + o(n)$$

are in general sufficient (Marcellini) and necessary (Giaquinta and Marcellini) for regularity of minima.

# One moment back to uniform ellipticty

- **Classical fact 1.** For solutions to

$$-\mathrm{div}\,(\mathrm{c}(x)|Du|^{p-2}Du) = 0$$

and, more in general, uniformly elliptic equations with $p$-growth we have

$$\mathrm{c}(\cdot) \text{ is Hölder} \Longrightarrow Du \text{ is Hölder.}$$

- **Classical fact 2.** For minima of non-differentiable functionals

$$w \mapsto \int_\Omega [\mathrm{c}(x)|Dw|^p + \mathtt{h}(x,w)]\,dx$$

and, more in general, uniformly elliptic integrals with $p$-growth, we have

$$\mathrm{c}(\cdot), \mathtt{h}(x,\cdot) \text{ are Hölder} \Longrightarrow Du \text{ is Hölder.}$$

# Two classical issues

- **Open problem 1. Schauder for nonuniformlly elliptic equations**. For solutions to

$$-\mathrm{div}\,(\mathrm{c}(x)A(Du)) = 0\,, \qquad \mathcal{R}_A(z) \approx |z|^{q-p}$$

and more general, equations with polynomial nonuniform ellipticity

$$\mathrm{c}(\cdot) \text{ is Hölder} \Longrightarrow Du \text{ is Hölder.}$$

- **Open problem 2. Non-differentiable functionals**. For minima of non-differentiable functionals

$$w \mapsto \int_\Omega [F(Dw) + \mathtt{h}(x, w)]\, dx$$

and, more in general, of integrals with polynomial nonuniform ellipticity, it holds that

$$\text{coefficients (like } \mathtt{h}(\cdot,\cdot)\text{) are Hölder} \Longrightarrow Du \text{ is Hölder.}$$

# Lieberman's review.

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*Publications results for "Author=(giaquinta) AND Reviewer=(lieberman)"*

**MR0749677 (85k:35077)** Reviewed

Giaquinta, M. (I-FRNZ); Giusti, E. (I-FRNZ)

**Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form.**

*J. Reine Angew. Math.* 351 (1984), 55–65.

35J60 (35B65 49A22)

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It is by now classical that solutions of the Dirichlet problem for a divergence form elliptic equation: $\operatorname{div} A(x,u,Du) = B(x,u,Du)$ in $\Omega$, $u = \varphi$ on $\partial\Omega$, are $C^{k,\alpha}$ if $\varphi \in C^{k,\alpha}$ for any nonnegative integer $k \neq 1$, under suitable hypotheses on the coefficients $A$ and $B$. Moreover, the reviewer has proved this result for $k = 1$ [Comm. Partial Differential Equations 6 (1981), no. 4, 437–497; MR0612553] assuming, among other things, that $A$ has Holder continuous first derivatives and that $B$ is Holder continuous. The present paper provides an alternative proof of this regularity result for $k = 1$ by means of some interesting techniques developed by the authors to study the regularity of minima of functionals [Invent. Math. 72 (1983), no. 2, 285–298; MR0700772]. Basically, they freeze the coefficient vector $A$ at a point and then use a perturbation argument.

As well as being applicable to minimization problems, their method allows weaker smoothness hypotheses, namely, $A$ is $C^1$ in $Du$ and $C^{0,\alpha}$ in $x$ and $u$, and $B$ is bounded and measurable. In addition, bounded weak solutions of the Dirichlet problem can be studied directly when certain growth properties are imposed on the coefficients for large $Du$.

A comment needs to be made concerning their brief application to equations when their growth properties fail. As they point out, such equations fall under their considerations provided a global gradient bound has been established; however, this gradient bound has only been proved when $A$ is differentiable with respect to all its arguments, and in many cases more smoothness of the coefficients is needed. The results of this paper are thus much more striking when applied to uniformly elliptic equations than to nonuniformly elliptic ones.

Reviewed by Gary M. Lieberman

THEOREM OF LADYZHENSKAYA AND URAL'TSEVA [83]. *Suppose that a function* $u \in C^2(\bar{\Omega})$ *satisfying the condition*

$$\max_{\bar{\Omega}}|u| \leqslant m, \quad \max_{\bar{\Omega}}|\nabla u| \leqslant M, \tag{2.5}$$

*is a solution of* (1.1) *in a bounded domain* $\Omega \subset \mathbf{R}^n$, $n \geqslant 2$, *and that equation* (1.1) *is elliptic at this solution in the sense that*

$$a^{ij}(x, u(x), \nabla u(x))\xi_i\xi_j \geqslant \nu\xi^2, \qquad \nu = \text{const} > 0, \xi \in \mathbf{R}^n, x \in \bar{\Omega}. \tag{2.6}$$

*Suppose that on the set* $\mathscr{F}_{\Omega,m,M} \equiv \bar{\Omega} \times \{|u| \leqslant m\} \times \{|p| \leqslant M\}$ *the functions* $a^{ij}(x, u, p)$, $i, j = 1, \dots, n$, *and* $a(x, u, p)$ *satisfy the condition*

$$|a^{ij}| + \left|\frac{\partial a^{ij}}{\partial x}\right| + \left|\frac{\partial a^{ij}}{\partial u}\right| + \left|\frac{\partial a^{ij}}{\partial p}\right| + |a| \leqslant M_1 \equiv \text{const} > 0 \quad \text{on } \mathscr{F}_{\Omega,m,M}. \tag{2.7}$$

*Then there exists a number* $\gamma \in (0,1)$, *depending only on* $n$, $\nu$, $M$ *and* $M_1$, *such that for any subdomain* $\Omega'$, $\bar{\Omega}' \subset \Omega$,

$$\|\nabla u\|_{C^\gamma(\bar{\Omega}')} \leqslant c_1, \tag{2.8}$$

*where* $c_1$ *depends only on* $n$, $\nu$, $M$, $M_1$, *and the distance from* $\Omega'$ *to* $\partial\Omega$. *If the domain* $\Omega$ *belongs to the class* $C^2$ *and* $u = \varphi(x)$ *on* $\partial\Omega$, *where* $\varphi \in C^2(\bar{\Omega})$, *then*

$$\|\nabla u\|_{C^\gamma(\bar{\Omega})} \leqslant c_2, \tag{2.9}$$

terms of the majorants $\mathscr{E}_1$ and $\mathscr{E}_2$. Here it is also important to note that the structure of these conditions and the character of the basic a priori estimates established for solutions of (2) do not depend on the "parabolicity constant" of the equation. This determines at the outset the possibility of using the results obtained to study in addition boundary value problems for quasilinear degenerate parabolic equations. In view of the results of Ladyzhenskaya and Ural'tseva (see [80]), the proof of classical solvability of the first boundary value problem for equations of the form (2) can be reduced to establishing an a priori estimate of $\max_Q|\nabla u|$, where $\nabla u$ is the spatial gradient, for solutions of a one-parameter family of equations (2) having the same structure as the original equation (see §2.1).

# Giaquinta & Giusti's comments

Our technique is a nonlinear version of the well-known method of freezing the coefficients $A^j$ at a point $x_0$, and then using a perturbation argument. A special form of De Giorgi's theorem is needed that requires linear growth for the $A^i$ and at most a quadratic growth for $B$. However, the general case of coefficients $A^i$ and $B$ of arbitrary growth can easily be reduced to this once a gradient bound has been proved. This happens for instance for the minimal surface equation

$$\operatorname{div}\left\{\frac{Du}{\sqrt{1+|Du|^2}}\right\}=0$$

for which a gradient estimate for $C^{1,\alpha}$ boundary values has been proved in [10] (see also [13]).

# Solutions - Nonuniformly elliptic Schauder theory

- Solutions in a paper by D. & Mingione 2021.
- Catches both cases of non-differentiable functionals and equations with Hölder continuous coefficients.
- Introduces a hybrid perturbation approach which is direct and does not rely on **freezing and comparing**.
- Crucial point in the proof is to get **$L^\infty$-bounds for the gradient**.

# Schauder's theory does not always hold...

$$w \mapsto \int_\Omega \left[|Dw|^p + a(x)|Dw|^q\right] \ dx$$

$$1 < p \le q, \qquad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega)$$

To hope for regularity, the ellipticity ratios cannot blow up too fast:

$$\frac{q}{p} < 1 + \mathrm{o}(n, \alpha).$$

This is confirmed by **counterexamples**.

- If $1 < p < n < n + \alpha < q$, functionals with $(p, q)$-growth may admit a minimizer $u \notin W^{1,q}_{loc}(B_1)$. ⇜ Esposito, Leonetti & Mingione JDE (2004).
- For any $\varepsilon > 0$, there exists a minimizer $u \in W^{1,p}(\Omega)$ of functional with $(p, q)$-growth, such that $\dim_{\mathcal{H}}(\Sigma(u)) > n - p - \varepsilon$. The singular set is a fractal of Cantor type. ⇜ Fonseca, Malý & Mingione ARMA (2004); Balci, Diening & Surnachev Calc. Var. 2020.

# ...but sometimes yes

**Theorem (Baroni, Colombo & Mingione Calc. Var. 2018)**

*Let u be a minimizer of the Double Phase energy*

$$w \mapsto \int_\Omega \left[|Dw|^p + a(x)|Dw|^q\right] \, dx$$

$$1 < p \le q, \qquad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega)$$

*Then:*

$$\frac{q}{p} \le 1 + \frac{\alpha}{n} \implies u \in C^{1,\beta_0}_{\text{loc}}(\Omega).$$

- Previous results by Colombo & Mingione, ARMA (2 papers) 2015 .
- The bound on $q/p$ is sharp, thanks to counterexamples of Esposito, Leonetti & Mingione (JDE 2004), Fonseca, Malý & Mingione (ARMA 2004).

# Half-way nonuniformly elliptic

The double phase functional

$$w \mapsto \int_\Omega (|Dw|^p + a(x)|Dw|^q)\, dx \equiv \int_\Omega F(x, Dw)\, dx$$

allows for treating Hölder coefficients but is pointwise uniformly elliptic, in the sense that

$$\mathcal{R}(x,z) = \frac{\text{highest eigenvalue of } \partial_{zz}F(x,z)}{\text{lowest eigenvalue of } \partial_{zz}F(x,z)} < \infty$$

$$\color{red}{\mathcal{R}(z,B)} = \frac{\sup_{x\in B} \text{highest eigenvalue of } \partial_{zz}F(x,z)}{\inf_{x\in B} \text{lowest eigenvalue of } \partial_{zz}F(x,z)} \to \infty$$

# Pointwise uniform ellipticity - special structures

Regularity results available for special structures:

$$w \mapsto \int_\Omega \tilde{F}(x, |Dw|)\, dx$$

- Acerbi & Mingione ARMA 2001.
- Baasandorj, Byun & Oh JFA 2020; Calc. Var. 2021.
- Baroni, Colombo & Mingione ARMA 2015; Calc. Var. 2018.
- D. & Oh JDE 2019.
- Hästo & Ok JEMS 2021.
- Karppinen & Lee *IMRN* 2021.

*In all such cases we have that the frozen integrand*

$$z \mapsto \tilde{F}(x_0, |z|)$$

*is uniformly elliptic.*

- This does not happen for basic model examples as

$$w \mapsto \int_\Omega c(x) F(Dw)\, dx \equiv \int_\Omega \bar{F}(x, Dw)\, dx$$

under genuine nonuniform ellipticity

$$|z|^{p-2}\mathbb{I}_d \le \partial_{zz} F(z) \le |z|^{q-2}\mathbb{I}_d$$

- Freezing yields

$$\mathcal{R}_{\partial_z \bar{F}(x_0,\cdot)} \approx \frac{\text{highest eigenvalue of } \partial_{zz}\bar{F}(x_0, z)}{\text{lowest eigenvalue of } \partial_{zz}\bar{F}(x_0, z)} \approx \frac{\text{highest eigenvalue of } \partial_{zz}F(z)}{\text{lowest eigenvalue of } \partial_{zz}F(z)} \approx |z|^{q-p}\,.$$

**Theorem (D. & Mingione, 2021)**

*Let $u \in W^{1,1}$ be a minimizer of the functional*

$$w \mapsto \int_\Omega c(x) F(Dw)\, dx$$

*where*

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz} F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $c(\cdot) \in C^{0,\alpha}(\Omega)$
- *and*

$$\frac{q}{p} \le 1 + \frac{1}{5}\left(\frac{\alpha}{n}\right)^2.$$

*Then $Du$ is locally Hölder continuous in $\Omega$.*

For solutions to suitable Dirichlet problems

$$\begin{cases} -\operatorname{div} A(x, Du) = 0 & \text{in } \Omega \\ u \equiv u_0 & \text{on } \partial\Omega\,, \end{cases} \qquad u_0 \in W^{1,\frac{p(q-1)}{p-1}}(\Omega)\,,$$

where

$$|z|^{p-2}\mathbb{I}_d \lesssim \partial_z A(x,z) \lesssim |z|^{q-2}\mathbb{I}_d$$

we have

**Theorem (D. & Mingione, 2021)**

*If*

$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(\frac{\alpha}{n}\right)^{2},$$

*then there exists a solution to the above Dirichlet problem such that $Du$ is locally Hölder continuous in $\Omega$.*

# Non-differentiable functionals #1

**Theorem (D. & Mingione, 2021)**

*Let $u \in W^{1,1}$ be a minimizer of the functional*

$$w \mapsto \int_\Omega [F(Dw) + h(x,w)]\,dx$$

*where*

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d, \qquad 1 < p < q$
- $|h(x,v_1) - h(x,v_2)| \lesssim |v_1 - v_2|^\alpha, \quad \alpha \in (0,1]$
- *and*

$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(1 - \frac{\alpha}{p}\right)\frac{\alpha}{n}.$$

*Then $Du$ is locally Hölder continuous in $\Omega$.*

# Non-differentiable functionals #2

**Theorem (D. & Mingione, 2021)**

*Let $u \in W^{1,1}$ be a minimizer of the functional*

$$w \mapsto \int_\Omega [F(Dw) + g(x, w, Dw) + h(x, w)]\, dx$$

*where $F(\cdot)$ and $h(\cdot)$ are in Theorem 1, and*

- $z \mapsto g(\cdot, z)$ *is convex and* $|\partial_{zz} g(\cdot, z)| \lesssim |z|^{\gamma-2}$
- $|g(x, v_1, z) - g(x, v_2, z)| \lesssim \left(|x_1 - x_2|^\alpha + |v_1 - v_2|^\alpha\right) |z|^\gamma$
- $\alpha + \gamma < p$
- *and*
$$\frac{q}{p} \le 1 + \frac{1}{5}\left(1 - \frac{\alpha + \gamma}{p}\right)\frac{\alpha}{n}.$$

*Then $Du$ is locally Hölder continuous in $\Omega$.*

# Non-differentiable functionals #3

**Theorem (D. & Mingione, 2021)**

*Let $u \in W^{1,1}$ be a minimizer of the functional*

$$w \mapsto \int_\Omega c(x,w)F(Dw)\,dx$$

*where*

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $c(\cdot) \in C^{0,\alpha}(\Omega)\,, \quad \alpha \in (0,1]$
- $p > n$
- *and*

$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(1 - \frac{n}{p}\right)\left(\frac{\alpha}{n}\right)^2.$$

*Then $Du$ is locally Hölder continuous in $\Omega$.*

# Some comments

- The role of the assumption

$$p > n$$

compensates the lack of a priori continuity of $u$ which is known in the case $p = q$.

- When $p = q$ by De Giorgi's theory the local Hölder continuity of $u$ is just implied by

$$|z|^p \lesssim F(x, v, z) \lesssim |z|^p + 1$$

with no convexity used.

- $p > n$ implies that $u$ is Hölder continuous, but this fact is not used in the proof.
- This assumption could be optimal.

# A classical theorem of Stein

**Theorem (Stein Ann. Math. 1981)**

$$Dw \in L(n,1) \implies w \text{ is continuous}$$

- Recall that

$$f \in L(t,\gamma) \iff \int_0^\infty \left(\lambda^t |\{x : |f(x)| > \lambda\}|\right)^{\gamma/t} \frac{d\lambda}{\lambda} < \infty$$

- This is the borderline case of Sobolev-Morrey theorem.
- Combining Stein's theorem with standard Calderón-Zygmund theory we have

**Theorem (Stein Ann. Math. 1981)**

$$-\Delta u \in L(n,1) \implies Du \;\; \text{is continuous},$$

- Cianchi J. Geom. Anal. (1993).

# A Nonlinear Stein theorems

**Theorem (Kuusi & Mingione, Calc. Var. 2014)**

*If $u \in W^{1,p}$ solves*

$$-\mathrm{div}\,(c(x)|Du|^{p-2}Du) = f$$

*where $c(\cdot)$ is Dini-continuous and bounded away from zero. Then*

$$f \in L(n,1) \implies Du \text{ is continuous}.$$

See also Kuusi & Mingione, ARMA 2013, JEMS 2018.

# Stein theorems for polynomial nonuniform ellipticity

Consider the functional

$$w \mapsto \int_\Omega [F(x, |Dw|) - f(x) \cdot w] \, dx$$

with

$$|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz} F(x, |z|) \lesssim |z|^{q-2}\mathbb{I}_d \, .$$

**Theorem (D. & Mingione, ARMA 2021)**

*Let $u \in W^{1,1}(\Omega)$ be a vector-valued minimizer of the functional and assume that*

$$x \mapsto \partial_z F(x, \cdot) \in W^{1,d} \qquad \text{and} \qquad \frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{d} \, .$$

*Then*

$$f \in L(n,1) \implies Du \text{ is continuous} \, .$$

See also Beck & Mingione, CPAM 2020.

# Non-differentiable functionals #1

**Theorem (D. & Mingione, 2021)**

*Let $u \in W^{1,1}$ be a minimizer of the functional*

$$w \mapsto \int_\Omega [F(Dw) + \hbar(x,w)]\,dx$$

*where*

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $|\hbar(x,v_1) - \hbar(x,v_2)| \lesssim |v_1 - v_2|^\alpha\,, \quad \alpha \in (0,1]$
- *and*
$$\frac{q}{p} \le 1 + \frac{1}{5}\left(1 - \frac{\alpha}{p}\right)\frac{\alpha}{n}\,.$$

*Then $Du$ is locally Hölder continuous in $\Omega$.*

# A new Lorentz borderline condition

**Theorem (D. & Mingione, 2021)**

*Let $u \in W^{1,1}$ be a minimizer of the functional*

$$w \mapsto \int_\Omega [F(Dw) + h(x,w)]\, dx$$

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $|h(x,v_1) - h(x,v_2)| \lesssim f(x)|v_1 - v_2|^\alpha, \quad \alpha \in (0,1]$
- $f \in L(n/\alpha, 1/2)$
- *and*

$$\frac{q}{p} \le 1 + \frac{1}{5}\left(1 - \frac{\alpha}{p}\right)\frac{\alpha}{n}.$$

*Then $Du$ is locally bounded in $\Omega$.*

# Escaping polynomial nonuniform ellipticity

We consider nonuniformly elliptic functionals with fast growth of the type

$$v \mapsto \int_\Omega \exp(|Dv|^p)\, dx\,, \quad p > 1\,,$$

and, more in general, of the type

$$v \mapsto \int_\Omega \exp(\exp(\ldots \exp(|Dv|^p)\ldots))\, dx\,, \quad p > 1\,.$$

These have been considered at length in the literature:

- Duc & Eels (Harmonic type mappings)
- Lieberman
- Marcellini
- L. C. Evans (weak KAM theory).

# Escaping polynomial nonuniform ellipticity

Consider functional

$$w \mapsto \int_\Omega \gamma(x) \left[ \exp(\gamma_1(x) \exp(\ldots \exp(\gamma_2(x)|Dw|^{p(x)}) \ldots)) - f(x) \cdot w \right] dx$$

**Theorem (D. & Mingione, ARMA 2021)**

*Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a vector-valued local minimizer, under assumptions*

- $D\gamma, D\gamma_1, D\gamma_2, Dp \in L^{n+\varepsilon}$ *for some* $\varepsilon > 0$
- $f \in L(n,1)$.

*Then $Du$ is locally bounded in $\Omega$.*

Remark: Any rate of Hölder continuity exponent of coefficients suffices.

# Applications to obstacle problems

$$w \mapsto \min_{\mathcal{K}_{\psi}(\Omega)} \int_{\Omega} \gamma_1(x) \exp(\exp(\ldots \exp(\gamma_2(x)|Dw|^{p(x)}) \ldots)) \, dx$$

$$\mathcal{K}_{\psi}(\Omega) := \left\{ w \in W^{1,1}_{\mathrm{loc}}(\Omega) \colon w(x) \geq \psi(x) \text{ in } \Omega \right\} .$$

After approximation and linearization, solutions to obstacle problems can be rearranged as a solutions to

$$-\mathrm{div}\, \partial_z F(x, |Du|) = f \qquad |f| \lesssim |D^2\psi| + |D\psi| + 1,$$

$$u \to \min \int_{\Omega} [F(x, |Dw|) - f \cdot w] \, dx .$$

- Fuchs, Nonlinear Anal. 1990.
- Fuchs & Mingione, Manuscripta Math. 2000.

Consider functional

$$w \mapsto \int_{\Omega} \gamma(x) \exp(\gamma_1(x) \exp(\ldots \exp(\gamma_2(x)|Dw|^{p(x)}) \ldots)) \, dx \, .$$

**Theorem (D. & Mingione, ARMA 2021)**

*Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a scalar constrained local minimizer (in the class $\mathcal{K}_{\psi}(\Omega)$ of functions lying above the obstacle $\psi$) under assumptions*

- $D\gamma, D\gamma_1, D\gamma_2, Dp \in L^{n+\varepsilon}$ *for some* $\varepsilon > 0$
- $D^2\psi \in L(n,1)$.

*Then $Du$ is locally bounded in $\Omega$.*

# Sharpness of $D^2\psi \in L(n,1)$

In the simplest, linear case

$$u \to \min_{\mathcal{K}_{\psi}(\Omega)} \int_{\Omega} |Dw|^2 \, dx$$

the linearization procedure leads to

$$-\triangle u = -1_{\{u=\psi\}} \triangle \psi =: f \, .$$

By standard Calderón-Zygmund theory we have

$$\triangle\psi \in L(n,1) \Longleftrightarrow D^2\psi \in L(n,1) \, .$$

- Cianchi J. Geom. Anal. 1993.

# Further references


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