A heat equation approach to some problems in conformal geometry

Nicola Garofalo
University of Padova

July 19, 2021
Asia-Pacific Analysis & PDE Seminar

I want to thank Enrico for the kind invitation to speak in the Asia-Pacific seminar. I have never been to Australia (I had to sadly cancel at the last moment an invitation from Neil Trudinger because one of our kids was about to be born), and my first (and last) trip to Asia was a beautiful five week visit to China back in September 1978 (Beijing$\rightarrow$ Guangzhou$\rightarrow$ Hangzhou$\rightarrow$ Shanghai $\rightarrow$ Nanjing$\rightarrow$Beijing)! So it is very nice to be "visiting" the Asia-Pacific region again after such a long time.

I want to thank Enrico for the kind invitation to speak in the Asia-Pacific seminar. I have never been to Australia (I had to sadly cancel at the last moment an invitation from Neil Trudinger because one of our kids was about to be born), and my first (and last) trip to Asia was a beautiful five week visit to China back in September 1978 (Beijing→ Guangzhou→ Hangzhou→ Shanghai → Nanjing→Beijing)! So it is very nice to be "visiting" the Asia-Pacific region again after such a long time.

Sunset in Perth

Preface

[1] E. Cartan, 1928 "Sur la represéntation géométrique des systèmes matériels non holonomes", Proc. Int. Congr. Math., Bologna, 4: 253-261

Preface

In this talk I present recent joint works with Giulio Tralli which revolve around the heat equation in a class of geometric ambients which, besides their mathematical relevance, are of considerable interest in the applied sciences: quantum mechanics; physics of semi-flexible polymers; non-holonomic mechanics (e.g. control of the motion of the arms of a robot); physiology of neurovision; formation of crystalline structures...

These geometric ambients model physical systems with constrained dynamics, in which motion is only possible in a prescribed set of directions in the tangent space (sub-Riemannian [1], versus Riemannian geometry).

The key redeeming feature is that the missing directions in the tangent space are recovered by taking a sufficiently large number of commutators of the vector fields which describe the PDEs of interest.

[1] E. Cartan, 1928 "Sur la representéntation géométrique des systèmes matériels non holonomes", Proc. Int. Congr. Math., Bologna, 4: 253-261

[2] P. Woit, *Quantum theory, groups and representations. An introduction*. Springer, Cham, 2017. xxii+668 pp.

The relevant framework for these PDEs are non-Abelian Lie groups $\mathbb{G}$ (Riemannian manifolds with a smooth non-commutative group law) whose Lie algebra (the tangent space at the group identity) possesses a special layered structure suggested by the physical problem at hand.

The most important of these Lie groups is the ubiquitous $2n+1$-dimensional Heisenberg group $\mathbb{H}^n$, first introduced by H. Weyl in his group representation theory approach to quantum mechanics [2].

$\mathbb{H}^n$ is equipped with a conformally invariant PDO, the so-called horizontal Laplacian $\mathscr{L}$. Given $s \in (0,1)$, I will indicate by $\mathscr{L}^s \overset{def}{=} (-\mathscr{L})^s$ the fractional powers of this operator. A natural question is whether these nonlocal operators retain the geometric properties of $-\mathscr{L}$.

Unfortunately, unlike what happens for $(-\Delta)^s$, the pseudodifferential operators $\mathscr{L}^s$ do not preserve the conformal invariances of the local operator $\mathscr{L}$!

[2] P. Woit, *Quantum theory, groups and representations. An introduction*. Springer, Cham, 2017. xxii+668 pp.

[3] C. Fefferman & C. R. Graham, *Conformal invariants*. In "The mathematical heritage of Élie Cartan (Lyon, 1984)", Astérisque (1985), 95-116

My focus in this talk will instead be on a class of nonlocal operators that are conformally invariant in $\mathbb{H}^n$, or more in general in Lie groups of Heisenberg type. These nonlocal operators, which I will denote by $\mathscr{L}_s$ to distinguish them from $\mathscr{L}^s$, come from CR (Cauchy-Riemann) geometry [3].

The objective of my talk is to present a new approach, based on the heat equation and some of its variants, to:

[3] C. Fefferman & C. R. Graham, *Conformal invariants*. In "The mathematical heritage of Élie Cartan (Lyon, 1984)", Astérisque (1985), 95-116

My focus in this talk will instead be on a class of nonlocal operators that are conformally invariant in $\mathbb{H}^n$, or more in general in Lie groups of Heisenberg type. These nonlocal operators, which I will denote by $\mathscr{L}_s$ to distinguish them from $\mathscr{L}^s$, come from CR (Cauchy-Riemann) geometry [3].

The objective of my talk is to present a new approach, based on the heat equation and some of its variants, to:

(1) prove the invertibility of the fractional powers $\mathscr{L}_s$;

[3] C. Fefferman & C. R. Graham, *Conformal invariants*. In "The mathematical heritage of Élie Cartan (Lyon, 1984)", Astérisque (1985), 95-116

My focus in this talk will instead be on a class of nonlocal operators that are conformally invariant in $\mathbb{H}^n$, or more in general in Lie groups of Heisenberg type. These nonlocal operators, which I will denote by $\mathscr{L}_s$ to distinguish them from $\mathscr{L}^s$, come from CR (Cauchy-Riemann) geometry [3].

The objective of my talk is to present a new approach, based on the heat equation and some of its variants, to:

(1) prove the invertibility of the fractional powers $\mathscr{L}_s$;

(2) find explicit formulas for the fundamental solutions of $\mathscr{L}_s$;

[3] C. Fefferman & C. R. Graham, *Conformal invariants*. In "The mathematical heritage of Élie Cartan (Lyon, 1984)", Astérisque (1985), 95-116

My focus in this talk will instead be on a class of nonlocal operators that are conformally invariant in $\mathbb{H}^n$, or more in general in Lie groups of Heisenberg type. These nonlocal operators, which I will denote by $\mathscr{L}_s$ to distinguish them from $\mathscr{L}^s$, come from CR (Cauchy-Riemann) geometry [3].

The objective of my talk is to present a new approach, based on the heat equation and some of its variants, to:

(1) prove the invertibility of the fractional powers $\mathscr{L}_s$;

(2) find explicit formulas for the fundamental solutions of $\mathscr{L}_s$;

(3) prove some intertwining formulas for $\mathscr{L}_s$ which are connected to the conformal fractional CR Yamabe problem

$$\mathscr{L}_s u = u^{\frac{Q+2s}{Q-2s}}.$$

(the meaning of the word "intertwining" and of the "dimension" Q will be clarified later in my talk).

[3] C. Fefferman & C. R. Graham, *Conformal invariants*. In "The mathematical heritage of Élie Cartan (Lyon, 1984)", Astérisque (1985), 95-116

...I dragged up from the dark abyss things of strange aspect and strange beauty...(Rabindranath Tagore)

[4] This is equivalent to M. Riesz' 1938 definition of the fractional Laplacian

$$(-\Delta)^s f(x) = -\frac{s 2^{2s}\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(1-s)} \mathrm{PV}\int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} dy$$

...I dragged up from the dark abyss things of strange aspect and strange beauty...(Rabindranath Tagore)

I will start with discussing a model question which, as I will show, encompasses parts (1)-(3) of the plan of my talk. In $\mathbb{R}^n$ with $n \geq 2$, for $0 < s < 1$ consider the pseudodifferential operator which in Fourier transform is given by $\widehat{(-\Delta)^s f}(\xi) = (2\pi|\xi|)^{2s}\hat{f}(\xi)$ [4]. Then, for every $x \in \mathbb{R}^n$, and $y > 0$ the following nonlocal equation holds:

[4] This is equivalent to M. Riesz' 1938 definition of the fractional Laplacian

$$(-\Delta)^s f(x) = -\frac{s 2^{2s}\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(1-s)} \text{PV}\int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} dy$$

...I dragged up from the dark abyss things of strange aspect and strange beauty...(Rabindranath Tagore)

I will start with discussing a model question which, as I will show, encompasses parts (1)-(3) of the plan of my talk. In $\mathbb{R}^n$ with $n \geq 2$, for $0 < s < 1$ consider the pseudodifferential operator which in Fourier transform is given by $\widehat{(-\Delta)^s f}(\xi) = (2\pi|\xi|)^{2s}\hat{f}(\xi)$ [4]. Then, for every $x \in \mathbb{R}^n$, and $y > 0$ the following nonlocal equation holds:

$$(-\Delta)^s \left((|x|^2 + y^2)^{-\frac{n-2s}{2}}\right) = \frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma\left(\frac{n}{2}-s\right)} (2y)^{2s} \left(|x|^2 + y^2\right)^{-\frac{n+2s}{2}}. \qquad (0.1)$$

[4] This is equivalent to M. Riesz' 1938 definition of the fractional Laplacian

$$(-\Delta)^s f(x) = -\frac{s 2^{2s}\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(1-s)} \mathrm{PV} \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} dy$$

[5] if you are interested you can check Section 8 of my survey article Fractional thoughts. *New developments in the analysis of nonlocal operators, 1-135, Contemp. Math., 723, Amer. Math. Soc., Providence, RI, 2019*

A direct proof of (0.1) is by Fourier transform and ultimately hinges on some integral formulas involving special functions [5]. Such proof is (implicitly) known at least since the celebrated 1983 work of E. Lieb concerning the best constants in the Hardy-Littlewood-Sobolev inequalities.

[5] if you are interested you can check Section 8 of my survey article Fractional thoughts. *New developments in the analysis of nonlocal operators, 1-135, Contemp. Math., 723, Amer. Math. Soc., Providence, RI, 2019*

A direct proof of (0.1) is by Fourier transform and ultimately hinges on some integral formulas involving special functions [5]. Such proof is (implicitly) known at least since the celebrated 1983 work of E. Lieb concerning the best constants in the Hardy-Littlewood-Sobolev inequalities.

Observe that a notable consequence of (0.1) is that it provides a family of positive solutions to the nonlocal Yamabe equation in $\mathbb{R}^n$

[5] if you are interested you can check Section 8 of my survey article Fractional thoughts. *New developments in the analysis of nonlocal operators, 1-135, Contemp. Math., 723, Amer. Math. Soc., Providence, RI, 2019*

A direct proof of (0.1) is by Fourier transform and ultimately hinges on some integral formulas involving special functions [5]. Such proof is (implicitly) known at least since the celebrated 1983 work of E. Lieb concerning the best constants in the Hardy-Littlewood-Sobolev inequalities.

Observe that a notable consequence of (0.1) is that it provides a family of positive solutions to the nonlocal Yamabe equation in $\mathbb{R}^n$

$$(-\Delta)^s f = f^{\frac{n+2s}{n-2s}}. \tag{0.2}$$

[5] if you are interested you can check Section 8 of my survey article Fractional thoughts. *New developments in the analysis of nonlocal operators, 1-135, Contemp. Math., 723, Amer. Math. Soc., Providence, RI, 2019*

[6] more recently, Nyström and Sande (2016) and Stinga and Torrea (2017) independently generalised Jones' results to the case $0 < s < 1$

To introduce the main theme of my talk let me present an alternative heat equation proof of the intertwining formula (0.1) which does not use the Fourier transform and/or any formula from special functions.

[6] more recently, Nyström and Sande (2016) and Stinga and Torrea (2017) independently generalised Jones' results to the case $0 < s < 1$

To introduce the main theme of my talk let me present an alternative heat equation proof of the intertwining formula (0.1) which does not use the Fourier transform and/or any formula from special functions. Instead of looking at $(-\Delta)^s$, consider the fractional heat operator $(\partial_t - \Delta)^s$ and its extension problem:

[6] more recently, Nyström and Sande (2016) and Stinga and Torrea (2017) independently generalised Jones' results to the case $0 < s < 1$

To introduce the main theme of my talk let me present an alternative heat equation proof of the intertwining formula (0.1) which does not use the Fourier transform and/or any formula from special functions. Instead of looking at $(-\Delta)^s$, consider the fractional heat operator $(\partial_t - \Delta)^s$ and its extension problem: given a function $f \in C_0^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$, find $U \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t \times \mathbb{R}^+_y)$ such that

[6]more recently, Nyström and Sande (2016) and Stinga and Torrea (2017) independently generalised Jones' results to the case $0 < s < 1$

To introduce the main theme of my talk let me present an alternative heat equation proof of the intertwining formula (0.1) which does not use the Fourier transform and/or any formula from special functions. Instead of looking at $(-\Delta)^s$, consider the fractional heat operator $(\partial_t - \Delta)^s$ and its extension problem: given a function $f \in C_0^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$, find $U \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t \times \mathbb{R}^+_y)$ such that

$$\begin{cases} \mathfrak{P}^{(s)} U \overset{def}{=} \frac{\partial^2 U}{\partial y^2} + \frac{1-2s}{y} \frac{\partial U}{\partial y} + \Delta_x U - \frac{\partial U}{\partial t} = 0, \\ U(x,t,0) = f(x,t). \end{cases} \tag{0.3}$$

[6] more recently, Nyström and Sande (2016) and Stinga and Torrea (2017) independently generalised Jones' results to the case $0 < s < 1$

To introduce the main theme of my talk let me present an alternative heat equation proof of the intertwining formula (0.1) which does not use the Fourier transform and/or any formula from special functions. Instead of looking at $(-\Delta)^s$, consider the fractional heat operator $(\partial_t - \Delta)^s$ and its extension problem: given a function $f \in C_0^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$, find $U \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t \times \mathbb{R}^+_y)$ such that

$$\begin{cases} \mathfrak{P}^{(s)} U \overset{def}{=} \frac{\partial^2 U}{\partial y^2} + \frac{1-2s}{y} \frac{\partial U}{\partial y} + \Delta_x U - \frac{\partial U}{\partial t} = 0, \\ U(x,t,0) = f(x,t). \end{cases} \tag{0.3}$$

The "Dirichlet problem" (0.3) was first introduced when $s = 1/2$ in a beautiful (but not so well-known) 1968 pioneering paper by F. Jones, who first constructed an explicit Poisson kernel [6].

[6] more recently, Nyström and Sande (2016) and Stinga and Torrea (2017) independently generalised Jones' results to the case $0 < s < 1$

To introduce the main theme of my talk let me present an alternative heat equation proof of the intertwining formula (0.1) which does not use the Fourier transform and/or any formula from special functions. Instead of looking at $(-\Delta)^s$, consider the fractional heat operator $(\partial_t - \Delta)^s$ and its extension problem: given a function $f \in C_0^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$, find $U \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t \times \mathbb{R}^+_y)$ such that

$$\begin{cases} \mathfrak{P}^{(s)} U \overset{def}{=} \frac{\partial^2 U}{\partial y^2} + \frac{1-2s}{y} \frac{\partial U}{\partial y} + \Delta_x U - \frac{\partial U}{\partial t} = 0, \\ U(x,t,0) = f(x,t). \end{cases} \tag{0.3}$$

The "Dirichlet problem" (0.3) was first introduced when $s = 1/2$ in a beautiful (but not so well-known) 1968 pioneering paper by F. Jones, who first constructed an explicit Poisson kernel [6].

I now make the claim that the conformal invariances of (0.1) are embedded in the fundamental solution $q^{(s)}(x,y,t)$ of the parabolic operator $\mathfrak{P}^{(s)}$ in (0.3).

[6] more recently, Nyström and Sande (2016) and Stinga and Torrea (2017) independently generalised Jones' results to the case $0 < s < 1$

9 / 46

To see this, note that if $w \in \mathbb{R}^{2(1-s)}$ and $y = |w|$, then $\mathfrak{P}^{(s)}$ represents the action on cylindrically symmetric functions $U(x, w, t) = \overline{U}(x, y, t)$ of the heat operator $\Delta_x + \Delta_w - \partial_t$ in the space with fractal dimension $\mathbb{R}^{n+2(1-s)} \times \mathbb{R}^+_t$.

To see this, note that if $w \in \mathbb{R}^{2(1-s)}$ and $y = |w|$, then $\mathfrak{P}^{(s)}$ represents the action on cylindrically symmetric functions $U(x, w, t) = \overline{U}(x, y, t)$ of the heat operator $\Delta_x + \Delta_w - \partial_t$ in the space with fractal dimension $\mathbb{R}^{n+2(1-s)} \times \mathbb{R}^+_t$. Its fundamental solution (with pole at the origin) is given by

To see this, note that if $w \in \mathbb{R}^{2(1-s)}$ and $y = |w|$, then $\mathfrak{P}^{(s)}$ represents the action on cylindrically symmetric functions $U(x, w, t) = \overline{U}(x, y, t)$ of the heat operator $\Delta_x + \Delta_w - \partial_t$ in the space with fractal dimension $\mathbb{R}^{n+2(1-s)} \times \mathbb{R}^+_t$. Its fundamental solution (with pole at the origin) is given by

$$q^{(s)}(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}+1-s}} e^{-\frac{|x|^2+y^2}{4t}} = \frac{1}{(4\pi t)^{1-s}} e^{-\frac{y^2}{4t}} p(x, 0, t). \tag{0.4}$$

To see this, note that if $w \in \mathbb{R}^{2(1-s)}$ and $y = |w|$, then $\mathfrak{P}^{(s)}$ represents the action on cylindrically symmetric functions $U(x, w, t) = \overline{U}(x, y, t)$ of the heat operator $\Delta_x + \Delta_w - \partial_t$ in the space with fractal dimension $\mathbb{R}^{n+2(1-s)} \times \mathbb{R}^+_t$. Its fundamental solution (with pole at the origin) is given by

$$q^{(s)}(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}+1-s}} e^{-\frac{|x|^2+y^2}{4t}} = \frac{1}{(4\pi t)^{1-s}} e^{-\frac{y^2}{4t}} p(x, 0, t). \qquad (0.4)$$

If we denote by $q^{(-s)}(x, y, t)$ the heat kernel obtained by replacing s into $-s$ in (0.4), then by Bochner's subordination the two functions

To see this, note that if $w \in \mathbb{R}^{2(1-s)}$ and $y = |w|$, then $\mathfrak{P}^{(s)}$ represents the action on cylindrically symmetric functions $U(x, w, t) = \overline{U}(x, y, t)$ of the heat operator $\Delta_x + \Delta_w - \partial_t$ in the space with fractal dimension $\mathbb{R}^{n+2(1-s)} \times \mathbb{R}^+_t$. Its fundamental solution (with pole at the origin) is given by

$$q^{(s)}(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}+1-s}} e^{-\frac{|x|^2+y^2}{4t}} = \frac{1}{(4\pi t)^{1-s}} e^{-\frac{y^2}{4t}} p(x, 0, t). \qquad (0.4)$$

If we denote by $q^{(-s)}(x, y, t)$ the heat kernel obtained by replacing s into $-s$ in (0.4), then by Bochner's subordination the two functions

$$E^{(\pm s)}(x, y) \stackrel{def}{=} \int_0^\infty q^{(\pm s)}(x, y, t)dt \qquad (0.5)$$

are the fundamental solutions of the time-independent differential operators $\frac{\partial^2}{\partial y^2} + \frac{1 \mp 2s}{y} \frac{\partial}{\partial y} + \Delta_x$.

To see this, note that if $w \in \mathbb{R}^{2(1-s)}$ and $y = |w|$, then $\mathfrak{P}^{(s)}$ represents the action on cylindrically symmetric functions $U(x, w, t) = \overline{U}(x, y, t)$ of the heat operator $\Delta_x + \Delta_w - \partial_t$ in the space with fractal dimension $\mathbb{R}^{n+2(1-s)} \times \mathbb{R}^+_t$. Its fundamental solution (with pole at the origin) is given by

$$q^{(s)}(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}+1-s}} e^{-\frac{|x|^2+y^2}{4t}} = \frac{1}{(4\pi t)^{1-s}} e^{-\frac{y^2}{4t}} p(x, 0, t). \qquad (0.4)$$

If we denote by $q^{(-s)}(x, y, t)$ the heat kernel obtained by replacing s into $-s$ in (0.4), then by Bochner's subordination the two functions

$$E^{(\pm s)}(x, y) \overset{def}{=} \int_0^\infty q^{(\pm s)}(x, y, t)dt \qquad (0.5)$$

are the fundamental solutions of the time-independent differential operators $\frac{\partial^2}{\partial y^2} + \frac{1 \mp 2s}{y} \frac{\partial}{\partial y} + \Delta_x$. Now we use the elementary (but very important) consequence of (0.4) that

To see this, note that if $w \in \mathbb{R}^{2(1-s)}$ and $y = |w|$, then $\mathfrak{P}^{(s)}$ represents the action on cylindrically symmetric functions $U(x,w,t) = \overline{U}(x,y,t)$ of the heat operator $\Delta_x + \Delta_w - \partial_t$ in the space with fractal dimension $\mathbb{R}^{n+2(1-s)} \times \mathbb{R}^+_t$. Its fundamental solution (with pole at the origin) is given by

$$q^{(s)}(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}+1-s}} e^{-\frac{|x|^2+y^2}{4t}} = \frac{1}{(4\pi t)^{1-s}} e^{-\frac{y^2}{4t}} p(x,0,t). \tag{0.4}$$

If we denote by $q^{(-s)}(x,y,t)$ the heat kernel obtained by replacing s into $-s$ in (0.4), then by Bochner's subordination the two functions

$$E^{(\pm s)}(x,y) \overset{def}{=} \int_0^\infty q^{(\pm s)}(x,y,t)dt \tag{0.5}$$

are the fundamental solutions of the time-independent differential operators $\frac{\partial^2}{\partial y^2} + \frac{1 \mp 2s}{y}\frac{\partial}{\partial y} + \Delta_x$. Now we use the elementary (but very important) consequence of (0.4) that

$$\int_0^\infty q^{(\pm s)}(x,y,t)dt = \frac{\Gamma(\frac{n \mp 2s}{2})}{\pi^{\frac{n}{2} \mp s}} (|x|^2 + y^2)^{-\frac{n \mp 2s}{2}}. \tag{0.6}$$

Then, in view of (0.5), (0.6) the problem of proving (0.1) is reduced to that of establishing the equivalent dimension-free intertwining relation

Then, in view of (0.5), (0.6) the problem of proving (0.1) is reduced to that of establishing the equivalent dimension-free intertwining relation

$$(-\Delta)^s \left(E^{(s)}(\cdot, y)\right)(x) \overset{?}{=} (2\pi y)^{2s}\, E^{(-s)}(x, y). \tag{0.7}$$

Then, in view of (0.5), (0.6) the problem of proving (0.1) is reduced to that of establishing the equivalent dimension-free intertwining relation

$$(-\Delta)^s \left(E^{(s)}(\cdot, y)\right)(x) \overset{?}{=} (2\pi y)^{2s}\, E^{(-s)}(x, y). \tag{0.7}$$

To prove (0.7) we once again bring to the center stage the heat semigroup $P_t = e^{-t\Delta}$.

Then, in view of (0.5), (0.6) the problem of proving (0.1) is reduced to that of establishing the equivalent dimension-free intertwining relation

$$(-\Delta)^s \left(E^{(s)}(\cdot, y)\right)(x) \overset{?}{=} (2\pi y)^{2s} E^{(-s)}(x, y). \tag{0.7}$$

To prove (0.7) we once again bring to the center stage the heat semigroup $P_t = e^{-t\Delta}$. Instead of the Fourier transform definition of $(-\Delta)^s$ we apply to the function $f = E^{(s)}(\cdot, y)$ the (equivalent) one given by the following beautiful formula of Balakrishnan (1959)

Then, in view of (0.5), (0.6) the problem of proving (0.1) is reduced to that of establishing the equivalent dimension-free intertwining relation

$$(-\Delta)^s \left(E^{(s)}(\cdot, y)\right)(x) \overset{?}{=} (2\pi y)^{2s}\, E^{(-s)}(x, y). \tag{0.7}$$

To prove (0.7) we once again bring to the center stage the heat semigroup $P_t = e^{-t\Delta}$. Instead of the Fourier transform definition of $(-\Delta)^s$ we apply to the function $f = E^{(s)}(\cdot, y)$ the (equivalent) one given by the following beautiful formula of Balakrishnan (1959)

$$\begin{aligned}(-\Delta)^s f(x) &= -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}}(P_t f(x) - f(x))dt \qquad (0.8)\\ &= -\frac{1}{\Gamma(1-s)} \int_0^\infty t^{-s}\partial_t P_t f(x)dt.\end{aligned}$$

Then, in view of (0.5), (0.6) the problem of proving (0.1) is reduced to that of establishing the equivalent dimension-free intertwining relation

$$(-\Delta)^s \left(E^{(s)}(\cdot, y)\right)(x) \overset{?}{=} (2\pi y)^{2s}\, E^{(-s)}(x, y). \tag{0.7}$$

To prove (0.7) we once again bring to the center stage the heat semigroup $P_t = e^{-t\Delta}$. Instead of the Fourier transform definition of $(-\Delta)^s$ we apply to the function $f = E^{(s)}(\cdot, y)$ the (equivalent) one given by the following beautiful formula of Balakrishnan (1959)

$$\begin{aligned}(-\Delta)^s f(x) &= -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}}(P_t f(x) - f(x))dt \qquad (0.8)\\ &= -\frac{1}{\Gamma(1-s)} \int_0^\infty t^{-s} \partial_t P_t f(x) dt.\end{aligned}$$

Now, the Chapman-Kolmogorov equation (semigroup property) gives

$$P_t(E^{(s)}(\cdot, y))(x) = \int_0^\infty \frac{1}{(4\pi\tau)^{1-s}} e^{-\frac{y^2}{4\tau}} p(x, 0, t+\tau) d\tau.$$

Substituting in (0.8), making the change of variable

$$(u, v) = (t + \tau, \frac{\tau}{t}),$$

and integrating by parts, we obtain **with elementary computations**

Substituting in (0.8), making the change of variable

$$(u, v) = (t+\tau, \frac{\tau}{t}),$$

and integrating by parts, we obtain **with elementary computations**

$$\begin{aligned}
&(-\Delta)^s(E^{(s)}(\cdot, y))(x) \\
&= -\frac{(4\pi)^{-(1-s)}}{\Gamma(1-s)} \int_0^\infty \int_0^\infty \left(\frac{\tau}{t}\right)^s e^{-\frac{y^2}{4\tau}} \partial_t p(x, 0, t+\tau) \frac{d\tau}{\tau} \frac{dt}{t} \\
&= -\frac{(4\pi)^{-(1-s)}}{\Gamma(1-s)} \int_0^\infty v^{s-1} \frac{1}{1+v} \int_0^\infty e^{-\frac{y^2}{4u}\frac{1+v}{v}} \partial_u p(x, 0, u) du dv \\
&= \frac{(4\pi)^{-(1-s)}}{\Gamma(1-s)} \int_0^\infty v^{s-1} \frac{1}{1+v} \int_0^\infty \partial_u \left(e^{-\frac{y^2}{4u}\frac{1+v}{v}}\right) p(x, 0, u) du dv \\
&= \frac{(4\pi)^{-(1-s)}}{\Gamma(1-s)} \int_0^\infty v^{s-1} \frac{1}{1+v} \int_0^\infty \frac{1+v}{4v} \frac{y^2}{u^2} e^{-\frac{y^2}{4u}\frac{1+v}{v}} p(x, 0, u) du dv \\
&= \frac{(4\pi)^{-(1-s)} y^2}{4\Gamma(1-s)} \int_0^\infty \frac{1}{u^2} e^{-\frac{y^2}{4u}} p(x, 0, u) \left(\int_0^\infty v^{s-1} e^{-\frac{y^2}{4uv}} \frac{dv}{v}\right) du.
\end{aligned}$$

Substituting in (0.8), making the change of variable

$$(u, v) = (t + \tau, \frac{\tau}{t}),$$

and integrating by parts, we obtain **with elementary computations**

$$\begin{aligned}
&(-\Delta)^s(E^{(s)}(\cdot, y))(x) \\
&= -\frac{(4\pi)^{-(1-s)}}{\Gamma(1-s)} \int_0^\infty \int_0^\infty \left(\frac{\tau}{t}\right)^s e^{-\frac{y^2}{4\tau}} \partial_t p(x, 0, t+\tau) \frac{d\tau}{\tau} \frac{dt}{t} \\
&= -\frac{(4\pi)^{-(1-s)}}{\Gamma(1-s)} \int_0^\infty v^{s-1} \frac{1}{1+v} \int_0^\infty e^{-\frac{y^2}{4u}\frac{1+v}{v}} \partial_u p(x, 0, u) du dv \\
&= \frac{(4\pi)^{-(1-s)}}{\Gamma(1-s)} \int_0^\infty v^{s-1} \frac{1}{1+v} \int_0^\infty \partial_u \left(e^{-\frac{y^2}{4u}\frac{1+v}{v}}\right) p(x, 0, u) du dv \\
&= \frac{(4\pi)^{-(1-s)}}{\Gamma(1-s)} \int_0^\infty v^{s-1} \frac{1}{1+v} \int_0^\infty \frac{1+v}{4v} \frac{y^2}{u^2} e^{-\frac{y^2}{4u}\frac{1+v}{v}} p(x, 0, u) du dv \\
&= \frac{(4\pi)^{-(1-s)} y^2}{4\Gamma(1-s)} \int_0^\infty \frac{1}{u^2} e^{-\frac{y^2}{4u}} p(x, 0, u) \left(\int_0^\infty v^{s-1} e^{-\frac{y^2}{4uv}} \frac{dv}{v}\right) du.
\end{aligned}$$

Noting that

$$\int_0^\infty v^{s-1} e^{-\frac{y^2}{4uv}} \frac{dv}{v} = \Gamma(1-s) \frac{y^{2s-2}}{4^{s-1} u^{s-1}},$$

we immediately reach the conclusion

$$(-\Delta)^s \left(E^{(s)}(\cdot, y) \right)(x) = (2\pi y)^{2s}\, E^{(-s)}(x, y),$$

which, as I have said, proves the intertwining formula (0.7), and therefore (0.1)!

Noting that

$$\int_0^\infty v^{s-1} e^{-\frac{y^2}{4uv}} \frac{dv}{v} = \Gamma(1-s) \frac{y^{2s-2}}{4^{s-1} u^{s-1}},$$

we immediately reach the conclusion

$$(-\Delta)^s \left(E^{(s)}(\cdot, y) \right)(x) = (2\pi y)^{2s} E^{(-s)}(x, y),$$

which, as I have said, proves the intertwining formula (0.7), and therefore (0.1)!

One of the objectives of my talk is to present a version of (0.1) in which $(-\Delta)^s$ is replaced by the conformal fractional horizontal Laplacian $\mathscr{L}_s$ in a Lie group of Heisenberg type $\mathbb{G}$ ...

Noting that
$$\int_0^\infty v^{s-1} e^{-\frac{y^2}{4uv}} \frac{dv}{v} = \Gamma(1-s)\frac{y^{2s-2}}{4^{s-1}u^{s-1}},$$
we immediately reach the conclusion

$$(-\Delta)^s \left(E^{(s)}(\cdot, y)\right)(x) = (2\pi y)^{2s} E^{(-s)}(x, y),$$

which, as I have said, proves the intertwining formula (0.7), and therefore (0.1)!

One of the objectives of my talk is to present a version of (0.1) in which $(-\Delta)^s$ is replaced by the conformal fractional horizontal Laplacian $\mathscr{L}_s$ in a Lie group of Heisenberg type $\mathbb{G}$ ... It's time to introduce the relevant geometric framework...

The Heisenberg group $\mathbb{H}^n$

This group arises in the description of n-dimensional quantum mechanical systems. Consider the $2n \times 2n$ symplectic matrix $J = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$. In $\mathbb{R}^{2n+1}$ introduce the non-Abelian group law (think of $\mathbb{R}^{2n+1} = \mathbb{R}^{2n}_z \oplus \mathbb{R}_\sigma$, with coordinates (z, σ), where $z = (x, y)$):

$$(z, \sigma) \circ (z', \sigma') = (z + z', \sigma + \sigma' + \frac{1}{2}\langle z, Jz' \rangle). \tag{0.9}$$

$\mathbb{H}^n$ denotes the Lie group $(\mathbb{R}^{2n+1}, \circ)$

- $g = (z, \sigma)$, $g' = (z', \sigma')$, etc. are generic points in $\mathbb{H}^n$
- the identity element with respect to $\circ$ is $e = (0, 0)$, $g^{-1} = (-z, -\sigma)$
- $L_g : \mathbb{H}^n \to \mathbb{H}^n$ denotes the left-translation $L_g(g') = g \circ g'$
- the Heisenberg algebra $\mathfrak{h}_n$ is generated by the $2n + 1$ vector fields: $X_j(g) = \partial_{x_j} - \frac{y_j}{2}\partial_\sigma$,...,$X_{n+j}(g) = \partial_{y_j} + \frac{x_j}{2}\partial_\sigma$, $T = \partial_\sigma$.

[7]when $n = 1$ this is Heisenberg's quantum mechanics commutation for position and momentum

The vector fields $X_1, ..., X_{2n}$ do not span the whole tangent space $T_e\mathbb{H}^n \cong \mathbb{R}^{2n+1}$!

[7] when $n = 1$ this is Heisenberg's quantum mechanics commutation for position and momentum

The vector fields $X_1, ..., X_{2n}$ do not span the whole tangent space $T_e\mathbb{H}^n \cong \mathbb{R}^{2n+1}$! However, they satisfy the commutation relation [7]

$$(\star) \qquad [X_i, X_{n+j}] = \delta_{ij} T, \qquad i, j = 1, ..., n.$$

[7] when $n = 1$ this is Heisenberg's quantum mechanics commutation for position and momentum

14 / 46

The vector fields $X_1, ..., X_{2n}$ do not span the whole tangent space $T_e\mathbb{H}^n \cong \mathbb{R}^{2n+1}$! However, they satisfy the commutation relation [7]

$$(\star) \qquad [X_i, X_{n+j}] = \delta_{ij} T, \qquad i, j = 1, ..., n.$$

The horizontal Laplacian on $\mathbb{H}^n$ is the second order pdo

$$\mathscr{L} = \sum_{j=1}^{2n} X_j^2 = \Delta_z + \frac{|z|^2}{4}\partial_{\sigma\sigma} + \partial_\sigma \sum_{j=1}^{n}(x_j\partial_{y_j} - y_j\partial_{x_j}).$$

[7] when $n = 1$ this is Heisenberg's quantum mechanics commutation for position and momentum

The vector fields $X_1, ..., X_{2n}$ do not span the whole tangent space $T_e\mathbb{H}^n \cong \mathbb{R}^{2n+1}$! However, they satisfy the commutation relation [7]

$$(\star) \qquad [X_i, X_{n+j}] = \delta_{ij} T, \qquad i, j = 1, ..., n.$$

The horizontal Laplacian on $\mathbb{H}^n$ is the second order pdo

$$\mathscr{L} = \sum_{j=1}^{2n} X_j^2 = \Delta_z + \frac{|z|^2}{4}\partial_{\sigma\sigma} + \partial_\sigma \sum_{j=1}^{n}(x_j\partial_{y_j} - y_j\partial_{x_j}).$$

This operator fails to be elliptic at every point $g = (z, \sigma) \in \mathbb{H}^n$, but because of $(\star)$ and a celebrated 1967 theorem by Hörmander, the operator $\mathscr{L}$ is hypoelliptic. By assigning the formal degree j to the corresponding layer of the Lie algebra spanned by commutators of order j, in view of $(\star)$ we can equip $\mathbb{H}^n$ with the non-isotropic dilations $\delta_\lambda(z, \sigma) = (\lambda z, \lambda^2\sigma)$. Similarly to what happens to Δ with the isotropic Euclidean dilations, one has $\mathscr{L} \circ \delta_\lambda = \lambda^2 \delta_\lambda \circ \mathscr{L}$.

[7] when $n = 1$ this is Heisenberg's quantum mechanics commutation for position and momentum

...A beautiful, classical formula...

It is immediate to verify that, if $Q = 2n + 2$, then Lebesgue measure interacts with the group dilations according to the law

...A beautiful, classical formula...

It is immediate to verify that, if $Q = 2n + 2$, then Lebesgue measure interacts with the group dilations according to the law $d(\delta_\lambda(z, \sigma)) = \lambda^Q dz d\sigma$.

15 / 46

...A beautiful, classical formula...

It is immediate to verify that, if $Q = 2n + 2$, then Lebesgue measure interacts with the group dilations according to the law $d(\delta_\lambda(z,\sigma)) = \lambda^Q dz d\sigma$. Thus, the anisotropic dilations determine a natural "dimension" associated with $\mathbb{H}^n$. The relevance of such homogeneous dimension is underscored by the following remarkable 1973 result of Folland. In what follows, $N(z,\sigma) = (|z|^4 + 16\sigma^2)^{1/4}$ is the so-called Koranyi gauge on $\mathbb{H}^n$.

...A beautiful, classical formula...

It is immediate to verify that, if $Q = 2n + 2$, then Lebesgue measure interacts with the group dilations according to the law $d(\delta_\lambda(z,\sigma)) = \lambda^Q dz d\sigma$. Thus, the anisotropic dilations determine a natural "dimension" associated with $\mathbb{H}^n$. The relevance of such homogeneous dimension is underscored by the following remarkable 1973 result of Folland. In what follows, $N(z,\sigma) = (|z|^4 + 16\sigma^2)^{1/4}$ is the so-called Koranyi gauge on $\mathbb{H}^n$.

Theorem. There exists a suitable explicit constant $C(n) > 0$ such that

$$\mathscr{E}(z,\sigma) = C(n) N(z,\sigma)^{2-Q}$$

is the fundamental solution with pole at the group identity of the horizontal Laplacian $-\mathscr{L}$ in $\mathbb{H}^n$.

One notable aspect of Folland's result is the resemblance with the fundamental solution of $-\Delta$ which for $n \geq 3$ is given by $c(n)|x|^{2-n}$.

[8] A horizontal Laplacian on $\mathbb{G}$ is the second order hypoelliptic operator defined by $\mathscr{L} = \sum_{j=1}^m X_j^2$, where $X_j(g) = dL_g(e_j)$, and $\{e_1, ..., e_m\}$ is an orthonormal basis of V_1

I will come back to Folland's result at the very end of my talk and show that, by running the heat flow on $\mathbb{H}^n$, we can actually "discover" the magic gauge function $N(z, \sigma)$ and recover Folland's theorem as the limiting case $s \nearrow 1$ of a family of fractional theorems!

[8] A horizontal Laplacian on $\mathbb{G}$ is the second order hypoelliptic operator defined by $\mathscr{L} = \sum_{j=1}^m X_j^2$, where $X_j(g) = dL_g(e_j)$, and $\{e_1, ..., e_m\}$ is an orthonormal basis of V_1

I will come back to Folland's result at the very end of my talk and show that, by running the heat flow on $\mathbb{H}^n$, we can actually "discover" the magic gauge function $N(z,\sigma)$ and recover Folland's theorem as the limiting case $s \nearrow 1$ of a family of fractional theorems!

Returning to the Heisenberg group $\mathbb{H}^n$ we see that its essential feature is that its Lie algebra is decomposed into two layers: $\mathfrak{h}_n = V_1 \oplus V_2$. The so-called horizontal layer $V_1 = R_z^{2n} \times \{0\}_\sigma$, and the vertical layer $V_2 = \{0\}_z \times \mathbb{R}_\sigma$. These two layers satisfy the properties: $[V_1, V_1] = V_2$ (bracket generating), $[V_1, V_2] = \{0\}$ (nilpotency). Because of this splitting of the Lie algebra in two layers, $\mathbb{H}^n$ is called a Carnot group of step two.

[8] A horizontal Laplacian on $\mathbb{G}$ is the second order hypoelliptic operator defined by $\mathscr{L} = \sum_{j=1}^m X_j^2$, where $X_j(g) = dL_g(e_j)$, and $\{e_1, ..., e_m\}$ is an orthonormal basis of V_1

I will come back to Folland's result at the very end of my talk and show that, by running the heat flow on $\mathbb{H}^n$, we can actually "discover" the magic gauge function $N(z, \sigma)$ and recover Folland's theorem as the limiting case $s \nearrow 1$ of a family of fractional theorems!

Returning to the Heisenberg group $\mathbb{H}^n$ we see that its essential feature is that its Lie algebra is decomposed into two layers: $\mathfrak{h}_n = V_1 \oplus V_2$. The so-called horizontal layer $V_1 = R_z^{2n} \times \{0\}_\sigma$, and the vertical layer $V_2 = \{0\}_z \times \mathbb{R}_\sigma$. These two layers satisfy the properties: $[V_1, V_1] = V_2$ (bracket generating), $[V_1, V_2] = \{0\}$ (nilpotency). Because of this splitting of the Lie algebra in two layers, $\mathbb{H}^n$ is called a Carnot group of step two. If the Lie algebra $\mathfrak{g}$ splits in r layers we talk of a Carnot group of step r $\mathbb{G}$.

[8] A horizontal Laplacian on $\mathbb{G}$ is the second order hypoelliptic operator defined by $\mathscr{L} = \sum_{j=1}^m X_j^2$, where $X_j(g) = dL_g(e_j)$, and $\{e_1, ..., e_m\}$ is an orthonormal basis of V_1

I will come back to Folland's result at the very end of my talk and show that, by running the heat flow on $\mathbb{H}^n$, we can actually "discover" the magic gauge function $N(z, \sigma)$ and recover Folland's theorem as the limiting case $s \nearrow 1$ of a family of fractional theorems!

Returning to the Heisenberg group $\mathbb{H}^n$ we see that its essential feature is that its Lie algebra is decomposed into two layers: $\mathfrak{h}_n = V_1 \oplus V_2$. The so-called horizontal layer $V_1 = R^{2n}_z \times \{0\}_\sigma$, and the vertical layer $V_2 = \{0\}_z \times \mathbb{R}_\sigma$. These two layers satisfy the properties: $[V_1, V_1] = V_2$ (bracket generating), $[V_1, V_2] = \{0\}$ (nilpotency). Because of this splitting of the Lie algebra in two layers, $\mathbb{H}^n$ is called a Carnot group of step two. If the Lie algebra $\mathfrak{g}$ splits in r layers we talk of a Carnot group of step r $\mathbb{G}$. This means that $\mathfrak{g} = V_1 \oplus ... \oplus V_r$, with $[V_1, V_i] = V_{i+1}$, $i = 1, ..., r-1$, and $[V_1, V_r] = \{0\}$. The bracket generating layer V_1 of $\mathfrak{g}$ is called the horizontal layer [8].

[8] A horizontal Laplacian on $\mathbb{G}$ is the second order hypoelliptic operator defined by $\mathscr{L} = \sum_{j=1}^m X_j^2$, where $X_j(g) = dL_g(e_j)$, and $\{e_1, ..., e_m\}$ is an orthonormal basis of V_1

To introduce our first result consider now a general Carnot group $\mathbb{G}$ with a fixed horizontal Laplacian $\mathscr{L}$, and let

17 / 46

To introduce our first result consider now a general Carnot group $\mathbb{G}$ with a fixed horizontal Laplacian $\mathscr{L}$, and let

$$P_t u(g) = e^{-t\mathscr{L}} u(g) = \int_{\mathbb{G}} p(g, g', t) u(g') dg'$$

be the heat semigroup constructed by Folland. I recall that such semigroup is stochastically complete, i.e., $P_t 1 = 1$.

The semigroup P_t is all that is needed to study the (non-conformal) fractional powers $\mathscr{L}^s \overset{def}{=} (-\mathscr{L})^s$, for $0 < s < 1$. One very effective way to specify the action of this nonlocal operator on a function $u \in C_0^\infty(\mathbb{G})$ is by resorting again to Balakrishnan's formula:

$$\mathscr{L}^s u(g) = -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}} (P_t u(g) - u(g)) dt.$$

With such formula in hands it is classical how to invert the pseudo-differential operators $\mathscr{L}^s$ using the heat equation based

With such formula in hands it is classical how to invert the pseudo-differential operators $\mathscr{L}^s$ using the heat equation based Riesz potentials

With such formula in hands it is classical how to invert the pseudo-differential operators $\mathscr{L}^s$ using the heat equation based Riesz potentials

$$\mathscr{I}^{(2s)}u(g) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} P_t u(g) dt, \qquad 0 < s < 1.$$

With such formula in hands it is classical how to invert the pseudo-differential operators $\mathscr{L}^s$ using the heat equation based Riesz potentials

$$\mathscr{I}^{(2s)}u(g) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}P_t u(g)dt, \qquad 0 < s < 1.$$

It is in fact relatively easy to prove the following basic result:

With such formula in hands it is classical how to invert the pseudo-differential operators $\mathscr{L}^s$ using the heat equation based Riesz potentials

$$\mathscr{I}^{(2s)}u(g) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} P_t u(g) dt, \qquad 0 < s < 1.$$

It is in fact relatively easy to prove the following basic result:

$$\mathscr{I}^{(2s)} \circ \mathscr{L}^s = \mathscr{L}^s \circ \mathscr{I}^{(2s)} = I. \tag{0.10}$$

With such formula in hands it is classical how to invert the pseudo-differential operators $\mathscr{L}^s$ using the heat equation based Riesz potentials

$$\mathscr{I}^{(2s)}u(g) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}P_t u(g)dt, \qquad 0 < s < 1.$$

It is in fact relatively easy to prove the following basic result:

$$\mathscr{I}^{(2s)} \circ \mathscr{L}^s = \mathscr{L}^s \circ \mathscr{I}^{(2s)} = I. \tag{0.10}$$

A direct important consequence of the inversion formula (0.10) is that the kernel

$$\mathscr{E}^{(s)}(g) \overset{def}{=} \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}p(g,t)dt \tag{0.11}$$

of the operator $\mathscr{I}^{(2s)}$ constitutes the fundamental solution of the nonlocal operator $\mathscr{L}^s$ with pole at the group identity.

When the Lie group $\mathbb{G} = \mathbb{R}^n$ is Abelian, then $\mathscr{L} = \Delta$ and (0.11) gives the classical formula of M. Riesz

$$E^{(s)}(x) = \frac{\Gamma(\frac{n}{2} - s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}|x|^{2s-n}. \tag{0.12}$$

When the Lie group $\mathbb{G} = \mathbb{R}^n$ is Abelian, then $\mathscr{L} = \Delta$ and (0.11) gives the classical formula of M. Riesz

$$E^{(s)}(x) = \frac{\Gamma(\frac{n}{2} - s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}|x|^{2s-n}. \tag{0.12}$$

This result, and Folland's theorem in the local case $s = 1$, might lead to believe that in the Heisenberg group $\mathbb{H}^n$ the fundamental solution (0.11) of the nonlocal operator $\mathscr{L}^s$ is given by the formula

$$\mathscr{E}^{(s)}(g) = C(n, s)N(z, \sigma)^{2s-Q}, \tag{0.13}$$

where as before $N(z, \sigma)$ is the Koranyi gauge.

When the Lie group $\mathbb{G} = \mathbb{R}^n$ is Abelian, then $\mathscr{L} = \Delta$ and (0.11) gives the classical formula of M. Riesz

$$E^{(s)}(x) = \frac{\Gamma(\frac{n}{2} - s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}|x|^{2s-n}. \tag{0.12}$$

This result, and Folland's theorem in the local case $s = 1$, might lead to believe that in the Heisenberg group $\mathbb{H}^n$ the fundamental solution (0.11) of the nonlocal operator $\mathscr{L}^s$ is given by the formula

$$\mathscr{E}^{(s)}(g) = C(n,s)N(z,\sigma)^{2s-Q}, \tag{0.13}$$

where as before $N(z,\sigma)$ is the Koranyi gauge. This guess is however completely wrong! The distribution defined by the right-hand side of (0.11) is **not** a function of the gauge $N(z,\sigma)$, except in the limiting (local) case when $s \nearrow 1$!

When the Lie group $\mathbb{G} = \mathbb{R}^n$ is Abelian, then $\mathscr{L} = \Delta$ and (0.11) gives the classical formula of M. Riesz

$$E^{(s)}(x) = \frac{\Gamma(\frac{n}{2} - s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}|x|^{2s-n}. \tag{0.12}$$

This result, and Folland's theorem in the local case $s = 1$, might lead to believe that in the Heisenberg group $\mathbb{H}^n$ the fundamental solution (0.11) of the nonlocal operator $\mathscr{L}^s$ is given by the formula

$$\mathscr{E}^{(s)}(g) = C(n,s)N(z,\sigma)^{2s-Q}, \tag{0.13}$$

where as before $N(z,\sigma)$ is the Koranyi gauge. This guess is however completely wrong! The distribution defined by the right-hand side of (0.11) is **not** a function of the gauge $N(z,\sigma)$, except in the limiting (local) case when $s \nearrow 1$!

As I have mentioned in the Preface, there is another pseudodifferential operator $\mathscr{L}_s$, very different from $\mathscr{L}^s$, and whose fundamental solution $\mathscr{E}_{(s)}(g)$ does instead satisfy a formula such as (0.13)!...This is where our story begins...

A small glimpse of things to come...fundamental solutions

A small glimpse of things to come...fundamental solutions

- Δ = Euclidean Laplacian in $\mathbb{R}^n$

$-\Delta$	$(-\Delta)^s$
$c(n)\|x\|^{2-n}$	$c(n,s)\|x\|^{2s-n}$
conformal	conformal

A small glimpse of things to come...fundamental solutions

- Δ = Euclidean Laplacian in $\mathbb{R}^n$

$-\Delta$	$(-\Delta)^s$
$c(n)\|x\|^{2-n}$	$c(n,s)\|x\|^{2s-n}$
conformal	conformal

- $\mathbb{H}^n$ = Heisenberg group
- $\mathscr{L}$ = horizontal Laplacian in $\mathbb{H}^n$
- $N(z,\sigma) = (\|z\|^4 + 16\sigma^2)^{1/4}$ = Koranyi gauge
- $Q = 2n+2$ homogeneous dimension

$-\mathscr{L}$	$\mathscr{L}^s$	$\mathscr{L}_s$
$C(n)N(z,\sigma)^{2-Q}$	no gauge symmetry	$C(n,s)N(z,\sigma)^{2s-Q}$
conformal	not conformal	conformal

[9] T. P. Branson, *Sharp inequalities, the functional determinant, and the complementary series*. Trans. Amer. Math. Soc. **347** (1995), no. 10, 3671-3742.

Unlike the operator $\mathscr{L}^s$, the definition of the conformal fractional powers $\mathscr{L}_s$ is all but "explicit" and fairly difficult to handle.

[9] T. P. Branson, *Sharp inequalities, the functional determinant, and the complementary series*. Trans. Amer. Math. Soc. **347** (1995), no. 10, 3671-3742.

Unlike the operator $\mathscr{L}^s$, the definition of the conformal fractional powers $\mathscr{L}_s$ is all but "explicit" and fairly difficult to handle. In their 2013 Annals paper Branson, Fontana and Morpurgo first introduced in the Heisenberg group $\mathbb{H}^n$ the pseudodifferential operator of order $2s$ given by the spectral formula

$$\mathscr{L}_s = 2^s |T|^s \frac{\Gamma(-\frac{1}{2}\mathscr{L}|T|^{-1} + \frac{1+s}{2})}{\Gamma(-\frac{1}{2}\mathscr{L}|T|^{-1} + \frac{1-s}{2})}, \qquad 0 < s < 1, \tag{0.14}$$

where $T = \partial_\sigma$ is the differentiation in the vertical direction. Notice that, using the property $\Gamma(x+1) = x\Gamma(x)$, we formally see that when $s = 1$, then $\mathscr{L}_1 = -\mathscr{L}$! The pseudodifferential operator (0.14) is the CR counterpart of the conformal fractional powers of the Laplacian on the sphere $\mathbb{S}^n$ introduced by Branson in 1995 [9].

[9] T. P. Branson, *Sharp inequalities, the functional determinant, and the complementary series*. Trans. Amer. Math. Soc. **347** (1995), no. 10, 3671-3742.

Unlike the operator $\mathscr{L}^s$, the definition of the conformal fractional powers $\mathscr{L}_s$ is all but "explicit" and fairly difficult to handle. In their 2013 Annals paper Branson, Fontana and Morpurgo first introduced in the Heisenberg group $\mathbb{H}^n$ the pseudodifferential operator of order $2s$ given by the spectral formula

$$\mathscr{L}_s = 2^s |T|^s \frac{\Gamma(-\frac{1}{2}\mathscr{L}|T|^{-1} + \frac{1+s}{2})}{\Gamma(-\frac{1}{2}\mathscr{L}|T|^{-1} + \frac{1-s}{2})}, \qquad 0 < s < 1, \tag{0.14}$$

where $T = \partial_\sigma$ is the differentiation in the vertical direction. Notice that, using the property $\Gamma(x+1) = x\Gamma(x)$, we formally see that when $s = 1$, then $\mathscr{L}_1 = -\mathscr{L}$! The pseudodifferential operator (0.14) is the CR counterpart of the conformal fractional powers of the Laplacian on the sphere $\mathbb{S}^n$ introduced by Branson in 1995 [9].

Since my talk is about the heat equation **I will not use** (0.14) as definition of $\mathscr{L}_s$!

[9] T. P. Branson, *Sharp inequalities, the functional determinant, and the complementary series*. Trans. Amer. Math. Soc. **347** (1995), no. 10, 3671-3742.

In their cited work Frank, Gonzalez, Monticelli and Tan have introduced the following CR extension problem for the fractional powers $\mathscr{L}_s$: given $f \in C_0^\infty(\mathbb{H}^n)$, find a function $F \in C^\infty(\mathbb{H}^n \times (0,\infty))$ such that

In their cited work Frank, Gonzalez, Monticelli and Tan have introduced the following CR extension problem for the fractional powers $\mathscr{L}_s$: given $f \in C_0^\infty(\mathbb{H}^n)$, find a function $F \in C^\infty(\mathbb{H}^n \times (0,\infty))$ such that

$$\begin{cases} \frac{\partial^2 F}{\partial y^2} + \frac{1-2s}{y}\frac{\partial F}{\partial y} + \frac{y^2}{4}\frac{\partial^2 F}{\partial \sigma^2} + \mathscr{L} F = 0, \\ F((z,\sigma),0) = f(z,\sigma), \end{cases} \tag{0.15}$$

where $\mathscr{L}$ is the horizontal Laplacian in $\mathbb{H}^n$.

In their cited work Frank, Gonzalez, Monticelli and Tan have introduced the following CR extension problem for the fractional powers $\mathscr{L}_s$: given $f \in C_0^\infty(\mathbb{H}^n)$, find a function $F \in C^\infty(\mathbb{H}^n \times (0, \infty))$ such that

$$\begin{cases} \frac{\partial^2 F}{\partial y^2} + \frac{1-2s}{y} \frac{\partial F}{\partial y} + \frac{y^2}{4} \frac{\partial^2 F}{\partial \sigma^2} + \mathscr{L} F = 0, \\ F((z, \sigma), 0) = f(z, \sigma), \end{cases} \tag{0.15}$$

where $\mathscr{L}$ is the horizontal Laplacian in $\mathbb{H}^n$. Without the term $\frac{y^2}{4} \frac{\partial^2 F}{\partial \sigma^2}$ the problem (0.15) would be the counterpart of the Caffarelli-Silvestre extension problem for $\mathscr{L}^s$ which I have recalled above. The additional term makes the above problem completely different from the Caffarelli-Silvestre type extension, but it introduces geometric meaning! For instance, Frank et al. proved the following fundamental weighted Dirichlet-to-Neumann relation for (0.15)

$$\mathscr{L}_s f(z, \sigma) = -\frac{2^{2s-1}\Gamma(1-s)}{\Gamma(1+s)} \lim_{y \to 0^+} y^{1-2s} \frac{\partial F}{\partial y}((z, \sigma), y).$$

Hyperbolic geometry and scattering

Hyperbolic geometry and scattering

One way to understand the problem (0.15) is to consider the Heisenberg group $\mathbb{H}^n$ as the boundary of the complex hyperbolic space

$$\mathbb{H}_n(\mathbb{C}) = \{((z,\sigma),y) \mid (x_1,\ldots,x_n,y_1,\ldots,y_n,\sigma) \in \mathbb{H}^n,\ \ y>0\}$$

endowed with the Riemannian metric $g_{\mathbb{H}_n(\mathbb{C})}$ with respect to which the $2n+2$ vector fields

$$V_i = yX_i,\ \ i=1,...,2n, \quad V_0 = -\frac{y^2}{2}\partial_\sigma \quad \text{and} \quad W_0 = y\partial_y$$

form an orthonormal frame of the tangent space. A computation of the connection shows that the Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}_n(\mathbb{C})} = \sum_{j=0}^{2n}\left(V_j^2 - \nabla_{V_j}V_j\right) = y^2\left(\mathscr{L} + \frac{y^2}{4}\partial_{\sigma\sigma} + \partial_{yy} - \frac{Q-1}{y}\partial_y\right),$$

where $\mathscr{L}$ is the horizontal Laplacian on the Heisenberg group $\mathbb{H}^n$.

24 / 46

One now has the following fact: consider a function $U(z,\sigma,y)$ that lives in $\mathbb{H}^n \times \mathbb{R}^+_y$, and define a function $u(z,\sigma,y)$ in $\mathbb{H}_n(\mathbb{C})$ by the formula

$$u(z,\sigma,y) = y^{\frac{Q}{2}-s}\ U(z,\sigma,y).$$

Then, one has for the scattering eigenvalue problem in $\mathbb{H}_n(\mathbb{C})$

24 / 46

One now has the following fact: consider a function $U(z,\sigma,y)$ that lives in $\mathbb{H}^n \times \mathbb{R}^+_y$, and define a function $u(z,\sigma,y)$ in $\mathbb{H}_n(\mathbb{C})$ by the formula

$$u(z,\sigma,y) = y^{\frac{Q}{2}-s}\ U(z,\sigma,y).$$

Then, one has for the scattering eigenvalue problem in $\mathbb{H}_n(\mathbb{C})$

$$\begin{aligned}
&\Delta_{\mathbb{H}_n(\mathbb{C})} u + \left(\frac{Q}{2} - s\right)\left(\frac{Q}{2} + s\right) u \\
&= y^{\frac{Q}{2}-s+2}\left\{\mathscr{L} U + \frac{y^2}{4}\partial_{\sigma\sigma} U + \partial_{yy} + \frac{1-2s}{y}\partial_y U\right\}.
\end{aligned}$$

Thus, *u* solves the eigenvalue problem in the complex hyperbolic space $\mathbb{H}_n(\mathbb{C}) \iff U$ is a solution of the extension problem of Frank et al. in $\mathbb{H}^n \times \mathbb{R}^+_y$!

25 / 46

In 2016-17 Roncal and Thangavelu used a parabolic version of the extension problem of Frank et al., which I will discuss below, combined with non-commutative harmonic analysis and group representation theory, to establish some beautiful sharp Hardy inequalities on the Heisenberg group, or more in general groups of Heisenberg type.

25 / 46

In 2016-17 Roncal and Thangavelu used a parabolic version of the extension problem of Frank et al., which I will discuss below, combined with non-commutative harmonic analysis and group representation theory, to establish some beautiful sharp Hardy inequalities on the Heisenberg group, or more in general groups of Heisenberg type.

Our works "Feeling the heat in a group of Heisenberg type" and "A heat equation approach to intertwining" were inspired by some of the ideas of Roncal and Thangavelu. Instead of non-commutative harmonic analysis and group representation theory we combine in a systematic way the parabolic extension problem with some ideas in our recent works on a (quite different) class of nonlocal hypoelliptic equations arising in the kinetic theory of gases.

[10] A Carnot group of step two is called of Heisenberg type if the Kaplan mapping $J : V_2 \to \text{End}(V_1)$, defined by

$$\langle J(\sigma)z, \zeta\rangle = \langle [z, \zeta], \sigma\rangle = -\langle J(\sigma)\zeta, z\rangle, \quad z, \zeta \in V_1,\ \sigma \in V_2,$$

satisfies $J(\sigma)^2 = -|\sigma|^2\ \mathbb{I}_{V_1}$ for every $\sigma \in V_2$. This assumption induces a complex structure on $\mathbb{G}$.

Henceforth, I will place my discussion in a Lie group of Heisenberg type $\mathbb{G}$. This class constitutes a nontrivial geometric extension of the Heisenberg group $\mathbb{H}^n$, except that now the vertical layer V_2 of the Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ can have arbitrary dimension. I henceforth denote $m = \dim V_1$, $k = \dim V_2$, and will routinely identify $\mathfrak{g} \cong \mathbb{R}^m \times \mathbb{R}^k$ (when the center of the Lie algebra has dimension $k = 1$, then we obtain back $\mathbb{H}^n$). The generic point $g \in \mathbb{G}$ will be identified with its coordinates $(z, \sigma) \in \mathbb{R}^m \times \mathbb{R}^k$. [10]. As before, the group anisotropic dilations are $\delta_\lambda(g) = (\lambda z, \lambda^2 \sigma)$. The homogeneous dimension of $\mathbb{G}$ associated with such dilations is now the number $Q = m + 2k$.

[10] A Carnot group of step two is called of Heisenberg type if the Kaplan mapping $J : V_2 \to \mathrm{End}(V_1)$, defined by

$$\langle J(\sigma)z, \zeta\rangle = \langle [z, \zeta], \sigma\rangle = -\langle J(\sigma)\zeta, z\rangle, \quad z, \zeta \in V_1,\ \sigma \in V_2,$$

satisfies $J(\sigma)^2 = -|\sigma|^2\, \mathbb{I}_{V_1}$ for every $\sigma \in V_2$. This assumption induces a complex structure on $\mathbb{G}$.

The Hulanicki-Gaveau-Cygan heat kernel

[11]for new self-contained PDE approach, see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. Preprint 2020 (ArXiv: 2007.10862)

27 / 46

The Hulanicki-Gaveau-Cygan heat kernel

In a group of Heisenberg type $\mathbb{G}$ consider the heat operator $\partial_t - \mathscr{L}$. The fundamental solution of this operator with pole at the group identity is given by

[11]for new self-contained PDE approach, see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. Preprint 2020 (ArXiv: 2007.10862)

The Hulanicki-Gaveau-Cygan heat kernel

In a group of Heisenberg type $\mathbb{G}$ consider the heat operator $\partial_t - \mathscr{L}$. The fundamental solution of this operator with pole at the group identity is given by

$$p(z,\sigma,t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle \sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda,$$

where as before I have identified a point $g \in \mathbb{G}$ with its logarithmic (Lie algebra) coordinates $(z,\sigma) \in \mathbb{R}^m \times \mathbb{R}^k$.

[11] for new self-contained PDE approach, see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. Preprint 2020 (ArXiv: 2007.10862)

The Hulanicki-Gaveau-Cygan heat kernel

In a group of Heisenberg type $\mathbb{G}$ consider the heat operator $\partial_t - \mathscr{L}$. The fundamental solution of this operator with pole at the group identity is given by

$$p(z,\sigma,t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda,$$

where as before I have identified a point $g \in \mathbb{G}$ with its logarithmic (Lie algebra) coordinates $(z,\sigma) \in \mathbb{R}^m \times \mathbb{R}^k$. This formula was independently obtained by Hulanicki and Gaveau for the Heisenberg group, and was subsequently generalised by Cygan to all groups of step two [11].

[11] for new self-contained PDE approach, see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. Preprint 2020 (ArXiv: 2007.10862)

The Hulanicki-Gaveau-Cygan heat kernel

In a group of Heisenberg type $\mathbb{G}$ consider the heat operator $\partial_t - \mathscr{L}$. The fundamental solution of this operator with pole at the group identity is given by

$$p(z,\sigma,t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle \sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh |\lambda|}\right)^{\frac{m}{2}} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh |\lambda|}} d\lambda,$$

where as before I have identified a point $g \in \mathbb{G}$ with its logarithmic (Lie algebra) coordinates $(z,\sigma) \in \mathbb{R}^m \times \mathbb{R}^k$. This formula was independently obtained by Hulanicki and Gaveau for the Heisenberg group, and was subsequently generalised by Cygan to all groups of step two [11].

We are going to build on variants of this formula to define $\mathscr{L}_s$.

[11] for new self-contained PDE approach, see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. Preprint 2020 (ArXiv: 2007.10862)

The conformal parabolic extension problem

The conformal parabolic extension problem

From the point of view of conformal geometry, the true counterpart of the parabolic extension problem for $(\partial_t - \Delta)^s$ described in the opening of my talk is as follows.

The conformal parabolic extension problem

From the point of view of conformal geometry, the true counterpart of the parabolic extension problem for $(\partial_t - \Delta)^s$ described in the opening of my talk is as follows. Let $\mathbb{G}$ be a group of Heisenberg type with a given horizontal Laplacian $\mathscr{L}$.

The conformal parabolic extension problem

From the point of view of conformal geometry, the true counterpart of the parabolic extension problem for $(\partial_t - \Delta)^s$ described in the opening of my talk is as follows. Let $\mathbb{G}$ be a group of Heisenberg type with a given horizontal Laplacian $\mathscr{L}$. Given a function $u \in C_0^\infty(\mathbb{G} \times \mathbb{R}_t)$, find a function $U \in C^\infty(\mathbb{G} \times \mathbb{R}_t \times \mathbb{R}_y^+)$ such that

$$\begin{cases} \mathfrak{P}_{(s)} U \overset{def}{=} \frac{\partial^2 U}{\partial y^2} + \frac{1-2s}{y}\frac{\partial U}{\partial y} + \frac{y^2}{4}\Delta_\sigma U + \mathscr{L} U - \frac{\partial U}{\partial t} = 0, \qquad \text{in } \mathbb{G} \times \mathbb{R}_t \times \mathbb{R}_y^+, \\ U(g,t,0) = f(g,t), \end{cases}$$

The conformal parabolic extension problem

From the point of view of conformal geometry, the true counterpart of the parabolic extension problem for $(\partial_t - \Delta)^s$ described in the opening of my talk is as follows. Let $\mathbb{G}$ be a group of Heisenberg type with a given horizontal Laplacian $\mathscr{L}$. Given a function $u \in C_0^\infty(\mathbb{G} \times \mathbb{R}_t)$, find a function $U \in C^\infty(\mathbb{G} \times \mathbb{R}_t \times \mathbb{R}_y^+)$ such that

$$\begin{cases} \mathfrak{P}_{(s)} U \overset{def}{=} \frac{\partial^2 U}{\partial y^2} + \frac{1-2s}{y}\frac{\partial U}{\partial y} + \frac{y^2}{4}\Delta_\sigma U + \mathscr{L} U - \frac{\partial U}{\partial t} = 0, & \text{in } \mathbb{G} \times \mathbb{R}_t \times \mathbb{R}_y^+, \\ U(g,t,0) = f(g,t), \end{cases}$$

The **big fact** is that the fundamental solution of the operator $\mathfrak{P}_{(s)}$ can be computed "explicitly". The reason for this is that $\mathfrak{P}_{(s)}$ must be considered as a parabolic Baouendi-Grushin operator in the space with fractal dimension $\mathbb{R}^{m+2(1-s)} \times \mathbb{R}^k \times (0,\infty)$.

To explain this comment let me make the crucial observation that, in a group of Heisenberg type $\mathbb{G}$, the relevant heat equation associated with the above parabolic extension problem is

29 / 46

To explain this comment let me make the crucial observation that, in a group of Heisenberg type $\mathbb{G}$, the relevant heat equation associated with the above parabolic extension problem is

$$\Delta_w U + \frac{|w|^2}{4}\Delta_\sigma U + \mathscr{L} U - \partial_t U = 0, \tag{0.16}$$

where now $(z,\sigma) \in \mathbb{G}$, $t > 0$. Here, as before, I am thinking of the variable w as running in the space with fractal dimension $\mathbb{R}^{2(1-s)}$.

29 / 46

To explain this comment let me make the crucial observation that, in a group of Heisenberg type $\mathbb{G}$, the relevant heat equation associated with the above parabolic extension problem is

$$\Delta_w U + \frac{|w|^2}{4}\Delta_\sigma U + \mathscr{L} U - \partial_t U = 0, \qquad (0.16)$$

where now $(z, \sigma) \in \mathbb{G}$, $t > 0$. Here, as before, I am thinking of the variable w as running in the space with fractal dimension $\mathbb{R}^{2(1-s)}$. The link between the PDE (0.16) and the one in the extension problem is seen by observing that, if $y = |w|$, then on a function $f(w) = \psi(y)$ we have $\Delta_w f = \partial_{yy}\psi + \frac{1-2s}{y}\partial_y \psi$.

To explain this comment let me make the crucial observation that, in a group of Heisenberg type $\mathbb{G}$, the relevant heat equation associated with the above parabolic extension problem is

$$\Delta_w U + \frac{|w|^2}{4}\Delta_\sigma U + \mathscr{L} U - \partial_t U = 0, \tag{0.16}$$

where now $(z,\sigma) \in \mathbb{G}$, $t > 0$. Here, as before, I am thinking of the variable w as running in the space with fractal dimension $\mathbb{R}^{2(1-s)}$. The link between the PDE (0.16) and the one in the extension problem is seen by observing that, if $y = |w|$, then on a function $f(w) = \psi(y)$ we have $\Delta_w f = \partial_{yy}\psi + \frac{1-2s}{y}\partial_y\psi$. Now, in a group of Heisenberg type the horizontal Laplacian is given by

$$\mathscr{L} = \Delta_z + \frac{|z|^2}{4}\Delta_\sigma + \sum_{\ell=1}^{k} \Theta_\ell \partial_{\sigma_\ell},$$

where $\Theta_\ell = \sum_{s=1}^m \langle J(\varepsilon_\ell)z, e_s\rangle \partial_{z_s}$ and $J : V_2 \to \mathrm{End}(V_1)$ is the Kaplan mapping.

[12] see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. ArXiv 2007.10862

If one looks for solutions U which are spherically symmetric in the horizontal variable $z \in \mathbb{R}^m$, then a calculation shows that $\Theta_\ell U = 0$ for every $\ell = 1, ..., k$, and the extension PDE (0.16) becomes

[12] see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. ArXiv 2007.10862

If one looks for solutions U which are spherically symmetric in the horizontal variable $z \in \mathbb{R}^m$, then a calculation shows that $\Theta_\ell U = 0$ for every $\ell = 1, ..., k$, and the extension PDE (0.16) becomes

$$\Delta_w U + \Delta_z U + \frac{|w|^2 + |z|^2}{4} \Delta_\sigma U - \partial_t U = 0.$$

Remarkably, this is a parabolic Baouendi-Grushin equation in $\mathbb{R}^{m+2(1-s)} \times \mathbb{R}^k \times (0, \infty)$ whose fundamental solution we can explicitly compute as a Fourier integral along the group center [12]:

[12]see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. ArXiv 2007.10862

If one looks for solutions U which are spherically symmetric in the horizontal variable $z \in \mathbb{R}^m$, then a calculation shows that $\Theta_\ell U = 0$ for every $\ell = 1, ..., k$, and the extension PDE (0.16) becomes

$$\Delta_w U + \Delta_z U + \frac{|w|^2 + |z|^2}{4} \Delta_\sigma U - \partial_t U = 0.$$

Remarkably, this is a parabolic Baouendi-Grushin equation in $\mathbb{R}^{m+2(1-s)} \times \mathbb{R}^k \times (0, \infty)$ whose fundamental solution we can explicitly compute as a Fourier integral along the group center [12]:

$$q_{(s)}((z,\sigma),t,y) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k+1-s}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle \sigma, \lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} \times e^{-\frac{|z|^2+y^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda. \tag{0.17}$$

[12] see N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. ArXiv 2007.10862

This function (0.17) plays a pervasive role in this talk. Similarly to the case of the standard heat equation, I now make the claim that the conformal invariances of the operator $\mathscr{L}_s$ are embedded in the function (0.17). To see this, consider the companion function $q_{(-s)}$, which is obtained by changing s into $-s$. Such function is the fundamental solution of the intertwined operator

31 / 46

This function (0.17) plays a pervasive role in this talk. Similarly to the case of the standard heat equation, I now make the claim that the conformal invariances of the operator $\mathscr{L}_s$ are embedded in the function (0.17). To see this, consider the companion function $q_{(-s)}$, which is obtained by changing s into $-s$. Such function is the fundamental solution of the intertwined operator

$$\mathfrak{P}_{(-s)} \overset{def}{=} \frac{\partial^2}{\partial y^2} + \frac{1+2s}{y}\frac{\partial}{\partial y} + \frac{y^2}{4}\Delta_\sigma + \mathscr{L} - \frac{\partial}{\partial t},$$

and the two operators $\mathfrak{P}_{(\pm s)}$ are linked by the Bessel intertwining relations

$$\begin{cases} \mathfrak{P}_{(s)}\left(y^{2s} q_{(-s)}\right) = y^{2s}\mathfrak{P}_{(-s)}\ q_{(-s)} = 0, \\ \mathfrak{P}_{(-s)}\left(y^{-2s} q_{(s)}\right) = y^{-2s}\mathfrak{P}_{(s)}\ q_{(s)} = 0. \end{cases}$$

32 / 46

To unravel the conformal geometry in the parabolic extension problem we note that, from Bochner's subordination principle, we know that

$$\mathfrak{e}_{(s)}((z,\sigma,y) \overset{def}{=} \int_0^\infty q_{(s)}((z,\sigma),t,y)dt \tag{0.18}$$

is a fundamental solution with pole at the origin of the time-independent part of $\mathfrak{P}_{(s)}$, i.e., the above discussed CR extension operator

To unravel the conformal geometry in the parabolic extension problem we note that, from Bochner's subordination principle, we know that

$$\mathfrak{e}_{(s)}((z,\sigma,y) \overset{def}{=} \int_0^\infty q_{(s)}((z,\sigma),t,y)dt \tag{0.18}$$

is a fundamental solution with pole at the origin of the time-independent part of $\mathfrak{P}_{(s)}$, i.e., the above discussed CR extension operator

$$\mathfrak{L}_{(s)} = \frac{\partial^2}{\partial y^2} + \frac{1-2s}{y}\frac{\partial}{\partial y} + \frac{y^2}{4}\Delta_\sigma + \mathscr{L}.$$

To unravel the conformal geometry in the parabolic extension problem we note that, from Bochner's subordination principle, we know that

$$\mathfrak{e}_{(s)}((z,\sigma,y) \overset{def}{=} \int_0^\infty q_{(s)}((z,\sigma),t,y)dt \tag{0.18}$$

is a fundamental solution with pole at the origin of the time-independent part of $\mathfrak{P}_{(s)}$, i.e., the above discussed CR extension operator

$$\mathfrak{L}_{(s)} = \frac{\partial^2}{\partial y^2} + \frac{1-2s}{y}\frac{\partial}{\partial y} + \frac{y^2}{4}\Delta_\sigma + \mathscr{L}.$$

This observation leads us to state our first main result. We introduce the constant

$$C_{(s)}(m,k) = \frac{2^{\frac{m}{2}+2k-3s-1}\Gamma(\frac{1}{2}(\frac{m}{2}+1-s))\Gamma(\frac{1}{2}(\frac{m}{2}+k-s))}{\pi^{\frac{m+k+1}{2}}\Gamma(s)}. \tag{0.19}$$

33 / 46

We have the following

33 / 46

We have the following

Theorem

Let $0 < s \le 1$. In any group of Heisenberg type $\mathbb{G}$, the distribution $\mathfrak{e}_{(s)}((z,\sigma,y)$ defined by (0.18), in the thick space $\mathbb{G} \times \mathbb{R}^+_y$, is given by

$$\mathfrak{e}_{(s)}((z,\sigma),y) = \frac{\Gamma(s)}{(4\pi)^{1-s}} C_{(s)}(m,k)\ \left((|z|^2 + y^2)^2 + 16|\sigma|^2\right)^{-\frac{1}{4}(m+2k-2s)},$$

where in $C_{(s)}(m,k)$ is given as in (0.19). An equation similar to this holds if we replace s with $-s$, provided that $\Gamma(s)$ is replaced by $|\Gamma(-s)|$.

We have the following

Theorem

Let $0 < s \le 1$. In any group of Heisenberg type $\mathbb{G}$, the distribution $\mathfrak{e}_{(s)}((z,\sigma,y)$ defined by (0.18), in the thick space $\mathbb{G} \times \mathbb{R}^+_y$, is given by

$$\mathfrak{e}_{(s)}((z,\sigma),y) = \frac{\Gamma(s)}{(4\pi)^{1-s}} C_{(s)}(m,k)\ \left((|z|^2 + y^2)^2 + 16|\sigma|^2\right)^{-\frac{1}{4}(m+2k-2s)},$$

where in $C_{(s)}(m,k)$ is given as in (0.19). An equation similar to this holds if we replace s with $-s$, provided that $\Gamma(s)$ is replaced by $|\Gamma(-s)|$.

I emphasise that this theorem represents the CR counterpart of the above mentioned simple formula

We have the following

Theorem

Let $0 < s \leq 1$. In any group of Heisenberg type $\mathbb{G}$, the distribution $\mathfrak{e}_{(s)}((z,\sigma,y)$ defined by (0.18), in the thick space $\mathbb{G}\times\mathbb{R}^+_y$, is given by

$$\mathfrak{e}_{(s)}((z,\sigma),y) = \frac{\Gamma(s)}{(4\pi)^{1-s}} C_{(s)}(m,k)\ \left((|z|^2+y^2)^2+16|\sigma|^2\right)^{-\frac{1}{4}(m+2k-2s)},$$

where in $C_{(s)}(m,k)$ is given as in (0.19). An equation similar to this holds if we replace s with $-s$, provided that $\Gamma(s)$ is replaced by $|\Gamma(-s)|$.

I emphasise that this theorem represents the CR counterpart of the above mentioned simple formula

$$\int_0^\infty q^{(\pm s)}(x,y,t)dt = \frac{\Gamma\left(\frac{n\mp 2s}{2}\right)}{\pi^{\frac{n}{2}\mp s}}(|x|^2+y^2)^{-\frac{n\mp 2s}{2}}. \qquad (0.20)$$

We have the following

Theorem

Let $0 < s \leq 1$. In any group of Heisenberg type $\mathbb{G}$, the distribution $\mathfrak{e}_{(s)}((z,\sigma,y)$ defined by (0.18), in the thick space $\mathbb{G}\times\mathbb{R}^+_y$, is given by

$$\mathfrak{e}_{(s)}((z,\sigma),y) = \frac{\Gamma(s)}{(4\pi)^{1-s}} C_{(s)}(m,k)\ \left((|z|^2+y^2)^2+16|\sigma|^2\right)^{-\frac{1}{4}(m+2k-2s)},$$

where in $C_{(s)}(m,k)$ is given as in (0.19). An equation similar to this holds if we replace s with $-s$, provided that $\Gamma(s)$ is replaced by $|\Gamma(-s)|$.

I emphasise that this theorem represents the CR counterpart of the above mentioned simple formula

$$\int_0^\infty q^{(\pm s)}(x,y,t)dt = \frac{\Gamma\left(\frac{n\mp 2s}{2}\right)}{\pi^{\frac{n}{2}\mp s}}(|x|^2+y^2)^{-\frac{n\mp 2s}{2}}. \tag{0.20}$$

Its proof, however, is not "simple".

34 / 46

With the previous theorem in hands, our second main result is the following CR counterpart of the dimension-free identity (0.7).

34 / 46

With the previous theorem in hands, our second main result is the following CR counterpart of the dimension-free identity (0.7).

Theorem (Geometric intertwining)

Let $\mathbb{G}$ be a group of Heisenberg type and let $s \in (0,1)$. For every $g \in \mathbb{G}$ and $y > 0$ one has

34 / 46

With the previous theorem in hands, our second main result is the following CR counterpart of the dimension-free identity (0.7).

Theorem (Geometric intertwining)

Let $\mathbb{G}$ be a group of Heisenberg type and let $s \in (0,1)$. For every $g \in \mathbb{G}$ and $y > 0$ one has

$$\mathscr{L}_s(\mathfrak{e}_{(s)}(\cdot, y))(g) = (2\pi y)^{2s}\mathfrak{e}_{(-s)}(g, y). \tag{0.21}$$

With the previous theorem in hands, our second main result is the following CR counterpart of the dimension-free identity (0.7).

Theorem (Geometric intertwining)

Let $\mathbb{G}$ be a group of Heisenberg type and let $s \in (0,1)$. For every $g \in \mathbb{G}$ and $y > 0$ one has

$$\mathscr{L}_s(\mathfrak{e}_{(s)}(\cdot, y))(g) = (2\pi y)^{2s} \mathfrak{e}_{(-s)}(g, y). \tag{0.21}$$

Before I discuss (0.21), let me stress that the combination of the latter two theorems $\Longrightarrow$ the following result.

35 / 46

Theorem

Let $\mathbb{G}$ be a Lie group of Heisenberg type. For every $(z, \sigma) \in \mathbb{G}$ and $y > 0$ one has

35 / 46

Theorem

Let $\mathbb{G}$ be a Lie group of Heisenberg type. For every $(z, \sigma) \in \mathbb{G}$ and $y > 0$ one has

$$\mathscr{L}_s\left(((|z|^2 + y^2)^2 + 16|\sigma|^2)^{-\frac{Q-2s}{4}}\right) \tag{0.22}$$

$$= \frac{\Gamma\left(\frac{m+2+2s}{4}\right)\Gamma\left(\frac{Q+2s}{4}\right)}{\Gamma\left(\frac{m+2-2s}{4}\right)\Gamma\left(\frac{Q-2s}{4}\right)}(4y)^{2s}((|z|^2 + y^2)^2 + 16|\sigma|^2)^{-\frac{Q+2s}{4}}.$$

Theorem

Let $\mathbb{G}$ be a Lie group of Heisenberg type. For every $(z,\sigma) \in \mathbb{G}$ and $y > 0$ one has

$$\mathscr{L}_s\left(((|z|^2+y^2)^2+16|\sigma|^2)^{-\frac{Q-2s}{4}}\right) \tag{0.22}$$
$$= \frac{\Gamma\left(\frac{m+2+2s}{4}\right)\Gamma\left(\frac{Q+2s}{4}\right)}{\Gamma\left(\frac{m+2-2s}{4}\right)\Gamma\left(\frac{Q-2s}{4}\right)}(4y)^{2s}((|z|^2+y^2)^2+16|\sigma|^2)^{-\frac{Q+2s}{4}}.$$

The nonlinear, nonlocal equation (0.22) represents the sub-Riemannian counterpart of (0.1). The operator $(-\Delta)^s$ has been replaced by the conformal fractional horizontal Laplacian $\mathscr{L}_s$. A remarkable byproduct of the equation (0.22) is that it immediately provides explicit global solutions to the nonlocal CR Yamabe equation

$$\mathscr{L}_s u = u^{\frac{Q+2s}{Q-2s}}.$$

Few words for the non-experts...

36 / 46

Few words for the non-experts...

If one looks at the functions in (0.22) it should be clear that they are **very different** from the corresponding ones found by Aubin, Talenti and Lieb in the Euclidean case. For instance, they **are not** functions of the gauge $N(z,\sigma) = (|z|^4 + 16|\sigma|^2)^{1/4}$, whereas their Euclidean counterparts are spherically symmetric! A glimpse into such discrepancy can be achieved by considering that in $\mathbb{H}^n$ there is a "stereographic projection", called the Cayley transform.

Few words for the non-experts...

If one looks at the functions in (0.22) it should be clear that they are **very different** from the corresponding ones found by Aubin, Talenti and Lieb in the Euclidean case. For instance, they **are not** functions of the gauge $N(z,\sigma) = (|z|^4 + 16|\sigma|^2)^{1/4}$, whereas their Euclidean counterparts are spherically symmetric! A glimpse into such discrepancy can be achieved by considering that in $\mathbb{H}^n$ there is a "stereographic projection", called the Cayley transform. Via such projection, the Heisenberg group $\mathbb{H}^n \cong \mathbb{C}^n \times \mathbb{R}$ with coordinates (z,σ) has a conformal identification with the boundary of the unit ball in $\mathbb{C}^{n+1}$, the sphere $\mathbb{S}^{2n+1}$ with its standard CR structure.

Few words for the non-experts...

If one looks at the functions in (0.22) it should be clear that they are **very different** from the corresponding ones found by Aubin, Talenti and Lieb in the Euclidean case. For instance, they **are not** functions of the gauge $N(z,\sigma) = (|z|^4 + 16|\sigma|^2)^{1/4}$, whereas their Euclidean counterparts are spherically symmetric! A glimpse into such discrepancy can be achieved by considering that in $\mathbb{H}^n$ there is a "stereographic projection", called the Cayley transform. Via such projection, the Heisenberg group $\mathbb{H}^n \cong \mathbb{C}^n \times \mathbb{R}$ with coordinates (z,σ) has a conformal identification with the boundary of the unit ball in $\mathbb{C}^{n+1}$, the sphere $\mathbb{S}^{2n+1}$ with its standard CR structure. Analogously to the term $(|x|^2+1)^{-1}$ in the stereographic projection, the function $|i + (4\sigma + i|z|^2)|^{-2} =$

Few words for the non-experts...

If one looks at the functions in (0.22) it should be clear that they are **very different** from the corresponding ones found by Aubin, Talenti and Lieb in the Euclidean case. For instance, they **are not** functions of the gauge $N(z,\sigma) = (|z|^4 + 16|\sigma|^2)^{1/4}$, whereas their Euclidean counterparts are spherically symmetric! A glimpse into such discrepancy can be achieved by considering that in $\mathbb{H}^n$ there is a "stereographic projection", called the Cayley transform. Via such projection, the Heisenberg group $\mathbb{H}^n \cong \mathbb{C}^n \times \mathbb{R}$ with coordinates (z,σ) has a conformal identification with the boundary of the unit ball in $\mathbb{C}^{n+1}$, the sphere $\mathbb{S}^{2n+1}$ with its standard CR structure. Analogously to the term $(|x|^2+1)^{-1}$ in the stereographic projection, the function $|i + (4\sigma + i|z|^2)|^{-2} = ((|z|^2+1)^2 + 16\sigma^2)^{-1}$ appears as the conformal factor in the Cayley transform, and in fact appropriate powers of such factor played a central role in the works of Jerison and Lee on the CR Yamabe problem, and Frank and Lieb on the best constant in the Sobolev embedding. It is not by chance that powers of the same conformal factor also appear in (0.22).

Similarly to the classical case discussed in the opening, to prove (0.21) we do not adopt the spectral definition (0.14) of $\mathscr{L}_s$.

37 / 46

Similarly to the classical case discussed in the opening, to prove (0.21) we do not adopt the spectral definition (0.14) of $\mathscr{L}_s$. Instead, we use the following equivalent notion which closely resembles Balakrishnan's formula:

37 / 46

Similarly to the classical case discussed in the opening, to prove (0.21) we do not adopt the spectral definition (0.14) of $\mathscr{L}_s$. Instead, we use the following equivalent notion which closely resembles Balakrishnan's formula:

$$\mathscr{L}_s u(g) \overset{def}{=} -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}} \left[P_{(-s),t} u(g) - u(g) \right] dt. \tag{0.23}$$

37 / 46

Similarly to the classical case discussed in the opening, to prove (0.21) we do not adopt the spectral definition (0.14) of $\mathscr{L}_s$. Instead, we use the following equivalent notion which closely resembles Balakrishnan's formula:

$$\mathscr{L}_s u(g) \overset{def}{=} -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}} \left[P_{(-s),t} u(g) - u(g)\right] dt. \tag{0.23}$$

The linear operator $P_{(-s),t}$ in (0.23) needs an explanation, but let me say right-away that the equivalence between the original definition (0.14) of $\mathscr{L}_s$ of Branson-Fontana-Morpurgo and that in (0.23) was established by Roncal and Thangavelu.

Similarly to the classical case discussed in the opening, to prove (0.21) we do not adopt the spectral definition (0.14) of $\mathscr{L}_s$. Instead, we use the following equivalent notion which closely resembles Balakrishnan's formula:

$$\mathscr{L}_s u(g) \overset{def}{=} -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}} \left[P_{(-s),t} u(g) - u(g)\right] dt. \tag{0.23}$$

The linear operator $P_{(-s),t}$ in (0.23) needs an explanation, but let me say right-away that the equivalence between the original definition (0.14) of $\mathscr{L}_s$ of Branson-Fontana-Morpurgo and that in (0.23) was established by Roncal and Thangavelu.

With the fundamental solution $q_{(s)}$ of the extension operator $\mathfrak{P}_{(s)}$, and its companion $q_{(-s)}$, we define

Similarly to the classical case discussed in the opening, to prove (0.21) we do not adopt the spectral definition (0.14) of $\mathscr{L}_s$. Instead, we use the following equivalent notion which closely resembles Balakrishnan's formula:

$$\mathscr{L}_s u(g) \overset{def}{=} -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}} \left[P_{(-s),t} u(g) - u(g)\right] dt. \tag{0.23}$$

The linear operator $P_{(-s),t}$ in (0.23) needs an explanation, but let me say right-away that the equivalence between the original definition (0.14) of $\mathscr{L}_s$ of Branson-Fontana-Morpurgo and that in (0.23) was established by Roncal and Thangavelu.

With the fundamental solution $q_{(s)}$ of the extension operator $\mathfrak{P}_{(s)}$, and its companion $q_{(-s)}$, we define

$$\begin{cases} \mathscr{K}_{(s)}((z,\sigma),t) = (4\pi t)^{1-s} q_{(s)}((z,\sigma),t,0), \\ \mathscr{K}_{(-s)}((z,\sigma),t) = (4\pi t)^{1+s} q_{(-s)}((z,\sigma),t,0). \end{cases}$$

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

$$\mathscr{K}_{(s)}((z,\sigma),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

$$\mathscr{K}_{(s)}((z,\sigma),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

Note that, in the local case $s=1$, the kernel $\mathscr{K}_{(s)}$ coincides with the Gaveau-Hulanicki-Cygan heat kernel $p((z,\sigma),t)$.

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

$$\mathscr{K}_{(s)}((z,\sigma),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

Note that, in the local case $s=1$, the kernel $\mathscr{K}_{(s)}$ coincides with the Gaveau-Hulanicki-Cygan heat kernel $p((z,\sigma),t)$. Denote now by $\mathscr{K}_{(-s)}((z,\sigma),t)$ the function obtained by changing s into $-s$ in the previous definition.

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

$$\mathscr{K}_{(s)}((z,\sigma),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

Note that, in the local case $s=1$, the kernel $\mathscr{K}_{(s)}$ coincides with the Gaveau-Hulanicki-Cygan heat kernel $p((z,\sigma),t)$. Denote now by $\mathscr{K}_{(-s)}((z,\sigma),t)$ the function obtained by changing s into $-s$ in the previous definition. With a slight abuse of notation we let

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

$$\mathscr{K}_{(s)}((z,\sigma),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle \sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

Note that, in the local case $s=1$, the kernel $\mathscr{K}_{(s)}$ coincides with the Gaveau-Hulanicki-Cygan heat kernel $p((z,\sigma),t)$. Denote now by $\mathscr{K}_{(-s)}((z,\sigma),t)$ the function obtained by changing s into $-s$ in the previous definition. With a slight abuse of notation we let

$$\mathscr{K}_{(\pm s)}(g,g',t) = \mathscr{K}_{(\pm s)}(g^{-1}\circ g',t),$$

and then consider the two linear operators on $L^p(\mathbb{G})$ defined by the formula

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

$$\mathscr{K}_{(s)}((z,\sigma),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

Note that, in the local case $s=1$, the kernel $\mathscr{K}_{(s)}$ coincides with the Gaveau-Hulanicki-Cygan heat kernel $p((z,\sigma),t)$. Denote now by $\mathscr{K}_{(-s)}((z,\sigma),t)$ the function obtained by changing s into $-s$ in the previous definition. With a slight abuse of notation we let

$$\mathscr{K}_{(\pm s)}(g,g',t) = \mathscr{K}_{(\pm s)}(g^{-1}\circ g',t),$$

and then consider the two linear operators on $L^p(\mathbb{G})$ defined by the formula

$$P_{(\pm s),t}u(g) = \int_{\mathbb{G}} \mathscr{K}_{(\pm s)}(g,g',t)u(g')dg'.$$

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

$$\mathscr{K}_{(s)}((z,\sigma),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

Note that, in the local case $s=1$, the kernel $\mathscr{K}_{(s)}$ coincides with the Gaveau-Hulanicki-Cygan heat kernel $p((z,\sigma),t)$. Denote now by $\mathscr{K}_{(-s)}((z,\sigma),t)$ the function obtained by changing s into $-s$ in the previous definition. With a slight abuse of notation we let

$$\mathscr{K}_{(\pm s)}(g,g',t) = \mathscr{K}_{(\pm s)}(g^{-1}\circ g',t),$$

and then consider the two linear operators on $L^p(\mathbb{G})$ defined by the formula

$$P_{(\pm s),t}u(g) = \int_{\mathbb{G}} \mathscr{K}_{(\pm s)}(g,g',t)u(g')dg'.$$

The motivation for the operator $P_{(-s),t}$ is provided by (0.23).

It is immediate to recognise from the formula (0.17) for $q_{(s)}((z,\sigma),t,y)$ that

$$\mathscr{K}_{(s)}((z,\sigma),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

Note that, in the local case $s=1$, the kernel $\mathscr{K}_{(s)}$ coincides with the Gaveau-Hulanicki-Cygan heat kernel $p((z,\sigma),t)$. Denote now by $\mathscr{K}_{(-s)}((z,\sigma),t)$ the function obtained by changing s into $-s$ in the previous definition. With a slight abuse of notation we let

$$\mathscr{K}_{(\pm s)}(g,g',t) = \mathscr{K}_{(\pm s)}(g^{-1}\circ g',t),$$

and then consider the two linear operators on $L^p(\mathbb{G})$ defined by the formula

$$P_{(\pm s),t}u(g) = \int_{\mathbb{G}} \mathscr{K}_{(\pm s)}(g,g',t)u(g')dg'.$$

The motivation for the operator $P_{(-s),t}$ is provided by (0.23). The one for $P_{(s),t}$ is unraveled by the next slide.

Conformal Riesz operators

Conformal Riesz operators

Now that we have the heat equation based definition (0.23) of the nonlocal operators $\mathscr{L}_s$, we use the operators $P_{(s),t}$, intertwined with $P_{(-s),t}$, to introduce the following conformal version of the Riesz operators

Conformal Riesz operators

Now that we have the heat equation based definition (0.23) of the nonlocal operators $\mathscr{L}_s$, we use the operators $P_{(s),t}$, intertwined with $P_{(-s),t}$, to introduce the following conformal version of the Riesz operators

$$\mathscr{I}_{(2s)}u(g) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} P_{(s),t} u(g) dt.$$

Conformal Riesz operators

Now that we have the heat equation based definition (0.23) of the nonlocal operators $\mathscr{L}_s$, we use the operators $P_{(s),t}$, intertwined with $P_{(-s),t}$, to introduce the following conformal version of the Riesz operators

$$\mathscr{I}_{(2s)}u(g) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}P_{(s),t}u(g)dt.$$

I can now state our third main result which represents the geometric counterpart of the non-conformal inversion formula (0.10) stated above.

Conformal Riesz operators

Now that we have the heat equation based definition (0.23) of the nonlocal operators $\mathscr{L}_s$, we use the operators $P_{(s),t}$, intertwined with $P_{(-s),t}$, to introduce the following conformal version of the Riesz operators

$$\mathscr{I}_{(2s)} u(g) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} P_{(s),t} u(g) dt.$$

I can now state our third main result which represents the geometric counterpart of the non-conformal inversion formula (0.10) stated above.

Theorem (Invertibility of $\mathscr{L}_s$)

For every $0 < s < 1$ and $u \in C_0^\infty(\mathbb{G})$ one has

$$\left(\mathscr{I}_{(2s)} \circ \mathscr{L}_s\right) u = \left(\mathscr{L}_s \circ \mathscr{I}_{(2s)}\right) u = u.$$

I emphasise that the proof of this theorem is not as simple as the one for the corresponding non-geometric case. Our approach is based on some lemmas of independent interest which are purely inspired to semigroup methods and does not use any non-commutative harmonic analysis. A key role is played by a representation formula for the group convolution of the intertwined kernels $\mathscr{K}_{(s)}(\cdot, t)$ and $\mathscr{K}_{(-s)}(\cdot, \tau)$ and also by a remarkable cancellation property.

I emphasise that the proof of this theorem is not as simple as the one for the corresponding non-geometric case. Our approach is based on some lemmas of independent interest which are purely inspired to semigroup methods and does not use any non-commutative harmonic analysis. A key role is played by a representation formula for the group convolution of the intertwined kernels $\mathscr{K}_{(s)}(\cdot,t)$ and $\mathscr{K}_{(-s)}(\cdot,\tau)$ and also by a remarkable cancellation property.

A related circle of ideas is at the core of our proof of the geometric intertwining formula (0.21), but some additional complications creep up.

40 / 46

I emphasise that the proof of this theorem is not as simple as the one for the corresponding non-geometric case. Our approach is based on some lemmas of independent interest which are purely inspired to semigroup methods and does not use any non-commutative harmonic analysis. A key role is played by a representation formula for the group convolution of the intertwined kernels $\mathscr{K}_{(s)}(\cdot, t)$ and $\mathscr{K}_{(-s)}(\cdot, \tau)$ and also by a remarkable cancellation property.

A related circle of ideas is at the core of our proof of the geometric intertwining formula (0.21), but some additional complications creep up.

Our plan is to proceed as closely as possible to the proof of the Euclidean case outlined above, but we immediately encounter some difficulties. In the Euclidean setting there are two aspects that play a crucial role: (i) the same heat kernel $p(\cdot, t)$ occurs both in the expression (0.8) of $(-\Delta)^s$ and in that of the function $E^{(s)}$; (ii) the Chapman-Kolmogorov identity (semigroup property) plays a critical role. Both facts (i) and (ii) fail to hold in the geometric setting!

A third more pervasive complication is represented by the very different nature of the fundamental solutions of the parabolic extension problems.

41 / 46

A third more pervasive complication is represented by the very different nature of the fundamental solutions of the parabolic extension problems. But, of course, I must skip technical details...

A third more pervasive complication is represented by the very different nature of the fundamental solutions of the parabolic extension problems. But, of course, I must skip technical details...

Let me mention that one crucial consequence of the above invertibility theorem is that the kernel

$$\mathscr{E}_{(s)}(z,\sigma) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathscr{K}_{(s)}((z,\sigma),t)dt$$

of the conformal Riesz operator $\mathscr{I}_{2s}$ provides a fundamental solution for the operator $\mathscr{L}_s$.

In the following theorem, which is our fourth main result, we compute such kernel explicitly.

In the following theorem, which is our fourth main result, we compute such kernel explicitly.

Theorem (Fundamental solution of $\mathscr{L}_s$)

Let $\mathbb{G}$ be a group of Heisenberg type. The following statements hold:

In the following theorem, which is our fourth main result, we compute such kernel explicitly.

Theorem (Fundamental solution of $\mathscr{L}_s$)

Let $\mathbb{G}$ be a group of Heisenberg type. The following statements hold:

(i) For any $0 < s \leq 1$ one has

In the following theorem, which is our fourth main result, we compute such kernel explicitly.

Theorem (Fundamental solution of $\mathscr{L}_s$)

Let $\mathbb{G}$ be a group of Heisenberg type. The following statements hold:

(i) For any $0 < s \leq 1$ one has

$$\mathscr{E}_{(s)}(z,\sigma) \overset{def}{=} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathscr{K}_{(s)}((z,\sigma),t)dt = \frac{C_{(s)}(m,k)}{N(z,\sigma)^{Q-2s}},$$

where $Q = m + 2k$ is the homogeneous dimension of $\mathbb{G}$, $N(z,\sigma) = (|z|^4 + 16|\sigma|^2)^{1/4}$ is the natural gauge, and the constant $C_{(s)}(m,k)$ is that defined in (0.19) above.

In the following theorem, which is our fourth main result, we compute such kernel explicitly.

Theorem (Fundamental solution of $\mathscr{L}_s$)

Let $\mathbb{G}$ be a group of Heisenberg type. The following statements hold:

(i) For any $0 < s \leq 1$ one has

$$\mathscr{E}_{(s)}(z,\sigma) \overset{def}{=} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathscr{K}_{(s)}((z,\sigma),t)dt = \frac{C_{(s)}(m,k)}{N(z,\sigma)^{Q-2s}},$$

where $Q = m + 2k$ is the homogeneous dimension of $\mathbb{G}$, $N(z,\sigma) = (|z|^4 + 16|\sigma|^2)^{1/4}$ is the natural gauge, and the constant $C_{(s)}(m,k)$ is that defined in (0.19) above.

(ii) The distribution $\mathscr{E}_{(s)} \in C^\infty(\mathbb{G} \setminus \{e\}) \cap L^1_{loc}(\mathbb{G})$, and it provides a fundamental solution of $\mathscr{L}_s$ with pole at the group identity $e \in \mathbb{G}$ and vanishing at infinity.

I note explicitly that in the limiting case $s = 1$ the above theorem provides a heat equation proof of Folland's remarkable formula for the fundamental solution of $\mathscr{L}$!

I note explicitly that in the limiting case $s = 1$ the above theorem provides a heat equation proof of Folland's remarkable formula for the fundamental solution of $\mathscr{L}$! But...I have already taken-up too much of your time...

43 / 46

I note explicitly that in the limiting case $s = 1$ the above theorem provides a heat equation proof of Folland's remarkable formula for the fundamental solution of $\mathscr{L}$! But...I have already taken-up too much of your time...

$\mathscr{THANK}\ \ \mathscr{YOU}\ \ \mathscr{FOR}\ \ \mathscr{BEING}\ \ \mathscr{HERE}$!

Essential bibliography

Essential bibliography

- G. B. Folland, *A fundamental solution for a subelliptic operator*. Bull. Amer. Math. Soc. **79** (1973), 373-376.

Essential bibliography

- G. B. Folland, *A fundamental solution for a subelliptic operator*. Bull. Amer. Math. Soc. **79** (1973), 373-376.
- M. Cowling & U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*. Invent. Math. **96** (1989), no. 3, 507-549.

44 / 46

Essential bibliography

- G. B. Folland, *A fundamental solution for a subelliptic operator*. Bull. Amer. Math. Soc. **79** (1973), 373-376.
- M. Cowling & U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*. Invent. Math. **96** (1989), no. 3, 507-549.
- R. L. Frank & E. H. Lieb, *Sharp constants in several inequalities on the Heisenberg group*. Ann. of Math. (2) **176** (2012), no. 1, 349-381.

Essential bibliography

- G. B. Folland, *A fundamental solution for a subelliptic operator*. Bull. Amer. Math. Soc. **79** (1973), 373-376.
- M. Cowling & U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*. Invent. Math. **96** (1989), no. 3, 507-549.
- R. L. Frank & E. H. Lieb, *Sharp constants in several inequalities on the Heisenberg group*. Ann. of Math. (2) **176** (2012), no. 1, 349-381.
- T. P. Branson, L. Fontana & C. Morpurgo, *Moser-Trudinger and Beckner-Onofri's inequalities on the* CR *sphere*. Ann. of Math. (2) **177** (2013), no. 1, 1-52.

44 / 46

Essential bibliography

- G. B. Folland, *A fundamental solution for a subelliptic operator*. Bull. Amer. Math. Soc. **79** (1973), 373-376.
- M. Cowling & U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*. Invent. Math. **96** (1989), no. 3, 507-549.
- R. L. Frank & E. H. Lieb, *Sharp constants in several inequalities on the Heisenberg group*. Ann. of Math. (2) **176** (2012), no. 1, 349-381.
- T. P. Branson, L. Fontana & C. Morpurgo, *Moser-Trudinger and Beckner-Onofri's inequalities on the* CR *sphere*. Ann. of Math. (2) **177** (2013), no. 1, 1-52.
- R. L. Frank, M. del Mar González, D. Monticelli & J. Tan, *An extension problem for the CR fractional Laplacian*. Adv. Math. **270** (2015), 97-137.

45 / 46

- L. Roncal & S. Thangavelu, *Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group*. Adv. Math. **302** (2016), 106-158.

45 / 46

- L. Roncal & S. Thangavelu, *Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group*. Adv. Math. **302** (2016), 106-158.
- L. Roncal & S. Thangavelu, *An extension problem and trace Hardy inequality for the sublaplacian on H-type groups*. Int. Math. Res. Not. IMRN **14** (2020), 4238-4294.

45 / 46

- L. Roncal & S. Thangavelu, *Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group*. Adv. Math. **302** (2016), 106-158.
- L. Roncal & S. Thangavelu, *An extension problem and trace Hardy inequality for the sublaplacian on H-type groups*. Int. Math. Res. Not. IMRN **14** (2020), 4238-4294.
- N. Garofalo & G. Tralli, *Feeling the heat in a group of Heisenberg type*. Adv. Math. **381** (2021), 2-42.

45 / 46

- L. Roncal & S. Thangavelu, *Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group*. Adv. Math. **302** (2016), 106-158.
- L. Roncal & S. Thangavelu, *An extension problem and trace Hardy inequality for the sublaplacian on H-type groups*. Int. Math. Res. Not. IMRN **14** (2020), 4238-4294.
- N. Garofalo & G. Tralli, *Feeling the heat in a group of Heisenberg type*. Adv. Math. **381** (2021), 2-42.
- N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. Preprint 2020 (ArXiv: 2007.10862)

- L. Roncal & S. Thangavelu, *Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group*. Adv. Math. **302** (2016), 106-158.
- L. Roncal & S. Thangavelu, *An extension problem and trace Hardy inequality for the sublaplacian on H-type groups*. Int. Math. Res. Not. IMRN **14** (2020), 4238-4294.
- N. Garofalo & G. Tralli, *Feeling the heat in a group of Heisenberg type*. Adv. Math. **381** (2021), 2-42.
- N. Garofalo & G. Tralli, *Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels*. Preprint 2020 (ArXiv: 2007.10862)
- N. Garofalo & G. Tralli, *A heat equation approach to intertwining*, Preprint 2020 (ArXiv:2011.10828)