A constructive approach to semilinear wave equations

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Nonlinear wave equation

We consider nonlinear wave equations of power type nonlineality:

$$(\partial_t^2 - \Delta_x)u(t,x) = |u(t,x)|^p, \qquad (t,x) \in \mathbf{R}_t \times \mathbf{R}_x^n \quad \text{(NLW)}$$

for $p > 1$ and $n \geq 2$.

The solution with given initial data can be given as a fixed point of the contraction mapping

$$X \ni u(t,x) \mapsto \cos(tD_x)u(0,x) + \frac{\sin(tD_x)}{|D_x|}u_t(0,x) + \int_0^t \frac{\sin((t-s)D_x)}{|D_x|}|u(s,\cdot)|^p \in X$$

in an appropriate Banach space X.

But what does the solution look like?

Self-similar solution

We will try to capture the most characteristic solution of (NLW) = self-similar solution.

Important observation:

$$u(t,x) : \text{sol. to (NLW)}$$

$$\Longrightarrow$$

$$u_\lambda(t,x) = \lambda^{2/(p-1)} u(\lambda t, \lambda x) : \text{sol. to (NLW)},$$

where $\lambda \neq 0$.

A solution $u(t,x)$ to (NLW) is said to be **self-similar** if

$$u(t,x) \equiv u_\lambda(t,x)$$

for any $\lambda \neq 0$.

First we discuss the *existence* of self-similar solution for as large range of p as possible.

We introduce some critical indices:

- $p_k(n) = \dfrac{n+1}{n-1}$

  : lower bound for the exist. of weak global sol.

- $p_{str}(n) = \dfrac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$

  : lower bound for the exist. of strong global sol.

- $p_{conf}(n) = \dfrac{n+3}{n-1}$: conformal critical

We note $p_k(n) < p_{str}(n) < p_{conf}(n)$.

Remark on $p_k(n)$

• T. Kato (1980): Assume $1 < p \le p_k(n)$. Then weak solution to (NLW) with compactly supported initial data satisfying

$$\int u_t(0,x)\,dx > 0$$

or

$$\int u_t(0,x)\,dx = 0, \qquad \int u(0,x)\,dx \neq 0$$

does not exist time globally.

Remark on $p_{str}(n)$

- It is the positive root of $(n-1)p^2 - (n+1)p - 2 = 0$.

- Strauss conjecture (1981): Time global solutions always exist for (NLW) if $p > p_{str}(n)$ and the size of the compactly supported smooth initial data is small. No such result can hold if $1 < p \leq p_{str}(n)$.

The case $n = 2, 3$ with $p \neq p_{str(n)}$ was solved by Glassey (1981) and John (1979), respectively, and $p = p_{str(n)}$ by Schaeffer (1985).

It is not a conjecture anymore:

$*$ $1 < p < p_{str}(n)$ blow-up : Sideris (1984).

$*$ $p > p_{str}(n)$ global : Georgiev-Lindblad-Sogge (1997).

$*$ $p = p_{str}(n)$ blow-up :

Yordanov-Zhang (2006), Zhou (2007).

Remark on $p_{conf}(n)$

• This index is associated with the conformal symmetry map:

$$u(t,x) \mapsto u_{conf}(t,x) = (t^2 - |x|^2)^{-\frac{n-1}{2}} u\left(\frac{t}{t^2 - |x|^2}, \frac{x}{t^2 - |x|^2}\right)$$

for $|x| < t$.

If u solves (NLW) then u_{conf} does when $p = p_{conf}(n)$.

Existence of self-similar solutions

- Pecher (2000) : $n = 3$, $p_{str}(3) < p < p_{conf}(3)$

- Hidano (2002) : $n = 2$, $p_{str}(2) < p < p_{conf}(2)$

- Kato-Ozawa (2003) : $n \geq 3$ odd, $p_{str}(n) < p < p_{conf}(n)$

- Kato-Ozawa (2004) : $n \geq 2$, $p_{str}(n) < p < p_{conf}(n)$

Attempts for $p \notin [p_{str}, p_{conf}]$

- Pecher (2000) : $n = 3$, $p = p_{conf}(3)$
- Planchon (2000) : $p_{conf}(n) < p \in \mathbf{N}$
- Ribaud-Youssfi (2002) : $2 \le n \le 5$, $p_{conf}(n) < p$
- Attempts for $n \ge 6$ with large p by Ribaud-Youssfi (2002), De Almeida-Ferreira (2017).
- An attempt for $p_k(n) < p < p_{str}(n)$ by Kusumoto (in preparation).

Their basic strategy is to show the *uniqueness* of sol. to (NLW) with the initial data

$$u(0,x) = \epsilon |x|^{-2/(p-1)}, \quad \partial_t u(0,x) = \epsilon |x|^{-2/(p-1)-1}$$

for small $\epsilon > 0$

$$\Longrightarrow \qquad u(0,x) = u_\lambda(0,x),\ \partial_t u(0,x) = \partial_t u_\lambda(0,x)$$

$$\Longrightarrow \qquad u(t,x) \equiv u_\lambda(t,x)$$

more precisely, uniqueness in appropriates Banach spaces X as a fixed point of the contraction mapping

$$X \ni u(t,x) \mapsto \cos(tD_x)u(0,x) + \frac{\sin(tD_x)}{|D_x|}u_t(0,x) + \int_0^t \frac{\sin((t-s)D_x)}{|D_x|}|u(s,\cdot)|^p \in X$$

- Pecher

$$\|u(t,x)\|_X = \sup_{t>0} t^{\mu} \|u(t,x)\|_{L^r(\mathbf{R}^n)}$$

$$\|u(t,x)\|_X = \sup_{|x|\neq t} (|x|+t)||x|-t|^{(3-p)/(p-1)} u(t,x)$$

- Kato-Ozawa

$$\|u(t,x)\|_X = \left\| |t^2-|x|^2|^{\gamma} u(t,x) \right\|_{L^{p,\infty}(\mathbf{R}^{1+n}_+)}$$

<u>Our goal is to give a constructive proof!</u>

Reduction to ODE

The definition of self-similarity

$$u(t,x) \equiv u_\lambda(t,x) := \lambda^{2/(p-1)} u(\lambda t, \lambda x)$$

with $\lambda = 1/t$ implies

$$u(t,x) = t^{-2/(p-1)} u(1, x/t).$$

Hence $u(t,x)$ has to be of the for form

$$u(t,x) = t^{-\beta} \varphi(x/t), \quad \beta = 2/(p-1).$$

Conversely, such $u(t,x)$ is self-similar.

Plugging it into (NLW), we have

$$\beta(\beta+1)\varphi(y)+2(\beta+1)y\cdot\nabla\varphi(y)-\Delta\varphi(y)+\langle\varphi'' y, y\rangle = |\varphi(y)|^p$$

$\Longrightarrow$

$$\left(r^2-1\right)\psi_{rr}+\left(2(\beta+1)r-\frac{n-1}{r}\right)\psi_r+\beta(\beta+1)\psi = |\psi|^p$$

with radially symmetric solution $\varphi(y) = \psi(|y|)$

$\Longrightarrow$

$$4s(s-1)f_{ss}+2\{(2\beta+3)s-n\}f_s+\beta(\beta+1)f = |f|^p$$

with $\psi(r) = f(r^2)$

$\Longrightarrow$

$$L_{a,b,c} f(s) = \frac{1}{4} |f(s)|^p \qquad \text{(NLHG)}$$

where $s(= r^2) \geq 0$ and

$$L_{a,b,c} = s(s-1)\frac{d^2}{ds^2} + \{(a+b+1)s - c\}\frac{d}{ds} + ab$$

with

$$a = \beta/2 = 1/(p-1),$$
$$b = (\beta+1)/2 = 1/(p-1) + 1/2,$$
$$c = n/2.$$

Hypergeometric function

(NLHG) can be regarded as a nonliner perturbation of **hypergeometric differential equation**:

$$L_{a,b,c}h = 0 \tag{HG}$$

Hypergeometric function

$$h_{a,b,c}(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}$$

is a solution to (HG). We remark $h_{a,b,c}(0) = 1$ and $h_{a,b,c}(s) > 0$ for $s \geq 0$.

We find a sol to (NLHG) of the form

$$f(s) = \varepsilon h(s) g(s),$$

where h is a sol to (HG) and $\varepsilon > 0$. Plugging it into (NLHG), we have

$$s(s-1)g_{ss} + 2\left\{(a+b+1)s - c + s(s-1)\frac{h_s}{h}\right\}g_s = (\varepsilon h)^p |g|^p. \quad \text{(G)}$$

We can construct a bounded function $g(s) > 0$ on $[0,1]$ for sufficiently small $\varepsilon > 0$ due to the symmetric property:

Symmetric property of (NLHG)

We set

$$T : f(s) \mapsto f(1-s), \qquad S : f(s) \mapsto -s^{-a} f(1/s)$$

then we have

$$\begin{array}{ccc} L_{a,b,c} f = \frac{1}{4}|f|^p & \xrightarrow{S} & L_{a,a-c+1,a-b+1} f = \frac{1}{4}|f|^p \\ {\scriptstyle T}\big\downarrow & & \big\downarrow {\scriptstyle R=S^{-1}TS} \\ L_{a,b,a+b-c+1} f = \frac{1}{4}|f|^p & \xrightarrow{S} & L_{a,c-b,a-b+1} f = \frac{1}{4}|f|^p. \end{array}$$

By this symmetry:

- Interval shift : $[0, 1] \overset{S}{\longleftrightarrow} [1, \infty]$

  $\Longrightarrow$ It suffices to consider (G) on $[0, 1]$.

- Singular points shift : $s = 0 \overset{T}{\longleftrightarrow} s = 1$

  $\Longrightarrow$ It suffices to consider (G) near $s = 0$.

Then we construct a positive power series solution

$$g(s) = \sum_{k=0}^{\infty} \frac{a_k}{k!} s^k; \qquad a_k = \frac{d^k g}{dt^k}(0)$$

to

$$s(s-1)g_{ss} + 2\left\{(a+b+1)s - c + s(s-1)\frac{h_s}{h}\right\}g_s = (\varepsilon h)^p g^p \quad \text{(G)}$$

for $|s| < 1$.

Noting $h(0) = 1$, we have

$$
\begin{aligned}
(G)_{|s=0} &: -2cg'(0) = \varepsilon^p g(0)^p \\
(G)'_{|s=0} &: -g''(0) + 2(a + b + 1 - h'(0))g'(0) - 2cg''(0) \\
&\qquad\qquad = p\varepsilon^p g(0)^{p-1}(h'(0)g(0) + g'(0)) \\
(G)''_{|s=0} &: \qquad \cdots \qquad = \qquad \cdots \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}
$$

Setting $g(0) = 1$, each $g^{(k)}(0) = O(\varepsilon^p)$ $(k = 1, 2, \ldots)$ is given iteratively!.

Tentative self-similar solution

Summarising the argument so far, a self-similar solution is "tentatively" constructed by

$$u(t,x) = \varepsilon\, t^{-2/(p-1)} h(|x|^2/t^2)\, g(|x|^2/t^2),$$

where h is a sol. to (HG) and g a bounded function.

Does the nonlinear term $|u(t,x)|^p$ has a meaning as a distribution?

- We have the *stable* self-similar sol. to (NLW):

$$u(t,x) = c_{n,p}|x|^{-2/(p-1)},$$

$$c_{n,p} = \left\{ \frac{2}{p-1}\left(n - 2 - \frac{2}{p-1} \right) \right\}^{1/(p-1)}.$$

Since

$$|u(t,x)|^p = (c_{n,p})^p |x|^{-2p/(p-1)} \in L^1_{loc}(\mathbf{R}^n),$$

for $p < n/(n-2)$, then it can be understood as a distribution.

• Unfortunately, it is not always the case for our construction. Indeed, when $p = (n+1)/(n-1)$, corresponding hypergeometric function is

$$h(s) = |1-s|^{-a-1}(1-s)$$

with $a = (n-1)/2$, and $g(s)$ is a constant. Then

$$u(t,x) = c_n|t^2 - |x|^2|^{-a-1}(t^2 - |x|^2)$$

we require $u \in L^p_{loc}(\mathbf{R}^{n+1})$ or equivalently

$$p\,(-a) = -p/(p-1) > -1 \iff \frac{p}{p-1} < 1.$$

The case $n = 3$

The hypergeometric function is given by an elementary function:

$$h(s) = \begin{cases} \dfrac{(1+\sqrt{s})^{\frac{p-3}{p-1}} - (1-\sqrt{s})^{\frac{p-3}{p-1}}}{\sqrt{s}} & \text{for } 0 < s < 1 \\ \\ \dfrac{(\sqrt{s}+1)^{\frac{p-3}{p-1}} + (\sqrt{s}-1)^{\frac{p-3}{p-1}}}{\sqrt{s}} & \text{for } s > 1 \end{cases}$$

is a sol to (HG) with $a = 1/(p-1)$, $b = 1/(p-1) + 1/2$, $c = n/2$ when $p \neq 3$.

When $p = 3$, we need a modification:

$$h(s) = \begin{cases} \frac{\log(1+\sqrt{s})-\log(1-\sqrt{s})}{\sqrt{s}} & \text{for } 0 < s < 1 \\ \\ \frac{\log(\sqrt{s}+1)+\log(\sqrt{s}-1)}{\sqrt{s}} & \text{for } s > 1 \end{cases}$$

When $1 < p \leq 3$, $h(s)$ is singular only at $s = 1$, while $h(s)$ has no singularity when $p > 3$.

Then a self-similar solution is constructed by

$$u(t,x) = \varepsilon\, t^{-2/(p-1)} \underbrace{h(|x|^2/t^2)}_{sing:|x|=|t|} \underbrace{g(|x|^2/t^2)}_{bdd}.$$

For the nonlinear term $|u(t,x)|^p$ to make sense, we require $u \in L^p_{loc}(\mathbf{R}^{n+1})$ or equivalently

$$\begin{aligned} h(|x|^2) \in L^p_{loc}(\mathbf{R}^n) &\Longleftrightarrow p\,\frac{p-3}{p-1} > -1 \\ &\Longleftrightarrow p > p_{str}(3) = 1 + \sqrt{2}. \end{aligned}$$

For $u(t, \cdot) \in L^p(\mathbf{R}^n)$, we further require

$$p\left(1 - \frac{p-3}{p-1}\right) > 3 \Longleftrightarrow p < p_{conf}(3) = 3.$$

The result by Pecher is recaptured!

Conclusions

- Hypergeometric functions are behind the nonliner wave equations. (Remark: Zhou (2007) and Zhou-Hua (2014) implicitly indicated this fact in different contexts.)

- By virtue of it, we can *construct* self-similar solutions.

Thank you very much for your attendance in this difficult time!