The Generalised Polyharmonic Curve Flow of Closed Planar Curves

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Outline

We are looking at the images of immersed (embedded) closed curves $\Gamma : \mathbb{S}^2 \to \mathbb{R}^2$ in $\mathbb{R}^2$. We have some basic tools which will need to be understood before any geometric analysis takes place. We define:

- $ds = |\Gamma_u|\ du$ as the Euclidean arc length element of Γ,
- $L(\Gamma) = \int_\Gamma ds$ as the length of Γ,
- $A(\Gamma) = -\frac{1}{2}\int_\Gamma \langle \Gamma, \nu \rangle\, ds$ as the enclosed area, and
- $I(\Gamma) = L^2(\Gamma)/4\pi A(\Gamma) \geq 1$ as the *isoperimetric ratio*. Equality holds for circles!
- Also, $K_{osc}(\Gamma) = L\int_\Gamma (\kappa - \bar{\kappa})^2\, ds$ is the *normalised oscillation of curvature* [1]. Here we have used $\bar{\kappa} = \frac{1}{L}\int_\Gamma \kappa\, ds$ for the *averaged total curvature*.

This last energy is the key for the methods we use today.

The Flow Equation

We looked at a general class of flows, which we dub the *polyharmonnic curve flows*. To be more specific, our curvature equation reads as:

$$\begin{cases} \frac{\partial \Gamma}{\partial t} = (-1)^p \kappa_{s^{2p}} \cdot \nu, \quad p \in \mathbb{N} \setminus \{0\}, t \in [0, T). \\ \Gamma_0 = \Gamma(\cdot, 0) \in C^\infty \text{ is closed.} \end{cases} \quad \text{(PCF)}$$

Here:

- $\kappa = \langle \Gamma_{ss}, \nu \rangle$ is the regular Euclidean curvature.
- The Subscript s^{2p} means that we are taking $2p$ repeated derivatives of κ w.r.t s for $p \in \mathbb{N}$ i.e $\kappa_{s^{2p}} := \frac{\partial^{2p}\kappa}{\partial s^{2p}}$.
- When $p = 1$ we have the curve diffusion flow.

(PCF) is a system of quasilinear parabolic eqns of order $2p + 2$.

A Note on Short Time Existence

Now, by Mantegazza and Martinazzi [4], any problem of the form

$$\begin{cases} \frac{\partial u}{\partial t} = Q\left[u\right] \\ u\left(\cdot, 0\right) = u_0 \in C^{\infty}\left(\Sigma\right), \end{cases}$$

(where Q is a smooth quasilinear, locally elliptic operator [5] of even order) admits a smooth solution on some time interval $[0, T)$. Moreover, the solution is unique and depends continuously on u_0. So...

- We split Γ up into its x and y components.
- The problem (PCF) becomes a pair of equations $\dot{x} = Q_1\left[x\right], \dot{y} = Q_2\left[y\right]$ where Q_1, Q_2 are both smooth, quasilinear, and locally elliptic.
- BINGO! Short time existence for (PCF).

Theorem 1 (Exponential Convergence to Circles)

Suppose that $\Gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ *is a* $1-$*parameter family of solutions to* (PCF) *with* $\int_{\Gamma_0} \kappa\, ds = 2\pi$ *and* $A(\Gamma_0) > 0$*. Then there exists an* $\varepsilon_0 = \varepsilon_0(p, \Gamma_0) > 0$ *such that if*

$$K_{osc}(\Gamma_0) < \varepsilon_0 \text{ and } I(\Gamma_0) < \exp\left(\varepsilon_0 / 8\pi^2\right)$$

then $T = \infty$ *and* $\Gamma(\mathbb{S}^1)$ *approaches a round circle with radius* $\sqrt{A(\Gamma_0)/\pi}$*. Moreover the convergence is exponentially fast.*

Notes on the main theorem and K_{osc}

- The condition $\int_{\Gamma_0} \kappa \, ds = 2\pi = 2\omega\pi$ implies that the turning number ω of Γ_0 is 1 (so, *no* loops!), but does not rule out figure 8's etc.
- Although not stated explicitly, the smallness of $K_{osc}(\Gamma_0)$ will imply that Γ_0 is simple. Indeed, Theorem 16 of Wheeler [1] implies that if $\int_{\Gamma} \kappa \, ds = 2\pi$ and $m(\Gamma)$ is the maximal number of self-intersections a curve has, then $K_{osc}(\Gamma) \geq 16m^2 - 4\pi^2$.
- Hence if $K_{osc}(\Gamma) < 64 - 4\pi^2 \approx 24.52$, then Γ is embedded.
- We will show that if $K_{osc}(\Gamma_0)$ is small enough, then it acts as a Lyupanov functional, and remains small for the entire of the flow (which implies embeddedness is preserved!)

The Evolution of Basic Geometric Quantities

Before we begin, we need to see how various quantities evolve under the flow (PCF). We present them without proof (for brevity):

(a) $\dot{L}(\Gamma) = -\int_\Gamma \kappa_{s^p}^2 \, ds = -\left\|\kappa_{s^p}\right\|_2^2 \leq 0,$

(b) $\dot{A}(\Gamma) = 0,$

(c) $\dot{I}(\Gamma) = -2I(\Gamma)/L(\Gamma) \cdot \int_\Gamma \kappa_{s^p}^2 \, ds \leq 0.$

(d) $\frac{d}{dt} \int_\Gamma \kappa \, ds = 0 \implies \int_\Gamma \kappa \, ds \equiv 2\omega\pi.$

Moreover, for a general function f with the same period as Γ, we have

(e) $\frac{d}{dt} \int_\Gamma f \, ds = \int_\Gamma \dot{f} + (-1)^{p+1} f \cdot \kappa \cdot \kappa_{s^{2p}} \, ds.$

Some Useful Supporting Lemmas

We will introduce a few lemmas that will be used ad nauseum throughout this talk:

Lemma 2

Let $f : \mathbb{R} \to \mathbb{R}$ *be an absolutely continuous and periodic function of period* P*. Then, if* $\int_0^P f\,dx = 0$ *we have*

$$\int_0^P f^2\,dx \leq P^2/4\pi^2 \int_0^P f_x^2\,dx.$$

Lemma 3

Let $f : \mathbb{R} \to \mathbb{R}$ *satisfy the same conditions as the previous lemma. Then*

$$\|f\|_\infty^2 \leq P/2\pi \int_0^P f_x^2\,dx.$$

Another Useful Supporting Lemma

Lemma 4 (Dziuk,Kuwert,Schätzle [3])

Let $\Gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be sufficiently smooth, and closed. Then

$$\int_\Gamma \left| P_i^{j,l-1}\left(\kappa - \bar{\kappa}\right)\right| ds$$

$$\leq c\left(i,j,l\right) L^{1-i-j}\left(K_{osc}\right)^{\frac{i-\eta}{2}}\left(L^{2l+1}\int_\Gamma \left(\kappa - \bar{\kappa}\right)_{s^l}^2 ds\right)^{\frac{\eta}{2}}.$$

Here $P_i^{j,l-1}\left(\phi\right)$ refers to a polynomial in ϕ that contains i terms with a total of j derivatives, of highest order $l-1$. Also, $\eta := \left(j + i/2 - 1\right)/l$.

Note that Lemma 2 helps us to establish an L^1 bound for K_{osc} in time. Indeed, by applying Lemma 2 and (a), we have

$$\boxed{\int_0^T K_{osc}\, d\tau \leq L^{2(p+1)}\left(0\right)/2\left(p+1\right)\left(2\pi\right)^{2p} < c\left(\Gamma_0, p\right).(1)}$$

Lemma 5

Suppose that $\Gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ *solves* (PCF). *Then*

$$\frac{d}{dt}\left(K_{osc} + 8\pi^2 \ln L\right) + \frac{\|\kappa_{s^p}\|_2^2}{L} K_{osc} + L\left(2 - c_1 K_{osc} - c_2\sqrt{K_{osc}}\right)\|\kappa_{s^{p+1}}\|_2^2 \leq 0$$

for some constant $c_1, c_2 > 0$. *Moreover if for* $t \in [0, T^*)$ *we have*

$$K_{osc}(t) \leq \left(8c_1 + 2c_2^2 - 2c_2\sqrt{8c_1 + c_2^2}\right)/4c_1^2 =: 2K^\star,$$

then during this time the following inequality holds:

$$K_{osc} + 8\pi^2 \ln L + \leq K_{osc}(\Gamma_0) + 8\pi^2 \ln L(\Gamma_0). \tag{2}$$

Proof.

Applying (e) from our evolution equations, and performing integration by parts (A LOT!!), we get

$$\frac{d}{dt}\left(K_{osc} + 8\pi^2 \ln L\right) + \frac{\|\kappa_{s^p}\|_2^2}{L} K_{osc} + 2L\,\|\kappa_{s^{p+1}}\|_2^2 = L\int_\Gamma \left[(\kappa-\bar{\kappa})^3 + \bar{\kappa}(\kappa-\bar{\kappa})^2\right]_{s^p}(\kappa-\bar{\kappa})_{s^p}\,ds. \qquad (3)$$

Next we control the $P-$style terms via the last lemma, to get

$$L\int_\Gamma \left[(\kappa-\bar{\kappa})^3 + \bar{\kappa}(\kappa-\bar{\kappa})^2\right]_{s^p}(\kappa-\bar{\kappa})_{s^p}\,ds \leq L\int_\Gamma \left|P_4^{2p,p}(\kappa-\bar{\kappa})\right| ds + 2\int_\Gamma \left|P_3^{2p,p}(\kappa-\bar{\kappa})\right| ds \leq \left(c_1 K_{osc} + c_2\sqrt{K_{osc}}\right)\|\kappa_{s^{p+1}}\|_2^2 . \qquad (4)$$

Proof (Cont.)

Substituting this into (3) and absorbing into the LHS gives the first statement. The second statement follows immediately by integrating the first.

Corollary 6

Let $\Gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ *solve* (*PCF*), *and suppose that* Γ_0 *is a simple closed curve satisfying*

$$K_{osc}(\Gamma_0) \leq K^\star \text{ and } I(\Gamma_0) \leq \exp\left(K^\star/8\pi^2\right), \tag{5}$$

where $K^\star$ *is the parameter from the previous lemma. Then*

$$K_{osc}(\Gamma_t) \leq 2K^\star \text{ for } t \in [0, T).$$

Proof (Rough.)

This is proved by contradiction:

Proof (Cont.)

- We assume there is a *maximal* $T^* < T$ such that $K_{osc} \leq 2K^\star$ on $[0, T^\star)$.
- The circularity assumptions and (2) can be combined to show that for $t \in [0, T^*)$,

$$K_{osc} \leq K_{osc}(\Gamma_0) + 8\pi^2 \ln \sqrt{I(\Gamma_0)} \leq 3K^\star/2 < 2K^\star.$$

- This contradicts the maximality of T^*. We conclude the result.

This corollary gives us control over K_{osc} for the entirety of the flow!

Lemma 7

Let $\Gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ *be a maximal solution to* (PCF). *If* $T < \infty$, *then*

$$\int_\Gamma \kappa^2 \, ds \geq c \, (T - t)^{-1/2(p+1)} \tag{6}$$

Proof.

Assume $T < \infty$. By using the $P-$style estimation as before, it is possible to arrive at the inequality

$$\frac{d}{dt} \int_\Gamma \kappa_{s^m}^2 \, ds + \int_\Gamma \kappa_{s^{m+p+1}}^2 \, ds \leq c \, (m, p) \left(\int_\Gamma \kappa^2 \, ds \right)^{2(m+p)+3} \tag{7}$$

for any $m \in \mathbb{N}_0$. In particular, for $m = 0$ we conclude that

$$\frac{d}{dt} \int_\Gamma \kappa^2 \, ds \leq c \, (p) \left(\int_\Gamma \kappa^2 \, ds \right)^{2p+3} . \tag{8}$$

Proof (Cont.)

Now if $\limsup_{t\to T}\int_\Gamma \kappa^2\, ds = \infty$ then integrating (8) and taking $t \nearrow T$ would prove the lemma. So for the sake of contradiction, we assume

$$\limsup_{t\to T}\int_\Gamma \kappa^2\, ds < \varrho < \infty. \tag{9}$$

Roughly, we proceed as follows:

- Note that assuming (9) and integrating (7) gives the estimate $\int_\Gamma \kappa_{s^m}^2\, ds \leq c_m(\Gamma_0, \varrho, T)$ up until time T.
- By writing derivatives of κ in terms of derivatives of the immersion Γ (and with a little bootstrapping), it is possible to show that $\|\partial_u^m \Gamma\|_\infty \leq d_m(\Gamma_0, \varrho, T)$ up until time T.
- Hence Γ is smooth right up until time T, and by short time existence results the flow can be extended to some interval $[0, T+\delta)$. This contradicts the maximality of T!

Corollary 8 (Long Time Existence)

Suppose $\Gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ *solves* (PCF)*, as well as the initial "circularity" conditions* (5)*. Then* $T = \infty$*.*

Proof (Rough.)

This is proved by contradiction:

- We assume that $T < \infty$. By the previous lemma, this implies that $\|\kappa\|_2^2 \nearrow \infty$ as $t \nearrow T$,
- Expanding K_{osc} and using the isoperimetric inequality, this implies that

$$K_{osc} \geq \sqrt{4\pi A(\Gamma_0)} \int_\Gamma \kappa^2 \, ds - 4\pi^2 \nearrow \infty.$$

- This contradicts Corollary 6 (where we showed $K_{osc} \leq 2K^\star$). Hence T can not be less than ∞.

Now to prove the main theorem!

Proof of the Main Theorem.

Remember that by Corollary 8 if Γ_0 satisfies the circularity conditions (2) then $T = \infty$. Hence by (1) we have

$$\int_0^\infty K_{osc}\, d\tau < c\,(\Gamma_0, p)\,. \tag{10}$$

To show convergence to a circle, we aim to show that $K_{osc} \searrow 0$.

- From (10) it will be enough to establish an absolute bound on $|K'_{osc}|$ (to rule out the possibility of “spikes” in time from occurring!). This is relatively easy:
- It is possible to show that $\frac{d}{dt}\,\|\kappa_{s^p}\|_2^2 \leq c\,(p, \Gamma_0)\,K_{osc}$. Integrating and using (10) gives $\|\kappa_{s^p}\|_2^2 \leq c^\star\,(p, \Gamma_0)$.
- Hence by Lemma 5, for $\varepsilon_0 \leq K^\star$, we have

$$\left|\frac{d}{dt}K_{osc}\right| \leq \frac{8\pi^2 - K_{osc}}{L}\,\|\kappa_{s^p}\|_2^2 \leq \frac{8\pi^2}{\sqrt{4\pi A\,(\Gamma_0)}}c^\star\,(\Gamma_0, p) \ll \infty.$$

Proof (Exponential Convergence.)

- Hence $K_{osc} \searrow 0$, and $\Gamma\left(\mathbb{S}^1, t\right) \to \mathbb{S}_{\sqrt{A(\Gamma_0)/\pi}}$.
- The exponential convergence result is a bit fiddly. The key idea is that if Γ_0 satisfies the circularity conditions for ε_0 sufficiently small then the following holds:

$$\begin{aligned}\frac{d}{dt}\int_\Gamma \kappa_{s^m}^2\,ds &\leq -\int_\Gamma \kappa_{s^{m+p+1}}^2\,ds\\ &\leq -\left(2\pi/L\left(\Gamma_0\right)\right)^{2(p+1)}\int_\Gamma \kappa_{s^m}^2\,ds.\end{aligned}$$

- Integrating gives exponential decay in L^2.
- To get exponential decay in L^∞, simply combine the exponential L^2 result with Lemma 3.

FIN

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