SUMS PROBLEM COMPETITION, 1998

The organizers of the SUMS Problem Competition would like to acknowledge the many contributions made by David Jackson to the competition over the last 10 years. He was a prize winner in each of the 1989, 1990 and 1991 competitions, and supplied many problems since then, including three in this year's competition. He died in August this year at the age of 27.

SOLUTIONS

1. Consider the general case immediately. Suppose that we can form a palindrome using m_i a_i's for $i = 1, \ldots, r$. The total number of letters in the palindrome is $m_1 + \cdots + m_r$.

Case 1: $m_1 + \cdots + m_r$ is even, $2n$ say. Let the number of a_i's in the first half of the palindrome be k_i. Then the number of a_i's in the second half of the palindrome must also be k_i. Hence $m_i = 2k_i$ for each i. Thus each m_i must be even.

Case 2: $m_1 + \cdots + m_r$ is odd, $2n + 1$ say. Let the $n + 1$-st letter of the palindrome be a_j. For each i, let the number of a_i's in the first n letters of the palindrome be k_i. Then the number of a_i's in the last n letters of the palindrome must also be k_i. Hence $m_i = 2k_i$ for each $i \neq j$, and $m_j = 2k_j + 1$. Thus exactly one of the m_i's is odd.

Thus a palindrome can only exist when at most one of the m_i's is odd.

To count the number of palindromes, we again consider two cases separately.

Case 1: each m_i is even, $m_i = 2k_i$, say. Then to form a palindrome, we must first form a word of length $n = k_1 + \cdots + k_r$ consisting of k_i a_i's for each i. The second half of the palindrome must be this word repeated in reverse order. So the palindrome is determined by the first half, and the number of possiblities is the multinomial coefficient

$$\binom{k_1 + \cdots + k_r}{k_1, \ldots, k_r} = \frac{(k_1 + \cdots + k_r)!}{k_1!k_2! \cdots k_r!} = \frac{\frac{m_1 + \cdots + m_r}{2}!}{\frac{m_1}{2}!\frac{m_2}{2}! \ldots \frac{m_r}{2}!}.$$

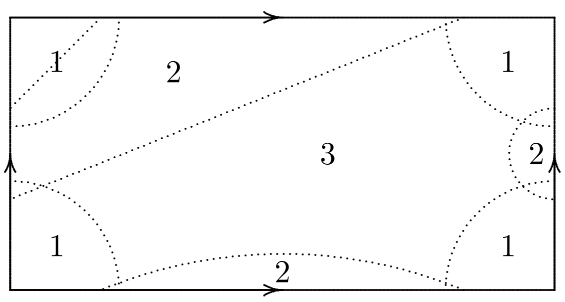
Case 2: m_j is odd, $m_j = 2k_j + 1$, say, and the other m_i's are even, $m_i = 2k_i$, say. Then to form a palindrome, we must first form a word of length $n = k_1 + \cdots + k_r$ consisting of k_i a_i's for each i. Then the $n + 1$-st letter of the palindrome must be a_j. The last n letters of the palindrome must be the same as the first n letters, repeated in reverse order. So the palindrome is determined by the first n letters, and the number of possibilities is

$$\binom{k_1 + \cdots + k_r}{k_1, \ldots, k_r} = \frac{(k_1 + \cdots + k_r)!}{k_1!k_2! \cdots k_r!} = \frac{\frac{m_1 + \cdots + m_r - 1}{2}!}{\frac{m_1}{2}! \ldots \frac{m_j - 1}{2}! \ldots \frac{m_r}{2}!}.$$

2. Let $f(x) = x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$. Then if the curve $y = f(x)$ meets the line $y = \alpha x + \beta$ in 4 distinct points, then $f(x) - \alpha x - \beta = 0$ for 4 distinct points, and so $f'(x) - \alpha = 0$ for 3 distinct points, by Rolles's theorem. Hence $f''(x) = 0$ for 2 distinct points, again by Rolles's theorem. But $f''(x) = 12x^2 + 6c_3x + 2c_2$, and so $(6c_3)^2 > 4 \cdot 12 \cdot 2c_2$. That is, $c_3^2 > 8c_2/3$. Conversely, if $c_3^2 > 8c_2/3$, let $\xi_1 < \xi_2$ be the two roots of $f''(x) = 0$.

Then $f''(x) > 0$ for $x < \xi_1$ and for $x > \xi_2$, and $f''(x) < 0$ for $\xi_1 < x < \xi_2$. So $f'(x)$ is increasing on $(-\infty, \xi_1]$ decreasing on $[\xi_1, \xi_2]$, and increasing on $[\xi_2, \infty)$. So ξ_1 is a local maximum for $f'(x)$ and ξ_2 is a local minimum for $f'(x)$, and $f'(\xi_2) < f'(\xi_1)$. Choose any $\alpha \in (f'(\xi_2), f'(\xi_1))$. Then $f'(x) - \alpha < 0$ for large negative x, $f'(x) - \alpha > 0$ for x near ξ_1, $f'(x) - \alpha < 0$ for x near ξ_2, and $f'(x) - \alpha > 0$ for large positive x. Hence $f'(x) - \alpha = 0$ holds for three distinct x's, say for $x = t_1, t_2, t_3$, where $t_1 < t_2 < t_3$. Then $f(x) - \alpha x$ is decreasing on $(-\infty, t_1]$, increasing on $[t_1, t_2]$, decreasing on $[t_2, t_3]$, and increasing on $[t_3, \infty)$. Thus t_2 is a local maximum for $f(x) - \alpha x$, and $f(t_2) - \alpha t_2 > f(t_1) - \alpha t_1$ and $f(t_2) - \alpha t_2 > f(t_3) - \alpha t_3$. Choose β less than $f(t_2) - \alpha t_2$, but greater than both $f(t_1) - \alpha t_1$ and $f(t_3) - \alpha t_3$. Then $f(x) - \alpha x - \beta$ is positive for large negative x, negative for x near t_1, positive for x near t_2 , negative for x near t_3, and positive again for large positive x. Hence $f(x) - \alpha x - \beta = 0$ for 4 x's, say $x = u_1, u_2, u_3, u_4$, where $u_1 < t_1 < u_2 < t_2 < u_3 < t_3 < u_4$.

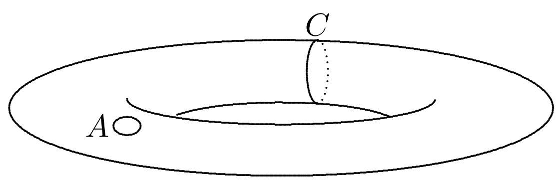
3. The smallest number of charts needed to cover the torus, T, is 3. It is easy to exhibit three charts which cover T. Imagine T as the image of the following rectangle, in which the opposite sides have been glued together along the sides indicated by the parallel arrows. The three charts are indicated by the numbers 1, 2 and 3. For example, chart 1 is the image of the quarter discs in the four corners of the rectangle.



We now show that it is not possible to cover T with less than three charts. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open disc of radius 1 in the plane, centred on the origin. A chart is the image of a continuous one to one map $\varphi : \mathbb{D} \to T$ such that $\varphi(\mathbb{D})$ is open in T and such that $\varphi : \mathbb{D} \to \varphi(\mathbb{D})$ is a homeomorphism.

It is certainly not possible to cover T with a single chart. For then we would have a homeomorphism φ of $\mathbb{D}$ onto T. This is impossible because T is compact and $\mathbb{D}$ is not.

To show that it is not possible to cover the torus with two charts, A and B, say, first suppose that A is small, as in the following diagram. Choose any loop C going around the torus and not meeting A:



Then because $C \subset T = A \cup B$ and $C \cap A = \emptyset$, we have $C \subset B$. But any loop in a disc, and hence in a chart, must be contractible to a single point. The given loop is not contractible.

When A' and B' are two charts covering T, neither of which is small, we shrink one of them, say A'. What we precisely need here is the following theorem (see Proposition A2.2.6 on page 100 of "A first course in geometric topology and differential geometry" by Ethan D. Bloch): Let A and A' be two charts in T. Then there is a homeomorphism $H : T \to T$ such that $H(A') = A$. (This is valid if T is replaced by any path connected surface).

When we apply this result, we see that $T = H(T) = H(A' \cup B') = H(A') \cup H(B') = A \cup B$, where $B = H(B')$. Both A and B are charts, and A is small. Note that in the notation of the cited book, a chart is the the homeomorphic image of a closed disc. But if T is written as the union of two charts in our sense, then a simple compactness argument (just shrink our discs a little) shows that T is also expressible as the union of two charts in the sense of Bloch.

4. The proof goes by induction on the minimum number of marbles in the three boxes. Let $a \le b \le c$ be the numbers of marbles in the boxes. Assume that if the numbers of marbles in the boxes are $a' \le b' \le c'$, where $a' < a$, then the process can be chosen to lead to an empty box. Label the boxes containing a, b and c marbles B_1, B_2 and B_3, respectively.

Divide a into b: $b = aq + r$, where $0 \le r < a$ and $q \ge 1$. Write q in the binary system:

$$q = m_0 + 2m_1 + \cdots + 2^k m_k,$$

where each m_i is 0 or 1, and $m_k = 1$. Place in the first box successively a, $2a, \ldots, 2^k a$ marbles, such that for $i = 0, \ldots, k$, if $m_i = 1$, the $2^i a$ marbles are taken from B_2 and if $m_i = 0$, the $2^i a$ marbles are taken from B_3. In this fashion, we have taken at most $(1 + 2 + \cdots + 2^{k-1})a = (2^k - 1)a < qa \le b \le c$ marbles from B_3. We have taken exactly qa marbles from B_2, leaving $r < a$ there. By the induction hypothesis, the process can be continued until one of the boxes is empty.

5. Let $u_1, u_2, \ldots$ be a sequence of positive terms such that $u_n \to 0$ as $n \to \infty$, but such that $\sum_{k=1}^{\infty} u_k^3 = \infty$. For example, take $u_k = 1/k^{1/3}$. For simplicity, we assume also that $u_k \le \pi$ for all k. The series

$$a_1 + a_2 + a_3 + \cdots = \frac{u_1}{2} + \frac{u_1}{2} - u_1 + \frac{u_2}{2} + \frac{u_2}{2} - u_2 + \frac{u_3}{2} + \frac{u_3}{2} - u_3 + \cdots$$

converges to 0, because for each n its $3n$-th partial sum s_{3n} equals 0, while $s_{3n+1} = u_{n+1}/2$ and $s_{3n+2} = u_{n+1}$. The $3n$-th partial sum S_{3n} of the series $\sin(a_1) + \sin(a_2) + \cdots$ is the sum of the n terms

$$2\sin(u_k/2) - \sin(u_k) = 2\sin(u_k/2)(1 - \cos(u_k/2)), \quad k = 1, \ldots, n.$$

Now $\sin(x)/x \to 1$ and $(1 - \cos(x))/x^2 \to 1/2$ as $x \to 0$ by l'Hôpital's rule. So the function $f(x)$, defined on $[0, \pi/2]$ by $f(0) = 1$ and $f(x) = 2\sin(x)(1 - \cos(x))/x^3$ for $0 < x \le \pi/2$, is continuous and positive throughout $[0, \pi/2]$. So its minimum value c on $[0, \pi/2]$ is positive. So $2\sin(x)(1-\cos(x))/x^3 \geq c$ for all $x \in (0, \pi/2]$. Since we assumed that $0 < u_k \leq \pi$ for all k,

$$2\sin(u_k/2)(1-\cos(u_k/2)) \geq \frac{c}{8}u_k^3.$$

Hence

$$S_{3n} \geq \frac{c}{8}\sum_{k=1}^{n} u_k^3,$$

which tends to ∞ as $n \to \infty$ by hypothesis. Hence the series $\sum_{k=1}^{\infty}\sin(a_k)$ diverges.

It is not possible to give a convergent series $\sum_{k=1}^{\infty} a_k$ with $|a_1| \geq |a_2| \geq \cdots$ such that $\sum_{k=1}^{\infty}\sin(a_k)$ is divergent. To see this, we set $b_k = \sin(a_k)/a_k$. Then $b_k \to 1$ as $k \to \infty$ because $a_k \to 0$. Also, $b_k = |b_k| = \sin(|a_k|)/|a_k|$ increases with k (once k is so large that $|a_k| \leq \pi/2$), because $\sin(x)/x$ is a decreasing function on $(0, \pi/2]$ (since $\tan(x) > x$ for $0 < x < \pi/2$).

Since $\sum_{k=1}^{\infty} a_k$ converges and $b_1, b_2, \ldots$ is a monotone sequence converging to a limit, the series $\sum_{k=1}^{\infty} a_k b_k$ converges by Abel's Test (see, for example, Bartle and Sherbert "Introduction to Real Analysis"). That is, $\sum_{k=1}^{\infty}\sin(a_k)$ converges.

6. For $n = 1, 2, \ldots$, let

$$f_n(t) = \Big(\frac{e^t + e^{2t} + e^{-3t}}{3}\Big)^{n^2} e^{-tn}.$$

It is easy to see that $f_n(t) \to \infty$ as $t \to \pm\infty$. For example, $f_n(t) \geq (e^{2t}/3)^{n^2}e^{-tn} = e^{tn(2n-1)}/3^{n^2} \to \infty$ as $t \to +\infty$. So $f_n(t)$ has a minimum value at some point of $\mathbb{R}$. At this point, $f_n'(t)$ must equal 0. A routine calculation shows that $f_n'(t) = 0$ if and only if

$$e^t + e^{2t} + e^{-3t} = n(e^t + 2e^{2t} - 3e^{-3t}). \tag{1}$$

Now

$$e^t + 2e^{2t} - 3e^{-3t} = e^{-3t}(e^t - 1)(2e^{4t} + 3e^{3t} + 3e^{2t} + 3e^t + 3)$$

is negative if $t < 0$, and so (1) has no solutions for $t < 0$, and clearly, $t = 0$ is not a solution. For $t > 0$,

$$\frac{d}{dt}\Big(\frac{e^t + e^{2t} + e^{-3t}}{e^t + 2e^{2t} - 3e^{-3t}}\Big) = -\frac{e^{3t} + 16e^{-2t} + 25e^{-t}}{(e^t + 2e^{2t} - 3e^{-3t})^2} < 0,$$

and so $(e^t+e^{2t}+e^{-3t})/(e^t+2e^{2t}-3e^{-3t})$ is strictly decreasing on $(0, \infty)$. As this expression clearly tends to ∞ as $t \to 0$ from the right, we see that for each n, (1) has a unique solution t_n, which is positive. Clearly $t_n \to 0$ as $n \to \infty$, and since

$$t\frac{e^t + e^{2t} + e^{-3t}}{e^t + 2e^{2t} - 3e^{-3t}} \to \frac{3}{14} \quad \text{as } t \to 0+$$

by l'Hôpital's rule, we see that $t_n n \to 3/14$ as $n \to \infty$.

By Taylor's theorem, $e^t = 1 + t + \frac{t^2}{2}(1+\epsilon(t))$, where $\epsilon(t) \to 0$ as $t \to 0$. Hence

$$\frac{e^t + e^{2t} + e^{-3t}}{3} = 1 + \frac{7}{3}t^2\big(1+\delta(t)\big),$$

where $\delta(t) = (1/14)(\epsilon(t) + 4\epsilon(2t) + 9\epsilon(-3t)) \to 0$ as $t \to 0$. Hence

$$\min_t f_n(t) = f_n(t_n) = \left(1 + \frac{7}{3}t_n^2\big(1+\delta(t_n)\big)\right)^{n^2} e^{-t_n n}$$

which equals

$$\left(1 + \frac{3}{28}\frac{1}{n^2}\big(1+\epsilon_n\big)\right)^{n^2} e^{-3/14+\epsilon_n'},$$

where $\epsilon_n, \epsilon_n' \to 0$ as $n \to \infty$. Now $(1+x/n)^n \to e^x$ as $n \to \infty$, and it is easy to see from this that if $x_n \to x$, then $(1+x_n/n)^n \to e^x$ as $n \to \infty$. Applying this to $x_n = (3/28)(1+\epsilon_n)$, we see that

$$\lim_{n\to\infty} \min_t f_n(t) = e^{3/28}e^{-3/14} = e^{-3/28}.$$

7. We work in the field $\mathbb{F}_p$ consisting of the set $\{0, 1, \ldots, p-1\}$, where addition and multiplication are taken modulo p. When $p = 2$, then 0 is the only number in $\mathbb{F}_p$ expressible in the form $x^3 - 3x$, and $1 = (2p-1)/3$ in this case. If $p = 3$, then all of 0, 1 and 2 are so expressible. So assume below that $p \geq 5$. Suppose that $\alpha \in \mathbb{F}_p$. Then

$$X^3 - 3X - (\alpha^3 - 3\alpha) = (X-\alpha)(X^2 + \alpha X + \alpha^2 - 3). \tag{1}$$

If $3(4-\alpha^2)$ has a square root σ in $\mathbb{F}_p$, then by the quadratic formula the quadratic factor on the right in (1) has roots $X = (-\alpha \pm \sigma)/2$. These are distinct iff $\alpha \neq \pm 2$, and are distinct from α iff $\alpha \neq \pm 1$. When $\alpha = \pm 1$ or ± 2, we can give explicit factorizations of the cubic polynomials in (1):

$$X^3 - 3X - 2 = (X-2)(X+1)^2 \quad \text{and} \quad X^3 - 3X + 2 = (X+2)(X-1)^2.$$

When $\alpha \neq \pm 1, \pm 2$ and when $3(4-\alpha^2)$ has a square root in $\mathbb{F}_p$, then the cubic polynomial in (1) is the product of three distinct linear factors:

$$X^3 - 3X - (\alpha^3 - 3\alpha) = (X-\alpha)(X-\beta)(X-\gamma), \text{ say.}$$

The elements $\alpha^3 - 3\alpha$, $\beta^3 - 3\beta$ and $\gamma^3 - 3\gamma$ are all the same, and α, β and γ are all different from ± 1 and ± 2, and all of $3(4-\alpha^2)$, $3(4-\beta^2)$ and $3(4-\gamma^2)$ have non-zero square roots in $\mathbb{F}_p$.

If $\alpha^3 - 3\alpha$ does not have a square root in $\mathbb{F}_p$, then the quadratic factor in (1) has no roots, and so $X^3 - X - (\alpha^3 - 3\alpha) = 0$ has no solutions in $\mathbb{F}_p$ other than α.

So we can divide the set $\mathbb{F}_p$ into 3 disjoint subsets: the set S_3 of $\alpha \in \mathbb{F}_p \setminus \{\pm 1, \pm 2\}$ such that $3(4-\alpha^2)$ has a square root in $\mathbb{F}_p$, the set $S_2 = \{\pm 1, \pm 2\}$, and the set S_1 of $\alpha \in \mathbb{F}_p$ such that $3(4-\alpha^2)$ has no square root in $\mathbb{F}_p$. The map $f : \mathbb{F}_p \to \mathbb{F}_p$ defined by

$f(\alpha) = \alpha^3 - 3\alpha$ is, because of the above discussion, one to one on S_1, two to one on S_2 and three to one on S_3. The set S of elements of $\mathbb{F}_p$ which are expressible in the form $\alpha^3 - 3\alpha$ is the image of f, and has order $|S| = |S_1| + |S_2|/2 + |S_3|/3$, which equals $2 + |S_1| + |S_3|/3$ because $p \geq 5$.

We next show that if $p \equiv 1 \bmod 3$ then $|S_1| = (p-1)/2$ and $|S_3| = (p-7)/2$, while if $p \equiv -1 \bmod 3$ then $|S_1| = (p-3)/2$ and $|S_3| = (p-5)/2$. The result will follow.

Recall that if p is an odd prime and if n is an integer not divisible by p, then we write $\left(\frac{n}{p}\right) = 1$ if n modulo p has a square root in $\mathbb{F}_p$ and $\left(\frac{n}{p}\right) = -1$ otherwise. The following well-known facts can be found in Chapter 3 of Niven and Zuckerman, "The Theory of Numbers", for example:

(i). $\left(\frac{mn}{p}\right) = \left(\frac{m}{p}\right)\left(\frac{n}{p}\right)$.

(ii). -1 has a square root in $\mathbb{F}_p$ if and only if $p \equiv 1 \bmod 4$. That is, $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$.

(iii). For distinct odd primes p and q,

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{(p-1)(q-1)/4}.$$

This is called the quadratic reciprocity theorem.

Applying (iii) to $q = 3$, we see that if $p \equiv 1 \bmod 3$ (so that $\left(\frac{p}{3}\right) = 1$) then $\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}$. Hence by (i) and (ii), $\left(\frac{-3}{p}\right) = 1$. Similarly, if $p \equiv -1 \bmod 3$ (so that $\left(\frac{p}{3}\right) = -1$) then $\left(\frac{3}{p}\right) = -(-1)^{(p-1)/2}$. Hence by (i) and (ii), $\left(\frac{-3}{p}\right) = -1$.

In summary, $\left(\frac{-3}{p}\right) = \pm 1$ according as $p \equiv \pm 1 \bmod 3$.

So to count the number of $\alpha \notin \{2, -2\}$ for which $3(4 - \alpha^2)$ is a square, we need only count the number of α for which $\alpha^2 - 4$ is a non-zero square. If $\alpha^2 - 4 = \beta^2$, then $(\alpha + \beta)(\alpha - \beta) = 4$. We can solve the equation $xy = 4$ by taking any $x \neq 0$ and $y = 4/x$. So setting $\alpha = \frac{1}{2}\left(x + \frac{4}{x}\right)$ and $\beta = \frac{1}{2}\left(x - \frac{4}{x}\right)$, we get a solution (α, β) of the equation $\alpha^2 - 4 = \beta^2$, and each solution arises in this way. The non-zero elements of $\mathbb{F}_p$ can be listed in this way:

$$2, -2, x_1, \frac{4}{x_1}, \ldots, x_r, \frac{4}{x_r},$$

where $r = (p-3)/2$. Thus the elements $\alpha_j = \frac{1}{2}\left(x_j + \frac{4}{x_j}\right)$, $j = 1, \ldots, r$, are the distinct $\alpha \notin \{2, -2\}$ for which $\alpha^2 - 4$ is a square. When $p \equiv 1 \bmod 3$, then -3 is a square, and so ± 1 are two of the numbers $\alpha_1, \ldots, \alpha_r$. Thus $|S_3| = (p-3)/2 - 2 = (p-7)/2$ in this case. When $p \equiv -1 \bmod 3$, then -3 is not a square, and so ± 1 do not appear among the numbers $\alpha_1, \ldots, \alpha_r$. Thus there are exactly $(p-3)/2$ elements $\alpha \neq \pm 2$ such that $\alpha^2 - 4$ is a square, and so S_3, which in this case is the set of α such that $\alpha^2 - 4$ is not a square, has $p - 4 - (p-3)/2 = (p-5)/2$ elements. As $\mathbb{F}_p$ is the disjoint union of the three sets S_1, S_2 and S_3, we have $|S_1| + |S_2| + |S_3| = p$. Hence $|S_1| = (p-1)/2$ or $|S_1| = (p-3)/2$ according as $p \equiv 1$ or $p \equiv -1 \bmod 3$.

8. Let $S_n = \sum_{k=1}^n a_k$ and $S'_n = \sum_{k=n}^\infty 1/a_k$ for each $n \geq 1$.

(i). Suppose that there is a number C such that $a_n^{-1} \sum_{k=1}^n a_k \leq C$ for all n. Then $S_n \leq Ca_n$, and so $a_n \geq S_n/C$ for all n. For each n, $S_{n+1} = S_n + a_n \geq S_n + S_n/C = S_n(1+1/C)$. Repeating this, we find that $S_{n+j} \geq S_n(1+1/C)^j$ for $j = 0, 1, \ldots$. So, noting that $S_n \geq a_n$ for each n, $a_{n+j} \geq S_{n+j}/C \geq S_n(1+1/C)^j/C \geq a_n(1+1/C)^j/C$ for all $j \geq 1$. Thus

$$\sum_{k=n}^\infty \frac{1}{a_k} = \frac{1}{a_n} + \sum_{j=1}^\infty \frac{1}{a_{n+j}} \leq \frac{1}{a_n} + \sum_{j=1}^\infty \frac{C}{a_n(1+\frac{1}{C})^j} = \frac{1}{a_n}\Big(1 + C\sum_{j=1}^\infty \Big(\frac{C}{C+1}\Big)^j\Big) = \frac{1+C^2}{a_n}.$$

So $a_n S'_n \leq 1 + C^2$.

Conversely, suppose that there is a number C' such that $a_n S'_n \leq C'$ for all n. Clearly, $S'_n > 1/a_n$ for all n, and so $C' > 1$. Also, $1/a_n \geq S'_n/C'$, and so if $n > 1$,

$$S'_{n-1} = S'_n + \frac{1}{a_{n-1}} \geq S'_n + \frac{S'_{n-1}}{C'},$$

so that $(1 - 1/C')S'_{n-1} \geq S'_n$. Repeating this step, we see that $(1-1/C')^j S'_{n-j} \geq S'_n$ for $j = 0, 1, \ldots, n-1$. Thus

$$\frac{1}{a_n} < S'_n \leq \Big(1 - \frac{1}{C'}\Big)^j S'_{n-j} \leq \frac{C'}{a_{n-j}}\Big(1 - \frac{1}{C'}\Big)^j$$

for $j = 1, \ldots, n-1$. Hence

$$\begin{aligned}\sum_{k=1}^n a_k = a_n + \sum_{j=1}^{n-1} a_{n-j} &\leq a_n\Big(1 + C'\sum_{j=1}^{n-1}\Big(1 - \frac{1}{C'}\Big)^j\Big)\\ &< a_n\Big(1 + C'\sum_{j=1}^{\infty}\Big(1 - \frac{1}{C'}\Big)^j\Big)\\ &= a_n(1 + (C'-1)C').\end{aligned}$$

So $a_n^{-1} S_n \leq C$ for $C = 1 + (C'-1)C'$.

9. Let $F_i = \{f_{i-1}+1, \ldots, f_i\}$ for $i = 1, \ldots, k$, and let $G_j = \{g_{j-1}+1, \ldots, g_j\}$ for $j = 1, \ldots, m$. Let W_1 be the set of permutations σ such that $\sigma(F_i) = F_i$ for each i, and let W_2 be the set of permutations σ such that $\sigma(G_j) = G_j$ for each j. If σ is the composition $\sigma_1 \circ \sigma_2$ of a $\sigma_1 \in W_1$ and a $\sigma_2 \in W_2$, and if $f_{j-1} < g_i \leq g_j$ (or, indeed, if $f_{j-1} \leq g_i \leq f_j$), then

$$F_1 \cup \cdots \cup F_{j-1} \subset G_1 \cup \cdots \cup G_i \subset F_1 \cup \cdots \cup F_j.$$

Hence $\sigma(\{1, \ldots, g_i\}) = \sigma(G_1 \cup \cdots \cup G_i) = \sigma_1(\sigma_2(G_1 \cup \cdots \cup G_i))$ equals $\sigma_1(G_1 \cup \cdots \cup G_i)$, which contains $\sigma_1(F_1 \cup \cdots \cup F_{j-1}) = F_1 \cup \cdots \cup F_{j-1} = \{1, \ldots, f_{j-1}\}$ and is contained in $\sigma_1(F_1 \cup \cdots \cup F_j) = F_1 \cup \cdots \cup F_j = \{1, \ldots, f_j\}$. Hence the condition is necessary.

To prove the converse, we use induction on n. First suppose that $f_1 \le g_1$. Let a be the largest integer such that $f_a \le g_1$. If $a = k$, then $g_1 = n$ and $m = 1$, and so W_2 is the group of all permutations, and the result is trivial. So assume that $a < k$. If $f_a < g_1$, then by hypothesis, $F_1 \cup \cdots \cup F_a \subset \sigma(G_1) \subset F_1 \cup \cdots \cup F_{a+1}$. If $f_a = g_1$, then $f_{a-1} < g_1 \le f_a$, and the hypothesis implies that $\sigma(G_1) = F_1 \cup \cdots \cup F_a$. For $\nu = 1, \dots, a$, F_ν is contained in $\sigma(G_1)$, and so equals $\sigma(S_\nu)$ for some $S_\nu \subset G_1$. We can therefore define a permutation τ of G_1 by setting $\tau(x) = \sigma(x)$ if $x \in S_\nu$ for some $\nu \le a$, and mapping the remaining part $G_1 \setminus (S_1 \cup \cdots \cup S_a)$ of G_1 onto the set $G_1 \setminus (F_1 \cup \cdots \cup F_a)$. We then extend τ to a permutation of $\{1, \dots, n\}$ by setting $\tau(x) = x$ for all $x \in G_2 \cup \cdots \cup G_m$. Note that $\tau \in W_2$ and that σ and τ agree on $S_1 \cup \cdots \cup S_a$. Let $\sigma' = \sigma \circ \tau^{-1}$. If $y \in \{1, \dots, f_a\}$, then $y = \sigma(x) = \tau(x)$ for some $x \in S_1 \cup \cdots \cup S_a$. So $\sigma'(y) = y$. Thus σ' also permutes $\{f_a + 1, \dots, n\}$. Let $\tilde{\sigma}$ denote the restriction of σ' to that set.

Assume for the moment that $f_a < g_1$. Now $\tilde{G}_1 = G_1 \setminus (F_1 \cup \cdots \cup F_a) = \{f_a + 1, \dots, g_1\}$, together with $G_2, \dots, G_m$, partition $\{f_a + 1, \dots, n\}$. So do the sets $F_{a+1}, \dots, F_k$. If $f_{j-1} < g_i \le f_j$ for some $i \ge 1$ and $j \ge a + 1$, then by the hypothesis on σ,

$$\{1, \dots, f_{j-1}\} \subset \sigma(G_1 \cup \cdots \cup G_i) \subset \{1, \dots, f_j\}. \tag{1}$$

Also, $\tilde{\sigma}(\{f_a + 1, \dots, g_i\})$ equals $\tilde{\sigma}(\tilde{G}_1 \cup G_2 \cup \cdots \cup G_i)$. Now

$$\tilde{\sigma}(\tilde{G}_1) = \sigma'(G_1 \setminus (F_1 \cup \cdots \cup F_a)) = \sigma'(G_1) \setminus \sigma'(F_1 \cup \cdots \cup F_a)) = \sigma(G_1) \setminus (F_1 \cup \cdots \cup F_a).$$

So if we remove $F_1 \cup \cdots \cup F_a = \{1, \dots, f_a\}$ from the sets in (1), we get

$$\{f_a + 1, \dots, f_{j-1}\} \subset \tilde{\sigma}(\tilde{G}_1) \cup \sigma(G_2 \cup \cdots \cup G_i) \subset \{f_a + 1, \dots, f_j\}.$$

This means that the permutation $\tilde{\sigma}$ satisfies the same condition as σ, but for the new partitions $\tilde{G}_1, G_2, \dots, G_m$ and $F_{a+1}, \dots, F_k$ of the smaller set $\{f_a + 1, \dots, n\}$. So by the induction hypothesis, there are permutations $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ such that $\tilde{\sigma}_1(F_i) = F_i$ for $i = a + 1, \dots, k$ and $\tilde{\sigma}_2(\tilde{G}_1) = \tilde{G}_1$ and $\tilde{\sigma}_2(G_j) = G_j$ for $j = 2, \dots, m$, so that $\tilde{\sigma} = \tilde{\sigma}_1 \circ \tilde{\sigma}_2$. We then define $\sigma_1 \in W_1$ by setting $\sigma_1(x) = x$ if $x \in F_1 \cup \cdots \cup F_a$ and $\sigma_1(x) = \tilde{\sigma}_1(x)$ if $x \in F_{a+1} \cup \cdots \cup F_k$. We define $\sigma_2 \in W_2$ by setting $\sigma_2(x) = \tau(x)$ if $x \in F_1 \cup \cdots \cup F_a$ and $\sigma_2(x) = \tilde{\sigma}_2(x)$ if $x \in F_{a+1} \cup \cdots \cup F_k$. Then $\sigma = \sigma_1 \circ \sigma_2$.

If $f_a = g_1$, then we arrive at the same conclusion with a little less work, because we do not need to use the set $\tilde{G}_1$.

Finally, if $f_1 > g_1$, then interchange the roles of the f_i's and g_j's. It is routine to check that σ^{-1} satisfies

$$\{1, \dots, g_{j-1}\} \subset \sigma^{-1}(\{1, \dots, f_i\}) \subset \{1, \dots, g_j\}$$

if $g_{j-1} < f_i \le g_j$. Hence by the case just treated, $\sigma^{-1} = \sigma_2 \circ \sigma_1$ for some $\sigma_1 \in W_1$ and $\sigma_2 \in W_2$. Thus $\sigma = \sigma_1^{-1} \circ \sigma_2^{-1}$ has the desired form.

10. We first give a solution of the stated problem, and then give a "formula" for a_n for general n.

A. Formula for a_n when n is a power of 2.

Step (i): For each $n \geq 1$, $a_{n+1} = a_n$ or $a_{n+1} = a_n + 1$. This is proved by induction. It is clearly true if $n = 1$. Suppose that it is true for $n = 1, \ldots, N$. We show that it is true for $n = N + 1$. Now $a_{N+2} = a_{a_{N+1}} + a_{N+2-a_{N+1}}$ is either $a_{a_N} + a_{N+2-a_N}$ or $a_{a_N+1} + a_{N+1-a_N}$ by the induction hypothesis. In the first case, since $N + 1 - a_N \leq N$, we know that $a_{N+2-a_N} = a_{N+1-a_N} + \epsilon$ for $\epsilon = 0$ or 1. So

$$a_{N+2} = a_{a_N} + a_{N+2-a_N} = a_{a_N} + a_{N+1-a_N} + \epsilon = a_{N+1} + \epsilon.$$

In the second case, since $a_N \leq N$ is an obvious consequence of the induction hypothesis, $a_{a_N+1} = a_{a_N} + \epsilon$ for $\epsilon = 0$ or 1. So

$$a_{N+2} = a_{a_N+1} + a_{N+1-a_N} = a_{a_N} + \epsilon + a_{N+1-a_N} = a_{N+1} + \epsilon.$$

So the statement is true for $n = N + 1$.

Step (ii): $a_1 \leq a_2 \leq \cdots$, and $a_n \leq n$ for all n. This is an immediate consequence of Step (i).

Step (iii): $a_n \geq n/2$ for all $n \geq 1$. This is also proved by induction. It is clearly true if $n = 1$ and $n = 2$. Suppose that it is true for $n = 1, \ldots, N$. Then as $a_N \leq N$ and $N + 1 - a_N \leq N$, we find from the induction hypothesis that

$$a_{N+1} = a_{a_N} + a_{N+1-a_N} \geq \frac{a_N}{2} + \frac{N+1-a_N}{2} = \frac{N+1}{2}.$$

Final Step: We show that $a_{2^k} = 2^{k-1}$ for all integers $k \geq 1$. This is certainly true for $k = 1$. Suppose that it is true for a particular k, but not true when k is replaced by $k+1$. We know from Step (iii) that $a_{2^{k+1}} \geq 2^k$. So suppose that $a_{2^{k+1}} \geq 2^k + 1$. Let n be the smallest integer such that $a_n = 2^k + 1$. Since $a_1 \leq a_2 \leq \cdots$, we see that $n \leq 2^{k+1}$. As $a_{2^k} = 2^{k-1}$, we know from Step (i) that $n \geq 2^k + 2$. By the choice of n and by Step (i) we know that $a_{n-1} = 2^k$. Using $n - 2^k \leq 2^k$ and Step (ii),

$$2^k + 1 = a_n = a_{a_{n-1}} + a_{n-a_{n-1}} = a_{2^k} + a_{n-2^k} \leq a_{2^k} + a_{2^k} = 2^{k-1} + 2^{k-1} = 2^k.$$

This contradiction completes the induction step.

B. Formula for a_n for general n. First we show the following: Every positive integer n has a unique representation

$$n = 2^k + \sum_{\ell=1}^{k} \binom{\mu_\ell}{\ell}, \tag{1}$$

where the μ_ℓ are positive integers such that for a certain $j \in \{0, \ldots, k\}$, depending on n,

$$0 \leq \mu_1 < \mu_2 < \cdots < \mu_j < \mu_{j+1} = \cdots = \mu_k = k.$$

Example: $n = 720$, $k = 9$, $j = 5$. Then

$$720 = 2^9 + \binom{0}{1} + \binom{3}{2} + \binom{4}{3} + \binom{6}{4} + \binom{8}{5} + \binom{9}{6} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9}.$$

The existence and uniqueness of the representation (1) is shown as follows: The sum in (1) is at most

$$2^k + \sum_{\ell=1}^{k} \binom{k}{\ell} = 2^k + (2^k - 1) = 2^{k+1} - 1$$

and so k is uniquely determined by $2^k \le n \le 2^{k+1} - 1$. There is a unique integer $j \in \{0, \ldots, k\}$ such that

$$2^k + \sum_{\ell=j+1}^{k} \binom{k}{\ell} \le n < 2^k + \sum_{\ell=j}^{k} \binom{k}{\ell};$$

for example, take $j = k$ if $n = 2^k$, and take $j = 0$ if $n = 2^{k+1} - 1$.

Now write $n = m + 2^k + \sum_{\ell=j+1}^{k} \binom{k}{\ell}$, where $0 \le m < \binom{k}{j}$. We want to show that generally if $0 \le m < \binom{k}{j}$ for a given k and some $j \in \{0, \ldots, k\}$ then

$$m = \sum_{\ell=1}^{j} \binom{\mu_\ell}{\ell} \quad \text{where } 0 \le \mu_1 < \mu_2 < \cdots < \mu_j < k. \tag{2}$$

If $m = 0$, then $\mu_\ell = \ell - 1$ for $\ell = 1, \ldots, j$ gives the required representation. If $j = 0$, then $m = 0$ and again we have a (trivial) representation (2). So assume that $j, m > 0$. First determine μ_j by requiring

$$\binom{\mu_j}{j} \le m < \binom{\mu_j + 1}{j}.$$

Clearly such a μ_j exists and $\mu_j < k$ since $m < \binom{k}{j}$. Now $m < \binom{\mu_j+1}{j} = \binom{\mu_j}{j} + \binom{\mu_j}{j-1}$, and hence $0 \le m - \binom{\mu_j}{j} < \binom{\mu_j}{j-1}$. We may assume by induction on j that $m - \binom{\mu_j}{j}$ has a representation (2):

$$m - \binom{\mu_j}{j} = \sum_{\ell=1}^{j-1} \binom{\mu_\ell}{\ell} \quad \text{where } 0 \le \mu_1 < \mu_2 < \cdots < \mu_{j-1} < \mu_j$$

Thus m has the required representation (2). Uniqueness of the representation is easily seen from the above.

We shall call the representation (1) of n the c-representation of n (c for combinatorial).

Here now is a formula for a_n: If n has representation (1), then omitting the zero terms from (1), namely those with $\mu_\ell = \ell - 1$, we can write the c-representation

$$n = 2^k + \sum_{\ell=i+1}^{k} \binom{\mu_\ell}{\ell} \tag{3}$$

for some $i = i(n)$ in $\{0, 1, \ldots, k\}$. We claim that then a_n is given by

$$a_n = 2^{k-1} + \sum_{\ell=i+1}^{k} \binom{\mu_\ell - 1}{\ell - 1}, \tag{4}$$

(note that $\mu_\ell = \ell - 1$ for $\ell \leq i$ and $\mu_\ell \geq \ell$ for $\ell > i$).

In particular, if $n = 2^k$, then the sum in (4) is empty ($i = k$), and we obtain $a_{2^k} = 2^{k-1}$.

In the example $n = 720$ given above, $i = 1$, and

$$a_{720} = 2^8 + \binom{2}{1} + \binom{3}{2} + \binom{5}{3} + \binom{7}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} = 399.$$

The formula for a_n was first guessed from numerical evidence—a computer-produced table of a_n, $n \leq 1024$.

Write α_n for the right hand side in (4). Then we have to show that α_n satisfies the same recursion as a_n, namely

$$\alpha_{n+1} = \alpha_{\alpha_n} + \alpha_{n+1-\alpha_n}. \tag{5}$$

First we observe that if $i(n) > 0$, then the c-representation of $n + 1$ is

$$n + 1 = 2^k + \sum_{\ell=i}^{k} \binom{\mu_\ell}{\ell}, \quad \text{with } \mu_i = i. \tag{6}$$

This is no longer true if $i = 0$, since $\binom{0}{0}$ is not admitted in the c-representation. In that case, we write $\mu_1 = 1 + d$ for some $d \geq 0$ and define $i_1 \geq 1$ as the index for which $\mu_\ell = \ell + d$ for $1 \leq \ell \leq i_1$ and $\mu_\ell > \ell + d$ for $\ell > i_1$. Hence

$$n + 1 = 2^k + \sum_{\ell=0}^{i_1} \binom{\ell + d}{d} + \sum_{\ell=i_1+1}^{k} \binom{\mu_\ell}{\ell} = 2^k + \binom{d + i_1 + 1}{i_1} + \sum_{\ell=i_1+1}^{k} \binom{\mu_\ell}{\ell}, \tag{7}$$

and this last expression is indeed a c-representation of $n+1$ since $d + i_1 + 1 < \mu_{i_1+1}$. Here we have made use of the well-known combinatorial formula

$$\sum_{\ell=0}^{m} \binom{\ell + d}{\ell} = \binom{m + d + 1}{m},$$

which is easily proved by induction on m. It follows from (6) that

$$\alpha_{n+1} = 2^{k-1} + 1 + \sum_{\ell=i+1}^{k} \binom{\mu_\ell - 1}{\ell - 1} = \alpha_n + 1 \quad \text{if } i > 0$$

and from (7) that

$$\alpha_{n+1} = 2^{k-1} + \binom{d+i_1}{i_1-1} + \sum_{\ell=i_1+1}^{k} \binom{\mu_\ell - 1}{\ell - 1} = \alpha_n \quad \text{if } i = 0,$$

since

$$\sum_{\ell=1}^{i_1} \binom{\ell+d-1}{\ell-1} = \binom{d+i_1}{i_1-1}.$$

Thus

$$\alpha_{n+1} = \begin{cases} \alpha_n & \text{if } i(n) = 0; \\ \alpha_n + 1 & \text{if } i(n) > 0. \end{cases} \tag{8}$$

Next we observe that the c-representation of $n - \alpha_n$ is

$$n - \alpha_n = 2^{k-1} + \sum_{\ell=i+1}^{k} \binom{\mu_\ell - 1}{\ell} \tag{9}$$

since $\binom{\mu_\ell}{\ell} - \binom{\mu_\ell-1}{\ell-1} = \binom{\mu_\ell-1}{\ell}$ and $2^k - 2^{k-1} = 2^{k-1}$. Here if $\mu_{i+1} = i+1$, then there are some zero terms in (9), but in any case $i(n-\alpha_n) \geq i(n)$.

We are now ready to prove (5). Suppose first that $i(n) > 0$. Then by (8),

$$\alpha_{n+1} = \alpha_n + 1 \quad \text{and} \quad \alpha_{n+1-\alpha_n} = \alpha_{n-\alpha_n} + 1,$$

because $i(n-\alpha_n) \geq i(n) > 0$, and we have to show that

$$\alpha_n = \alpha_{\alpha_n} + \alpha_{n-\alpha_n}. \tag{10}$$

But

$$\alpha_{\alpha_n} = 2^{k-2} + \sum_{\ell=i+1}^{k} \binom{\mu_\ell - 2}{\ell - 2} \quad \text{by (4)},$$

and

$$\alpha_{n-\alpha_n} = 2^{k-2} + \sum_{\ell=i+1}^{k} \binom{\mu_\ell - 2}{\ell - 1} \quad \text{by (9)},$$

hence

$$\alpha_{\alpha_n} + \alpha_{n-\alpha_n} = 2^{k-1} + \sum_{\ell=i+1}^{k} \binom{\mu_\ell - 1}{\ell - 1} = \alpha_n$$

as required.

If $i(n) = 0$ and $\mu_1 > 1$, then by (8), $\alpha_{n+1} = \alpha_n$, $\alpha_{n+1-\alpha_n} = \alpha_{n-\alpha_n}$ and the statement to be proved is again (10). Write the c-representation of n in the form

$$n = 2^k + \sum_{\ell=1}^{i_1} \binom{d_1+\ell}{\ell} + \sum_{\ell=i_1+1}^{i_2} \binom{d_2+\ell}{\ell} + \sum_{\ell>i_2} \binom{\mu_\ell}{\ell}$$

for some $d_2 > d_1 > 0$, $i_2 > i_1 > 0$, where $\mu_\ell > d_2 + \ell$ for $\ell > i_2$. Then

$$\alpha_n - 2^{k-1} - \sum_{\ell > i_2} \binom{\mu_\ell - 1}{\ell - 1} = \sum_{\ell=1}^{i_1} \binom{d_1 + \ell - 1}{\ell - 1} + \sum_{\ell=i_1+1}^{i_2} \binom{d_2 + \ell - 1}{\ell - 1}.$$

If $i_1 = 1$, the right hand side becomes $\binom{d_2+i_2}{i_2-1}$, hence setting

$$n' = 2^k + \binom{d_2 + i_2 + 1}{i_2} + \sum_{\ell > i_2} \binom{\mu_\ell}{\ell},$$

we have $\alpha_{n'} = \alpha_n$, $i(n') = i_2 - 1 > 0$ and

$$n' - n = \binom{d_2 + i_2 + 1}{i_2} - \sum_{\ell=2}^{i_2} \binom{d_2 + \ell}{\ell} - \binom{d_1 + 1}{1} = d_2 - d_1 + 1,$$

and so $n' = n + d_2 - d_1 + 1$.

If $i_1 > 1$, we define

$$n' = 2^k + \binom{d_1 + i_1 + 1}{i_1} + \sum_{\ell=i_1+1}^{i_2} \binom{d_2 + \ell}{\ell} + \sum_{\ell > i_2} \binom{\mu_\ell}{\ell}. \tag{11}$$

The right hand side of (11) gives the c-representation of n', and we see that $\alpha_{n'} = \alpha_n$, $i(n') = i_1 - 1 > 0$ and

$$n' - n = \binom{d_1 + i_1 + 1}{i_1} - \sum_{\ell=1}^{i_1} \binom{d_1 + \ell}{\ell} = 1,$$

so that $n' = n+1$. From (9) we can verify easily (since $i_1(n - \alpha_n) = i_1(n)$) that $\alpha_{n'-\alpha_{n'}} = \alpha_{n-\alpha_n}$ and we can replace (10) by

$$\alpha_{n'} = \alpha_{\alpha_{n'}} + \alpha_{n'-\alpha_{n'}},$$

which is satisfied since $i(n') > 0$.

The only case left to be considered is when $i(n) = 0$ and $\mu_1 = 1$. The argument is similar to the previous one, and will not be repeated. The only modification is that the equation to be satisfied now is

$$\alpha_n = \alpha_{\alpha_n} + \alpha_{n-\alpha_n} + 1$$

instead of (10).