Sydney University Mathematical Society
Problems Competition 2003

This competition is open to all undergraduates at any Australian university or tertiary institution. Prizes ($50 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Entrants from the University of Sydney will also be eligible for the Norbert Quirk Prizes (one for each of 1st, 2nd and 3rd years). Entries from fourth year students will be considered. When prizewinners are being selected, if two or more entries to a problem are essentially equal, then preference may be given to the students in the earlier year of university.

Contestants may use any source of information except other people. Solutions are to be received by 4.00 pm on Friday, September 5, 2003. They may be given to Dr. Donald Cartwright, Room 620, Carslaw Building, or posted to him at the School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006. Entries must state name, university, student number, course and year, term address and telephone number, and be marked **2003 SUMS Competition**. The prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr. Cartwright, at the above address.

These problems will also be posted at the website

http://www.maths.usyd.edu.au/SUMS/

Problems

(Extensions and generalizations of any problem are invited and are taken into account when assessing solutions.)

1. Notice that $2^2 = 4$, $12^2 = 144$, $38^2 = 1444$ and $2538^2 = 6441444$.

(i) Show that there is no integer x such that $x^2 = \dots 4444$ in its decimal representation.

(ii) Find a number n such that the last four digits of n^2 are 4144.

(iii) Show that for each $k \geq 1$ there is a number x such that the last k digits of x^2 consist of 1's and 4's only.

(iv) What can one say if 1 and 4 are replaced by other pairs of digits?

2. Recall that the Fibonacci numbers $(f_n)_{n\geq 0}$ are defined by $f_0 = f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$ if $n \geq 0$.

(a) Let $\ell_n \in \{0, 1, \dots, 9\}$ be the last digit in f_n. Thus the sequence (ℓ_n) starts $1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, \dots$. Show that this sequence is periodic. What is its period?

(b) Notice that there are 6 n's such that f_n is only one digit long. Show that if $k \geq 2$, there are either 4 or 5 n's such that f_n has exactly k digits.

3. Oscar and Nicole are playing a game with matchsticks: They form two piles of matches, one with 42 matches, and the other with 86. They take turns removing matches from the piles, according to the following rules: at each stage the matches taken (at least 1) must all come from one pile, and the number taken must be a divisor of the number of matches in the other pile. The player who removes the last match wins. Nicole goes first. Describe a strategy for Oscar to adopt so that he wins the game, no matter what Nicole does. Show that if we instead start with piles of 40 and 86 matches, then Nicole can always win, if she adopts the correct strategy.

4. Let G be a finite group. If $x \in G$, then the conjugacy class of x is the set of elements of the form gxg^{-1}, where $g \in G$. Now suppose that H is a subgroup of G which contains an element from each conjugacy class in G. Show that $H = G$.

5. If u and v are distinct roots of the quadratic equation $x^2 - px + q = 0$, then we have $u+v = p$ and $uv = q$. Now suppose that P, Q, U and V are 2×2 matrices and that U and V are distinct solutions of the matrix equation $X^2 - PX + Q = 0$. For a square matrix A, let $\mathrm{Tr}(A)$ denote the sum of its diagonal terms, and let $\det(A)$ denote its determinant. Show that $\mathrm{Tr}(U+V) = \mathrm{Tr}(P)$ and $\det(UV) = \det(Q)$ is true if we add the extra hypothesis that $U - V$ is invertible.

6. Suppose that we have a curve C given by an equation in polar coordinates $r = f(\theta)$, $0 \le \theta \le 2\pi$. Here f is a continuous function, and $f(\theta) \ge 0$ for all θ. Assume that the region bounded by this curve is convex. Now consider a segment of length $a+b$, and a point P at distance a from one end of the segment. Imagine sliding this segment round inside the curve so that its ends are touching C. The point P traces a curve C' inside C. What is the area of the region between the two curves?

7. Suppose that f, c are integers such that $1 \le f \le c$. Under what conditions is it possible to find integers n, m so that $1 \le m < c$ and $(f-1)/c < n/m \le f/c$?

8. We consider strings of 0's and 1's, and modify them using the "substitution" rule $\sigma(0) = 01$, $\sigma(1) = 0$ as follows: if $\xi = x_1x_2\cdots x_n$ is a string of 0's and 1's, we define $\sigma(\xi)$ to be the concatenation $\sigma(x_1)\sigma(x_2)\cdots\sigma(x_n)$. For example, starting from the string 0, and applying σ repeatedly, we get

$$\sigma(0) = 01,\ \sigma^2(0) = \sigma(01) = 010,\ \sigma^3(0) = \sigma(010) = 01001,\ \text{etc.}$$

(a) Show that $\sigma^n(0)$ is a string of length f_{n+1}, where f_n is the n-th Fibonacci number (see Question 2).

(b) Show that the first f_n letters of $\sigma^n(0)$ are those of $\sigma^{n-1}(0)$.

Because of (b), there is a unique infinite string $\xi = 0100101\cdots$ of 0's and 1's with the property that, for each $n \ge 1$, the first f_n letters of ξ are those of $\sigma^{n-1}(0)$. Notice that the string 11 doesn't appear in ξ, but 00, 01 and 10 do occur. Show that more generally,

(c) For each $n \ge 1$, there are exactly $n+1$ different strings of length n occurring somewhere in ξ.

9. We consider a fixed finite alphabet of *symbols* and strings of these symbols. The strings have a fixed length n, and a string of length n will be called a *word*. We think of the symbols in a word as occurring in n places, so we can talk about the symbol in the first place in a word, the symbol in the second place, and so forth.

We define the *(Hamming) distance* between two words to be the number of places in which they differ.

Consider functions from the set of all words to itself which preserve the Hamming distance. Here are two examples of such functions.

(i) Apply a permutation to the places. For example if $n = 3$ and we swap the first two places, the word *abc* will be changed into the word *bac*.

(ii) Apply a permutation to the symbols in one place. For example, in the first place we might change a to b, b to c and c to a. The word *abc* would be changed into *bbc*.

Show that every function from the set of all words to itself which preserves the Hamming distance is a composite of functions of types (i) and (ii) above.

10. Let q be a prime power, $\mathbb{F}_q$ the finite field with q elements, and $\alpha \in \mathbb{F}_q$. Let $\mathbb{F}_q[X]$ denote the set of polynomials with coefficients in $\mathbb{F}_q$. A polynomial is called irreducible if it cannot be written as a product of two polynomials both have smaller degree. Each $f \in \mathbb{F}_q[X]$ can be written as a product of irreducible polynomials. For $n \ge 1$, the number of monic polynomials in $\mathbb{F}_q[X]$ of degree n with constant term α is clearly q^{n-1}. How many of these polynomials have distinct irreducible factors?