The University of Sydney
School of Mathematics and Statistics
NSW 2006 Australia

SUMS Problem Competition 2007

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Entrants may use any source of information except other people. The problems will also be posted on the web page **http://www.maths.usyd.edu.au/u/SUMS/**.

Entrants may submit solutions to as many problems as they wish. Prizes ($50 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry (one for each of 1st, 2nd and 3rd years). Extensions and generalizations of any problem are invited and are taken into account when assessing solutions. If two or more solutions to a problem are essentially equal, preference may be given to students in the earlier year of university.

Entries must be received by **Thursday, September 6, 2007**. They may be posted to Dr Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or delivered in person to Room 805, Carslaw Building. Please mark your entry SUMS Problem Competition 2007, and include your name, university, student number, course and year, term address and telephone number. Prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr Henderson at the above address.

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1. Among the numbers $2^1, 2^2, \cdots, 2^{10}$, there are 3 whose first digit is 1 (namely, $2^4 = 16$, $2^7 = 128$, and $2^{10} = 1024$). It turns out that among the numbers $2^1, 2^2, \cdots, 2^{100}$, there are 30 whose first digit is 1; and among the numbers $2^1, 2^2, \cdots, 2^{1000}$, there are 301 whose first digit is 1. For any positive integer N, define a_N by the rule that among the numbers 2^n with $1 \leq n \leq 10^N$, there are a_N whose first digit is 1. Prove that a_{N+1} is always obtained from a_N by adding a single digit at the end.

2. The sisters Alice, Bess, and Cath are fighting over a triangular pizza, which may be imagined as a triangle PQR. Their father David proposes the following procedure for sharing it between the four of them. Alice will select a point A on the edge PQ, then Bess will select a point B on the edge PR, then Cath will select a point C on the edge QR. David will then cut the pizza along the lines AB, BC, and AC, and take the centre piece ABC for himself, leaving three corner pieces (some possibly empty, if endpoints of edges have been chosen). The sisters will then either all take the corner piece to the left of the point they selected, or all take the corner piece to the right of their point; Alice (as the eldest) will get to choose left or right. As everyone knows, each sister will make her choices purely to maximize the area of her own share, except that Alice and Bess, if their own shares are unaffected, will act to the advantage of the youngest sister Cath. If they all reason perfectly, what will they do?

3. The members of a tennis club are planning a doubles carnival consisting of several rounds. In the spirit of social tennis, results don't matter, but participation does; so in each round, every member is to play in exactly one game. Each round is to be either a mixed doubles round, in which every game involves two male and two female players, or an ordinary doubles round, in which every game involves four players of the same gender. There is a further requirement that over the whole carnival, any two members play in the same game exactly once; whether they are partners or opponents in this game is immaterial. If there are 2^k male and 2^k female members, for what (positive integer) values of k is this possible?

4. Let T be a tree with n vertices. (A tree is a connected graph with no cycles.) Fix $1 \leq k \leq n$, and let $\mathcal{S}_k$ be the set of all k-element subsets of the set of vertices of T. For any $S \in \mathcal{S}_k$, let $c(S)$ be the number of connected components of the subgraph obtained by restricting to the vertices in S (i.e. deleting all of the tree except the vertices in S and the edges between them). Prove that $\sum_{S\in\mathcal{S}_k} c(S) = (n-k+1)\binom{n-1}{k-1}$.

5. Say that a rational number r is splittable if the cubic polynomial $x^3 - 3x - r$ factorizes as $(x-r_1)(x-r_2)(x-r_3)$ where r_1, r_2, r_3 are rational. Find polynomials f and g with integer coefficients such that r is splittable if and only if $r = \frac{f(t)}{g(t)}$ for some rational number t.

6. Let n be a positive integer. This question concerns sets of n integers $\{a_1, a_2, \cdots, a_n\}$ which have the property that the $\binom{n}{2}$ differences $|a_j - a_i|$, $1 \leq i < j \leq n$, are all distinct. For example, the set of the first n powers of 2, namely $\{1, 2, 4, 8, \cdots, 2^{n-1}\}$, has this property. Construct a set with this property for which the differences $|a_j - a_i|$ are all less than $e^{\frac{n+2}{2}}$.

7. Let $A = (a_{ij})_{i,j=1}^n$ be a square matrix of real numbers which is skew-symmetric, meaning that $a_{ij} = -a_{ji}$ for all i, j. Define a new skew-symmetric matrix $B = (b_{ij})_{i,j=1}^n$ as follows:

$$b_{ij} = \begin{cases} -a_{ij} + \sum_{p\geq 1} (-1)^{p+1} \sum_{\substack{i_1,i_2,\cdots,i_p\in\mathbb{Z} \\ i<i_1<i_2<\cdots<i_p<j}} a_{ii_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_pj}, & \text{if } i<j, \\ 0, & \text{if } i=j, \\ -b_{ji}, & \text{if } i>j. \end{cases}$$

Prove that $\det(B) = \det(A)$.

8. Take a cube with edges of length 1. Fix a length $0 < \ell < \frac{1}{\sqrt{2}}$, and attach a square to each face whose centre is the centre of the face, whose sides have length ℓ, and whose edges are (initially) parallel to the edges of the face. Now rotate each of these six squares anti-clockwise about the centre of its face, through some angle $0 < \theta < \frac{\pi}{4}$ (the same angle for all six). Let A be a vertex of the square on face F, let F' be the face which is closest to A of those adjacent to F, and let B and C be the vertices of the square on face F' which are closest to A. Prove that there are unique values of ℓ and θ (subject to the above bounds) for which ABC is an equilateral triangle, and that this value of ℓ is irrational.

9. Let $h(n)$ denote the number of permutations of the set $\{1, \cdots, n\}$ which do not preserve any two-element subset $\{i, j\}$. (In other words, there are no two elements i, j which the permutation leaves fixed, nor are there two elements i, j which the permutation swaps.) Show that

$$\lim_{n\to\infty} \frac{h(n)}{n!} = 2e^{-3/2}.$$

10. For any positive integers $m \leq n$, let $a_{n,m}$ denote the number of surjective functions from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, m\}$. Define a polynomial $p_n(x)$ by the formula

$$p_n(x) = a_{n,1}(x-1)^{n-1} + a_{n,2}(x-1)^{n-2} + \cdots + a_{n,n-1}(x-1) + a_{n,n}.$$

Let $b_{n,d}$ denote the coefficient of x^d in $p_n(x)$. Prove that $b_{n,d} \geq 0$ for all $0 \leq d \leq n-1$.