

# Topological invariants of groups and tropical geometry

Alex Suciu

Northeastern University

Recent Progress in Mathematics and Statistics Seminar

The University of Sydney

February 9, 2023

## 1 FINITENESS PROPERTIES OF ABELIAN COVERS

- The Bieri–Neumann–Strebel–Renz invariants
- Novikov–Sikorav homology

## 2 COHOMOLOGY JUMP LOCI

- Characteristic varieties
- Resonance varieties
- Bounding the Sigma-invariants

## 3 TROPICAL GEOMETRY

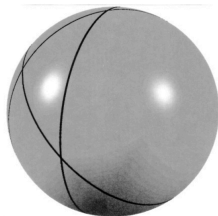
- Tropical varieties
- A tropical bound for the Sigma-invariants
- Alexander polynomial and BNS invariant

## 4 APPLICATIONS

- One-relator groups
- Compact 3-manifolds
- Kähler manifolds
- Hyperplane arrangements

## The Bieri–Neumann–Strebel–Renz invariants

- Let  $G$  be a finitely generated group. Let
  - $n = b_1(G) > 0$ .
  - $S(G) = S^{n-1}$ , the unit sphere in  $\text{Hom}(G, \mathbb{R}) = \mathbb{R}^n$ .
  - $\text{Cay}(G)$ , the Cayley graph of  $G$ .
- (Bieri–Neumann–Strebel 1987)



$$\Sigma^1(G) = \{\chi \in S(G) \mid \text{Cay}_\chi(G) \text{ is connected}\},$$

where  $\text{Cay}_\chi(G)$  is the induced subgraph of  $\text{Cay}(G)$  on vertex set  $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$ .

- (Bieri–Renz 1988)

$$\Sigma^q(G, \mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } FP_q\},$$

i.e., there is a projective  $\mathbb{Z}G_\chi$ -resolution  $P_\bullet \rightarrow \mathbb{Z}$ , with  $P_i$  finitely generated for all  $i \leq q$ . Moreover,  $\Sigma^1(G, \mathbb{Z}) = -\Sigma^1(G)$ .

- The BNSR-invariants of form a descending chain of open subsets,

$$S(G) \supseteq \Sigma^1(G, \mathbb{Z}) \supseteq \Sigma^2(G, \mathbb{Z}) \supseteq \dots$$

- The  $\Sigma$ -invariants control the finiteness properties of normal subgroups  $N \triangleleft G$  for which  $G/N$  is free abelian:

$$N \text{ is of type } FP_q \iff S(G, N) \subseteq \Sigma^q(G, \mathbb{Z})$$

where  $S(G, N) = \{\chi \in S(G) \mid \chi(N) = 0\}$ .

- In particular:  $\ker(\chi: G \rightarrow \mathbb{Z})$  is f.g.  $\iff \{\pm\chi\} \subseteq \Sigma^1(G)$ .

## Novikov–Sikorav homology

- For each  $\chi \in S(G)$ , the Novikov–Sikorav completion of  $\mathbb{Z}G$ ,

$$\widehat{\mathbb{Z}G}_\chi = \left\{ \lambda \in \mathbb{Z}G \mid \{g \in \text{supp } \lambda \mid \chi(g) \geq c\} \text{ is finite, } \forall c \in \mathbb{R} \right\}$$

is a ring containing  $\mathbb{Z}G$  as a subring.

- Alternatively, let  $U_m$  be the additive subgroup of  $\mathbb{Z}G$  (freely) generated by  $\{g \in G \mid \chi(g) \geq m\}$ .
- Requiring the decreasing filtration  $\{U_m\}_{m \in \mathbb{Z}}$  to form a basis of open neighborhoods of  $0$  defines a topology on  $\mathbb{Z}G$ , compatible with the ring structure. Then

$$\widehat{\mathbb{Z}G}_{-\chi} = \varprojlim_m \mathbb{Z}G/U_m.$$

### EXAMPLE

Let  $G = \mathbb{Z} = \langle t \rangle$  and  $\chi(t) = 1$ . Then

$$\widehat{\mathbb{Z}G}_\chi = \mathbb{Z}[[t^{-1}]] [t] = \left\{ \sum_{i \leq k} n_i t^i \mid n_i \in \mathbb{Z}, \text{ for some } k \in \mathbb{Z} \right\}.$$

- Now let  $X$  be a connected CW-complex with finite  $q$ -skeleton, for some  $q \geq 1$ . Write  $G := \pi_1(X)$  and  $S(X) := S(G)$ .
- (Farber–Geoghegan–Schütz 2010)

$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X; \widehat{\mathbb{Z}G}_{-\chi}) = 0, \forall i \leq q\}.$$

- (Bieri 2007) If  $G$  is  $FP_k$ , then  $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G, 1), \mathbb{Z}), \forall q \leq k$ .
- In particular, if  $G$  is finitely generated, the BNS set  $\Sigma^1(G) = -\Sigma^1(G, \mathbb{Z})$  consists of those characters  $\chi \in S(G)$  for which both  $H_0(G; \widehat{\mathbb{Z}G}_\chi)$  and  $H_1(G; \widehat{\mathbb{Z}G}_\chi)$  vanish.

## Characteristic varieties

- Let  $\mathbb{T}_G := \text{Hom}(G, \mathbb{C}^*)$  be the character group of  $G = \pi_1(X)$ , identified with  $\text{Char}(X) := H^1(X; \mathbb{C}^*)$ .
- The *characteristic varieties* of  $X$  are the sets

$$\mathcal{V}^i(X) = \{\rho \in \mathbb{T}_G \mid H_i(X; \mathbb{C}_\rho) \neq 0\}.$$

- If  $X$  has finite  $q$ -skeleton, then  $\mathcal{V}^i(X)$  is Zariski closed for all  $i \leq q$ .
- We may define similarly  $\mathcal{V}^i(X, \mathbb{k}) \subset H^1(X; \mathbb{k}^*)$  for any field  $\mathbb{k}$ .
- These constructions are compatible with restriction and extension of the base field. Namely, if  $\mathbb{k} \subset \mathbb{L}$  is a field extension, then

$$\mathcal{V}^i(X, \mathbb{k}) = \mathcal{V}^i(X, \mathbb{L}) \cap H^1(X; \mathbb{k}^*),$$

$$\mathcal{V}^i(X, \mathbb{L}) = \mathcal{V}^i(X, \mathbb{k}) \times_{\mathbb{k}} \mathbb{L}.$$

- Let  $X^{\text{ab}} \rightarrow X$  be the maximal abelian cover. View  $H_*(X^{\text{ab}}; \mathbb{C})$  as a module over  $\mathbb{C}[G_{\text{ab}}]$ . Then

$$\bigcup_{i \leq q} \mathcal{V}^i(X) = \bigcup_{i \leq q} V(\text{ann}(H_i(X^{\text{ab}}; \mathbb{C}))).$$

- Let  $\exp: \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n$ . Given a subvariety  $W \subset (\mathbb{C}^*)^n$ , define its *exponential tangent cone* at  $\mathbf{1}$  (the identity of  $(\mathbb{C}^*)^n$ ) as

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \forall \lambda \in \mathbb{C}\}.$$

- $\tau_1(W)$  depends only on the germ  $W_{(\mathbf{1})}$ ; it is non-empty iff  $\mathbf{1} \in W$ .
- If  $T \cong (\mathbb{C}^*)^r$  is an algebraic subtorus, then  $\tau_1(T) = T_{\mathbf{1}}(T) \cong \mathbb{C}^r$ .
- (Dimca–Papadima–S. 2009)  $\tau_1(W)$  is a finite union of rationally defined linear subspaces.
- For any subfield  $\mathbb{k} \subset \mathbb{C}$ , set  $\tau_1^{\mathbb{k}}(W) = \tau_1(W) \cap \mathbb{k}^n$ .



## Resonance varieties

- Let  $A = H^*(X; \mathbb{C})$ . For each  $a \in A^1$ , we have that  $a^2 = 0$ . Thus, there is a cochain complex

$$(A, \cdot a): A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \dots$$

- The *resonance varieties* of  $X$  are the homogeneous algebraic sets

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^i(A, a) \neq 0\}.$$

- Identify  $A^1 = H^1(X; \mathbb{C})$  with  $\mathbb{C}^n$ , where  $n = b_1(X)$ . The map  $\exp: H^1(X; \mathbb{C}) \rightarrow H^1(X; \mathbb{C}^*)$  has image  $\mathbb{T}_{\mathbb{C}}^0 = (\mathbb{C}^*)^n$ .

- (Dimca–Papadima–S. 2009)

$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X).$$

- (DPS-2009, DP-2014) If  $X$  is a  $q$ -formal space, then, for all  $i \leq q$ ,

$$\tau_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X).$$

## Bounding the $\Sigma$ -invariants

- Let  $\chi \in S(X)$ , and set  $\Gamma = \text{im}(\chi)$ ; then  $\Gamma \cong \mathbb{Z}^r$ , for some  $r \geq 1$ .
- A Laurent polynomial  $p = \sum_{\gamma} n_{\gamma} \gamma \in \mathbb{Z}\Gamma$  is  $\chi$ -*monic* if the greatest element in  $\chi(\text{supp}(p))$  is  $0$ , and  $n_0 = 1$ .
- Let  $\mathcal{R}\Gamma_{\chi}$  be the *Novikov ring*, i.e., the localization of  $\mathbb{Z}\Gamma$  at the multiplicative subset of all  $\chi$ -monic polynomials; then  $\mathcal{R}\Gamma_{\chi}$  is a PID.
- Let  $b_i(X, \chi) = \text{rank}_{\mathcal{R}\Gamma_{\chi}} H_i(X; \mathcal{R}\Gamma_{\chi})$  be the Novikov–Betti numbers.

### THEOREM (PAPADIMA–S. 2010)

Let  $X$  be a connected CW-complex with finite  $q$ -skeleton and let  $\chi: \pi_1(X) \rightarrow \mathbb{R}$  be a non-zero homomorphism. Then,

- $-\chi \in \Sigma^q(X, \mathbb{Z}) \implies b_i(X, \chi) = 0, \forall i \leq q.$
- $\chi \notin \tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)) \iff b_i(X, \chi) = 0, \forall i \leq q.$

### COROLLARY

$$\Sigma^q(X, \mathbb{Z}) \subseteq S \left( \tau_1^{\mathbb{R}} \left( \mathcal{V}^{\leq q}(X) \right) \right)^c$$

- Thus,  $\Sigma^q(X, \mathbb{Z})$  is contained in the complement of a finite union of rationally defined great subspheres.
- If  $X$  is  $q$ -formal, then  $\Sigma^i(X, \mathbb{Z}) \subseteq S(\mathcal{R}^{\leq i}(X))^c$  for all  $i \leq q$ .

### EXAMPLE

Let  $X$  be a nilmanifold. Then  $\Sigma^i(X, \mathbb{Z}) = S(X)$ , while  $\mathcal{V}^i(X) = \{1\}$ ,  $\forall i$ . Thus,  $\Sigma^q(X, \mathbb{Z}) = S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)))^c$ , for all  $q$ .

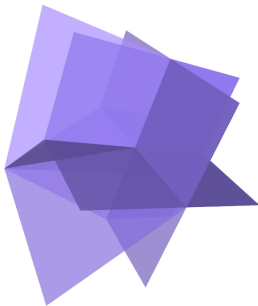
### EXAMPLE

- Let  $G = G_{\Gamma} = \langle v \in V \mid vw = wv \text{ if } \{v, w\} \in E \rangle$  be the right-angled Artin group associated to a finite simple graph  $\Gamma = (V, E)$ .
- There is a finite  $K(G, 1)$  which is formal.
- $\Sigma^q(G, \mathbb{R}) = S(\mathcal{R}^{\leq q}(G, \mathbb{R}))^c$  holds for all  $q$ .
- $\Sigma^q(G, \mathbb{Z}) = S(\mathcal{R}^{\leq q}(G, \mathbb{R}))^c$ , provided the homology groups of certain subcomplexes in the flag complex of  $\Gamma$  are torsion-free.
- This condition is always satisfied in degree  $q = 1$ , giving  $\Sigma^1(G) = S(\mathcal{R}^1(G, \mathbb{R}))^c$ .

## Tropical varieties

- Let  $\mathbb{K} = \mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}(\!(t^{1/n})\!)$  be the field of Puiseux series over  $\mathbb{C}$ .
- A non-zero element of  $\mathbb{K}$  has the form  $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \dots$ ,  $c_i \in \mathbb{C}^*$  and  $a_1 < a_2 < \dots$  rationals with a common denominator.
- The (algebraically closed) field  $\mathbb{K}$  admits a valuation  $v: \mathbb{K}^* \rightarrow \mathbb{Q}$ , given by  $v(c(t)) = a_1$ .
- Let  $v: (\mathbb{K}^*)^n \rightarrow \mathbb{Q}^n \subset \mathbb{R}^n$  be the  $n$ -fold product of the valuation.
- The *tropicalization* of a subvariety  $W \subset (\mathbb{K}^*)^n$ , denoted  $\text{Trop}(W)$ , is the closure (in the Euclidean topology) of  $v(W)$  in  $\mathbb{R}^n$ .
- This is a rational polyhedral complex in  $\mathbb{R}^n$ . For instance, if  $W$  is a curve, then  $\text{Trop}(W)$  is a graph with rational edge directions.

- If  $T$  is an algebraic subtorus of  $(\mathbb{K}^*)^n$ , then  $\text{Trop}(T)$  is the linear subspace  $\text{Hom}(\mathbb{K}^*, T) \otimes \mathbb{R} \subset \text{Hom}(\mathbb{K}^*, (\mathbb{K}^*)^n) \otimes \mathbb{R} = \mathbb{R}^n$ .
- Moreover, if  $z \in (\mathbb{K}^*)^n$ , then  $\text{Trop}(z \cdot T) = \text{Trop}(T) + v(z)$ .
- For a variety  $W \subset (\mathbb{C}^*)^n$ , we may define its tropicalization by setting  $\text{Trop}(W) = \text{Trop}(W \times_{\mathbb{C}} \mathbb{K})$ .
- In this case, the tropicalization is a polyhedral fan in  $\mathbb{R}^n$ .

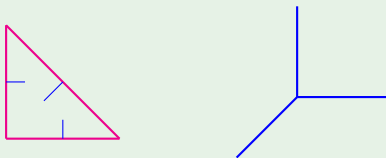


- For a polytope  $P \subset \mathbb{R}^n$ , let  $P^\circ = \{y \in (\mathbb{R}^n)^\vee : x \cdot y \leq 1, \forall x \in P\}$  be its polar dual, and set
  - $\mathcal{F}(P)$  the face fan (the set of cones spanned by the faces of  $P$ );
  - $\mathcal{N}(P)$  the (inner) normal fan.
- If  $0 \in \text{int}(P)$ , then  $\mathcal{N}(P) = \mathcal{F}(P^\circ)$ .
- If  $W = V(f)$  is a hypersurface defined by  $f = \sum_{u \in A} a_u t^u \in \mathbb{C}[t^{\pm 1}]$ , and  $\text{Newt}(f) = \text{conv}\{u \mid a_u \neq 0\} \subset \mathbb{R}^n$ , then

$$\text{Trop}(V(f)) = \mathcal{N}(\text{Newt}(f))^{\text{codim} > 0}.$$

## EXAMPLE

Let  $f = t_1 + t_2 + 1$ . Then  $\text{Newt}(f) = \text{conv}\{(1, 0), (0, 1), (0, 0)\}$  is a triangle, and so  $\text{Trop}(V(f))$  is a tripod.



## Tropicalizing the characteristic varieties

- Recall  $\mathbb{K} = \mathbb{C}\{\{t\}\}$  comes with a valuation map,  $\nu: \mathbb{K}^* \rightarrow \mathbb{Q}$ .
- Let  $\nu_X: \text{Char}_{\mathbb{K}}(X) \rightarrow \mathbb{Q}^n \subset \mathbb{R}^n$  be the composite

$$H^1(X; \mathbb{K}^*) \xrightarrow{\nu^*} H^1(X; \mathbb{Q}) \longrightarrow H^1(X; \mathbb{R}).$$

- I.e., if  $\rho: \pi_1(X) \rightarrow \mathbb{K}^*$  is a  $\mathbb{K}$ -valued character, then the morphism  $\nu \circ \rho: \pi_1(X) \rightarrow \mathbb{Q}$  defines  $\nu_X(\rho) \in H^1(X; \mathbb{Q}) = \mathbb{Q}^n \subset \mathbb{R}^n$ .
- Given an algebraic subvariety  $W \subset H^1(X; \mathbb{C}^*)$  we define its *tropicalization* as the closure in  $H^1(X; \mathbb{R}) \cong \mathbb{R}^n$  of the image of  $W \times_{\mathbb{C}} \mathbb{K} \subset H^1(X; \mathbb{K}^*)$  under  $\nu_X$ ,

$$\text{Trop}(W) := \overline{\nu_X(W \times_{\mathbb{C}} \mathbb{K})}.$$

- Applying this definition to the characteristic varieties  $\mathcal{V}^i(X)$ , and recalling that  $\mathcal{V}^i(X, \mathbb{K}) = \mathcal{V}^i(X) \times_{\mathbb{C}} \mathbb{K}$ , we have that

$$\text{Trop}(\mathcal{V}^i(X)) = \overline{\nu_X(\mathcal{V}^i(X, \mathbb{K}))}.$$

## LEMMA

Let  $W \subset (\mathbb{C}^*)^n$  be an algebraic variety. Then  $\tau_1^{\mathbb{R}}(W) \subseteq \text{Trop}(W)$ .

Sketch of proof.

- Every irreducible component of  $\tau_1^{\mathbb{R}}(W)$  is of the form  $L \otimes_{\mathbb{Q}} \mathbb{R}$ , for some linear subspace  $L \subset \mathbb{Q}^n$ .
- The complex torus  $T := \exp(L \otimes_{\mathbb{Q}} \mathbb{C})$  lies inside  $W$ .
- Thus,  $\text{Trop}(T) = L \otimes_{\mathbb{Q}} \mathbb{R}$  lies inside  $\text{Trop}(W)$ . □

## PROPOSITION

- $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subseteq \text{Trop}(\mathcal{V}^i(X))$ , for all  $i \leq q$ .
- If there is a subtorus  $T \subset \text{Char}^0(X)$  such that  $T \not\subset \mathcal{V}^i(X)$ , yet  $\rho T \subset \mathcal{V}^i(X)$  for some  $\rho \in \text{Char}(X)$ , then  $\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)) \subsetneq \text{Trop}(\mathcal{V}^i(X))$ .



## A tropical bound for the $\Sigma$ -invariants

### THEOREM (PS-2010, S-2021)

Let  $\rho: \pi_1(X) \rightarrow \mathbb{k}^*$  be a character such that  $\rho \in \mathcal{V}^{\leq q}(X, \mathbb{k})$ . Let  $v: \mathbb{k}^* \rightarrow \mathbb{R}$  be the homomorphism defined by a valuation on  $\mathbb{k}$ . If the homomorphism  $\chi := v \circ \rho: \pi_1(X) \rightarrow \mathbb{R}$  is non-zero, then  $\chi \notin \Sigma^q(X, \mathbb{Z})$ .

Sketch of proof.

- Let  $\widehat{\mathbb{k}}$  be the topological completion of  $\mathbb{k}$  with respect to the absolute value  $|c| = \exp(-v(c))$ . Get a field extension,  $\iota: \mathbb{k} \hookrightarrow \widehat{\mathbb{k}}$ .
- Let  $G = \pi_1(X)$ . Extend  $\rho: G \rightarrow \mathbb{k}^*$  to a ring map,  $\bar{\rho}: \mathbb{Z}G \rightarrow \mathbb{k}$ .
- Since  $\chi = v \circ \rho$ , we can extend  $\bar{\rho}$  to a morphism of topological rings,  $\hat{\rho}: \widehat{\mathbb{Z}G}_{-\chi} \rightarrow \widehat{\mathbb{k}}$ , making  $\widehat{\mathbb{k}}$  into a  $\widehat{\mathbb{Z}G}_{-\chi}$ -module, denoted  $\widehat{\mathbb{k}}_{\hat{\rho}}$ .
- Restricting scalars via the inclusion  $\mathbb{Z}G \hookrightarrow \widehat{\mathbb{Z}G}_{-\chi}$  yields the  $\mathbb{Z}G$ -module  $\widehat{\mathbb{k}}_{\iota \circ \rho}$ , defined by the character  $\iota \circ \rho: G \rightarrow \widehat{\mathbb{k}}^*$ .

- For a ring  $R$ , a bounded below chain complex of flat right  $R$ -modules  $K_*$ , and a left  $R$ -module  $M$ , there is a (right half-plane, boundedly converging) Künneth spectral sequence,

$$E_{ij}^2 = \operatorname{Tor}_i^R(H_j(K), M) \Rightarrow H_{i+j}(K \otimes_R M).$$

- Use ring  $R = \widehat{\mathbb{Z}G}_{-\chi}$ , chain complex of free  $R$ -modules  $K_* = C_*(\tilde{X}, \mathbb{Z}) \otimes_{\mathbb{Z}G} \widehat{\mathbb{Z}G}_{-\chi}$ , and  $R$ -module  $M = \hat{\mathbb{k}}_{\hat{\rho}}$ .
- Now let  $\rho \in \mathcal{V}^{\leq q}(X, \mathbb{k})$ , and suppose  $\chi = \nu \circ \rho \in \Sigma^q(X, \mathbb{Z})$ .
- This is equivalent to  $H_j(X; \widehat{\mathbb{Z}G}_{-\chi}) = 0$  for all  $j \leq q$ ; that is,  $H_j(K) = 0$  for  $j \leq q$ . Therefore,  $E_{ij}^2 = 0$  for  $j \leq q$ .
- Hence,  $H_{i+j}(X; \hat{\mathbb{k}}_{\nu \circ \rho}) = 0$  for  $j \leq q$ , and so  $H_j(X; \hat{\mathbb{k}}_{\nu \circ \rho}) = 0$  for  $j \leq q$ .
- This is equivalent to  $\nu \circ \rho \notin \mathcal{V}^{\leq q}(X, \hat{\mathbb{k}})$ . Hence,  $\rho \notin \mathcal{V}^{\leq q}(X, \mathbb{k})$ , contradicting our hypothesis on  $\rho$ .
- Therefore,  $\chi \notin \Sigma^q(X, \mathbb{Z})$ . □

## THEOREM (S-2021)

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(\mathrm{Trop}(\mathcal{V}^{\leq q}(X)))^c$$

Sketch of proof.

- Let  $\rho: \pi_1(X) \rightarrow \mathbb{K}^*$  and set  $\chi = v \circ \rho: \pi_1(X) \rightarrow \mathbb{Q}$ , a rational point on  $H^1(X; \mathbb{R})$ .
- Suppose  $\rho \in \mathcal{V}^{\leq q}(X, \mathbb{K}) = \mathcal{V}^{\leq q}(X) \times_{\mathbb{C}} \mathbb{K}$ .
- Then  $\chi$  is a rational point on  $\mathrm{Trop}(\mathcal{V}^{\leq q}(X)) = \overline{\nu_X(\mathcal{V}^{\leq q}(X; \mathbb{K}))}$ .
- Conversely, all rational points on  $\mathrm{Trop}(\mathcal{V}^{\leq q}(X))$  are of the form  $\nu_X(\rho) = v \circ \rho$ , for some  $\rho \in \mathcal{V}^{\leq q}(X, \mathbb{K})$ .
- Finally, assume that  $\chi \neq 0$ , so that  $\chi$  represents an (arbitrary) rational point in  $S(\mathrm{Trop}(\mathcal{V}^{\leq q}(X)))$ .
- By the previous theorem,  $\chi \in \Sigma^q(X, \mathbb{Z})^c$ .
- But the rational points are dense in  $S(\mathrm{Trop}(\mathcal{V}^{\leq q}(X)))$ , and  $\Sigma^q(X, \mathbb{Z})^c$  is closed in  $S(X)$ , and so we're done.  $\square$

## COROLLARY

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(\text{Trop}(\mathcal{V}^{\leq q}(X)))^c \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^{\leq q}(X)))^c.$$

$$\Sigma^1(G) \subseteq -S(\text{Trop}(\mathcal{V}^1(G)))^c \subseteq S(\tau_1^{\mathbb{R}}(\mathcal{V}^1(G)))^c.$$

## COROLLARY

If  $\mathcal{V}^{\leq q}(X)$  contains a component of  $\text{Char}(X)$ , then  $\Sigma^q(X, \mathbb{Z}) = \emptyset$ .

## THEOREM

Let  $f_\alpha: G \rightarrow G_\alpha$  be a finite collection of epimorphisms. If each  $\mathcal{V}^1(G_\alpha)$  contains a component of  $\mathbb{T}_{G_\alpha}$ , then

$$\Sigma^1(G) \subseteq \left( \bigcup_{\alpha} S(f_\alpha^*(H^1(G_\alpha; \mathbb{R}))) \right)^c.$$

## The Alexander polynomial

- Let  $H = G_{\text{ab}}/\text{tors}(G_{\text{ab}})$  be the maximal torsion-free abelian quotient of  $G = \pi_1(X)$  and  $q: X^H \rightarrow X$  the respective cover.
- Set  $A_X := H_1(X^H; q^{-1}(x_0), \mathbb{Z})$ , viewed as a  $\mathbb{Z}[H]$ -module.
- Let  $E_1(A_X) \subseteq \mathbb{Z}[H]$  be the ideal of codimension 1 minors in a presentation for  $A_X$ .
- $\Delta_X := \text{gcd}(E_1(A_X)) \in \mathbb{Z}[H]$  is the *Alexander polynomial* of  $X$ . It only depends on  $G$ , so also write it as  $\Delta_G$ .
- Suppose  $I_H^p \cdot (\Delta_G) \subseteq E_1(A_G)$ , for some  $p \geq 0$ . Then

$$\mathcal{V}^1(X) \cap \mathbb{T}_G^0 = \{1\} \cup V(\Delta_G).$$

- This condition is satisfied if  $G$  is a 1-relator group, or  $G = \pi_1(M)$ , where  $M$  is a closed, orientable 3-manifold with empty or toroidal boundary (C. McMullen, D. Eisenbud–W. Neumann).

- Let  $\text{Newt}(\Delta_G) \subset H_1(G; \mathbb{R})$  be the Newton polytope of  $\Delta_G$ .
- Given  $\phi \in H^1(G; \mathbb{Z}) \cong \text{Hom}(H, \mathbb{Z})$ , its *Alexander norm*,  $\|\phi\|_A$ , is the length of  $\phi(\text{Newt}(\Delta_G))$ .
- This defines a semi-norm on  $H^1(G; \mathbb{R})$ , with unit ball

$$B_A = \{\phi \in H^1(G; \mathbb{R}) \mid \|\phi\|_A \leq 1\}.$$

- If  $\Delta_G$  is symmetric (i.e., invariant under  $t_i \mapsto t_i^{-1}$ ), then  $B_A$  is, up to a scale factor of  $1/2$ , the polar dual of the Newton polytope of  $\Delta_G$ ,

$$2B_A = \text{Newt}(\Delta_G)^\circ.$$

## PROPOSITION

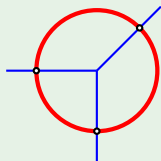
If  $\Delta_G$  is symmetric and  $I_H^p \cdot (\Delta_G) \subseteq E_1(A_G)$ , for some  $p \geq 0$ , then

$$\Sigma^1(G) \subseteq \bigcup_{F \text{ an open facet of } B_A} S(F).$$

## Two-generator, one-relator groups

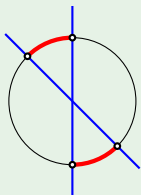
- Let  $G = \langle x, y \mid r \rangle$ , with  $b_1(G) = 2$ . In 1987, K. Brown gave a combinatorial algorithm for computing  $\Sigma^1(G)$ ; see Friedl–Tillmann (2020) for a modern approach.

### EXAMPLE



- Let  $G = \langle a, b \mid b^2(ab^{-1})^2a^{-2} \rangle$ .
- Then  $\Sigma^1(G) = S^1 \setminus \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, -1), (-1, 0)\}$ .
- On the other hand,  $\Delta_G = a + b + 1$ .
- Thus,  $\Sigma^1(G) = -S(\text{Trop}(V(\Delta_G)))^c$ , though  $\tau_1 \mathcal{V}^1(G) = \{0\}$ .

### EXAMPLE



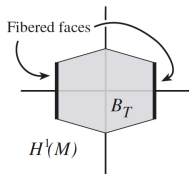
- Let  $G = \langle a, b \mid a^2ba^{-1}ba^2ba^{-1}b^{-3}a^{-1}ba^2ba^{-1}ba^{-1}a^{-2}b^{-1}ab^{-1}a^{-2}b^{-1}ab^3ab^{-1}a^{-2}b^{-1}ab^{-1}a^{-1}b \rangle$ .
- Then  $\Delta_G = (a-1)(ab-1)$ , and so  $S(\text{Trop}(V(\Delta_G)))$  consists of two pairs of points.
- Yet  $\Sigma^1(G)$  consists of two open arcs joining those points.

## Compact 3-manifolds

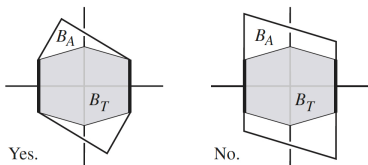
- Let  $M$  be a compact, connected, orientable 3-manifold with  $b_1(M) > 0$ .
- A non-zero class  $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z})$  is *fibred* if there exists a fibration  $p: M \rightarrow S^1$  such that the induced map  $p_*: \pi_1(M) \rightarrow \pi_1(S^1) = \mathbb{Z}$  coincides with  $\phi$ .
- The *Thurston norm*  $\|\phi\|_T$  of a class  $\phi \in H^1(M; \mathbb{Z})$  is the infimum of  $-\chi(\hat{S})$ , where  $S$  runs through all the properly embedded, oriented surfaces in  $M$  dual to  $\phi$ , and  $\hat{S}$  denotes the result of discarding all components of  $S$  which are disks or spheres.
- Thurston showed that  $\| - \|_T$  defines a seminorm on  $H^1(M; \mathbb{Z})$ , which can be extended to a continuous seminorm on  $H^1(M; \mathbb{R})$ .
- The unit norm ball,  $B_T = \{\phi \in H^1(M; \mathbb{R}) \mid \|\phi\|_T \leq 1\}$ , is a rational polyhedron with finitely many sides and symmetric in the origin.



There are facets of  $B_T$ , called the *fibred faces* (coming in antipodal pairs), so that a class  $\phi \in H^1(M; \mathbb{Z})$  fibers if and only if it lies in the cone over the interior of a fibred face.



- Bieri, Neumann, and Strebel showed that the BNS invariant of  $G = \pi_1(M)$  is the projection onto  $S(G)$  of the open fibred faces of the Thurston norm ball  $B_T$ ; in particular,  $\Sigma^1(G) = -\Sigma^1(G)$ .
- Under some mild assumptions, McMullen showed that  $\|\phi\|_A \leq \|\phi\|_T$ , leading to an upper bound for  $\Sigma_1(G)$  in terms of  $B_A$ , explained by Dunfield, as follows:



## THEOREM

Let  $M$  be a compact, connected, orientable, 3-manifold with empty or toroidal boundary. Set  $G = \pi_1(M)$  and assume  $b_1(M) \geq 2$ . Then

- (1)  $\text{Trop}(\mathcal{V}^1(G) \cap \mathbb{T}_G^0)$  is the positive-codimension skeleton of  $\mathcal{F}(B_A)$ , the face fan of the unit ball in the Alexander norm.
- (2) (Partly recovers McMullen's theorem)  $\Sigma^1(G)$  is contained in the union of the open cones on the facets of  $B_A$ .

Example: Let  $M$  be Seifert manifold with

- Orientable base surface of genus  $g$ .
- Exceptional fibers  $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ .
- Orbifold Euler number  $e = -\sum_{i=1}^r \beta_i / \alpha_i$ .
- $\theta(\alpha) = \alpha_1 \cdots \alpha_r / \text{lcm}(\alpha_1, \dots, \alpha_r)$ .

## PROPOSITION

Suppose  $e \neq 0$  and either  $g > 1$ , or  $g = 1$  and  $\theta(\alpha) > 1$ . Then  $\Sigma^1(M) = \emptyset$ .

## Kähler manifolds

- Let  $M$  be a compact Kähler manifold. Then  $M$  is formal.
- (Beauville, Catanese, Green–Lazarsfeld, Simpson, Arapura, B. Wang)  
 $\mathcal{V}^i(M)$  are finite unions of torsion translates of algebraic subtori of  $H^1(M, \mathbb{C}^*)$ .

THEOREM (DELZANT 2010)

$$\Sigma^1(M) = S(M) \setminus \bigcup_{\alpha} S(f_{\alpha}^*(H^1(C_{\alpha}; \mathbb{R}))),$$

where the union is taken over those orbifold fibrations  $f_{\alpha}: M \rightarrow C_{\alpha}$  with the property that either  $\chi(C_{\alpha}) < 0$ , or  $\chi(C_{\alpha}) = 0$  and  $f_{\alpha}$  has some multiple fiber.

In degree 1, we may recast this result in the tropical setting, as follows.

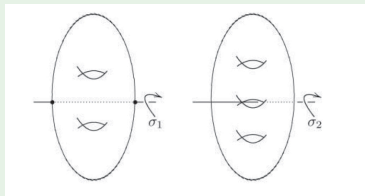
COROLLARY

$$\Sigma^1(M) = S(\text{Trop}(\mathcal{V}^1(M)))^c.$$

## EXAMPLE (THE CATANESE–CILIBERTO–MENDES LOPES SURFACE)

Let  $C_1$  be a smooth curve of genus 2 with an elliptic involution  $\sigma_1$ ,

- and  $C_2$  a curve of genus 3 with a free involution  $\sigma_2$ .



- Then  $\Sigma_1 = C_1/\sigma_1$  is a curve of genus 1,  $\Sigma_2 = C_2/\sigma_2$  is a curve of genus 2, and  $M = (C_1 \times C_2)/\sigma_1 \times \sigma_2$  is a smooth, complex projective surface with  $H_1(M; \mathbb{Z}) = \mathbb{Z}^6$ .
- Projection onto the first coordinate yields an orbifold fibration  $f_1: M \rightarrow \Sigma_1$  with two multiple fibers, each of multiplicity 2. The other projection defines a smooth fibration  $f_2: M \rightarrow \Sigma_2$ .
- We have  $\mathcal{V}^1(M) = \{t \mid t_1 = t_2 = 1\} \cup \{t_4 = t_5 = t_6 = 1, t_3 = -1\}$ , with the two components obtained by pullback along  $f_1$  and  $f_2$ .
- Thus,  $\Sigma^1(M) = S^5 \setminus \mathcal{S}(\{x_3 = \dots = x_6 = 0\} \cup \{x_1 = x_2 = 0\})$ .

## Hyperplane arrangements

- Let  $\mathcal{A} = \{H_1, \dots, H_n\}$  be an (essential, central) arrangement of hyperplanes in  $\mathbb{C}^d$ .
- Its complement,  $M(\mathcal{A}) \subset (\mathbb{C}^*)^d$ , is a smooth, quasi-projective Stein manifold; thus, it has the homotopy type of a finite,  $d$ -dimensional CW-complex.
- $H^*(M(\mathcal{A}); \mathbb{Z})$  is the Orlik–Solomon algebra of  $L(\mathcal{A})$ .
- (Arapura) The characteristic varieties  $\mathcal{V}^i(\mathcal{A}) := \mathcal{V}^i(M(\mathcal{A})) \subset (\mathbb{C}^*)^n$  are unions of translated subtori.
- Consequently,  $\text{Trop}(\mathcal{V}^i(\mathcal{A})) = -\text{Trop}(\mathcal{V}^i(\mathcal{A}))$ .
- (Denham–S.–Yuzvinsky 2016/17)  $M(\mathcal{A})$  is an “abelian duality space”; thus, its jump loci propagate:  $\mathcal{V}^1(\mathcal{A}) \subseteq \dots \subseteq \mathcal{V}^{d-1}(\mathcal{A})$ .
- (Arnol’d, Brieskorn)  $M(\mathcal{A})$  is formal. Thus,  $\tau_1(\mathcal{V}^i(\mathcal{A})) = \mathcal{R}^i(\mathcal{A})$ .

## THEOREM

Let  $M$  be the complement of an arrangement of  $n$  hyperplanes in  $\mathbb{C}^d$ .  
Then, for each  $1 \leq q \leq d - 1$ :

- $\text{Trop}(\mathcal{V}^q(M))$  is the union of a subspace arrangement in  $\mathbb{R}^n$ .
- $\Sigma^q(M, \mathbb{Z}) \subseteq S(\text{Trop}(\mathcal{V}^q(M)))^c$ .

## QUESTION (MFO MINIWORKSHOP 2007)

Given an arrangement  $\mathcal{A}$ , do we have

$$\Sigma^1(M(\mathcal{A})) = S(\mathcal{R}^1(\mathcal{A}, \mathbb{R}))^c? \quad (\star)$$

### EXAMPLE (KOBAN-McCAMMOND-MEIER 2013)

- Let  $\mathcal{A}$  be the braid arrangement in  $\mathbb{C}^n$ , defined by  $\prod_{1 \leq i < j \leq n} (z_i - z_j) = 0$ . Then  $M(\mathcal{A}) = \text{Conf}(n, \mathbb{C}) \simeq K(P_n, 1)$ .
- Answer to  $(\star)$  is yes:  $\Sigma^1(M(\mathcal{A}))$  is the complement of the union of  $\binom{n}{3} + \binom{n}{4}$  planes in  $\mathbb{C}^{\binom{n}{2}}$ , intersected with the unit sphere.

### EXAMPLE

- Let  $\mathcal{A}$  be the “deleted  $B_3$ ” arrangement, defined by  $z_1 z_2 (z_1^2 - z_2^2)(z_1^2 - z_3^2)(z_2^2 - z_3^2) = 0$ .
- (S. 2002)  $\mathcal{V}^1(\mathcal{A})$  contains a (1-dimensional) translated torus  $\rho \cdot T$ .
- Thus,  $\text{Trop}(\rho \cdot T) = \text{Trop}(T)$  is a line in  $\mathbb{C}^8$  which is *not* contained in  $\mathcal{R}^1(\mathcal{A}, \mathbb{R})$ . Hence, the answer to  $(\star)$  is no.

### QUESTION (REVISED)

$$\Sigma^1(M(\mathcal{A})) = S(\text{Trop}(\mathcal{V}^1(\mathcal{A}))^c? \quad (**)$$

# REFERENCE



A.I. Suciu, *Sigma-invariants and tropical varieties*, Math. Annalen **380** (2021), 1427–1463.