LATTICES IN COMPLETE RANK 2 KAC–MOODY GROUPS

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ABSTRACT. Let Λ be a minimal Kac–Moody group of rank 2 defined over the finite field $\mathbb{F}_q$, where $q = p^a$ with p prime. Let G be the topological Kac–Moody group obtained by completing Λ. An example is $G = SL_2(K)$, where K is the field of formal Laurent series over $\mathbb{F}_q$. The group G acts on its Bruhat–Tits building X, a tree, with quotient a single edge. We construct new examples of cocompact lattices in G, many of them edge-transitive. We then show that if cocompact lattices in G do not contain p–elements, the lattices we construct are the only edge-transitive lattices in G, and that our constructions include the cocompact lattice of minimal covolume in G. We also observe that, with an additional assumption on p–elements in G, the arguments of Lubotzky [L] for the case $G = SL_2(K)$ may be generalised to show that there is a positive lower bound on the covolumes of all lattices in G, and that this minimum is realised by a non-cocompact lattice, a maximal parabolic subgroup of Λ.

1. INTRODUCTION

A classical theorem of Siegel [Si] states that the minimum covolume among lattices in $G = SL_2(\mathbb{R})$ is $\frac{\pi}{21}$, and determines the lattice which realises this minimum. In the nonarchimedean setting, Lubotzky [L] determined the lattice of minimal covolume in $G = SL_2(K)$, where K is the field $\mathbb{F}_q((t^{-1}))$ of formal Laurent series over $\mathbb{F}_q$.

The group $G = SL_2(\mathbb{F}_q((t^{-1})))$ has, in recent developments, been viewed as the first example of a complete Kac–Moody group of rank 2 over the finite field $\mathbb{F}_q$. By definition, a complete Kac–Moody group is the completion of a minimal Kac–Moody group Λ over a finite field, with respect to some topology. We use the completion in the "building topology", as discussed in, for example, [CR]. Complete Kac–Moody groups are locally compact, totally disconnected topological groups, which may be thought of as infinite-dimensional analogues of semisimple algebraic groups (see Section 2.4 below for details).

1.1. **Constructions of cocompact lattices.** Our first main result, Theorem 1.1 below, constructs many new cocompact lattices in rank 2 complete Kac–Moody groups G. It is interesting that there exist any cocompact lattices in such groups G, since for $n \geq 3$, Kac–Moody groups of rank n do not admit cocompact lattices (with the possible exception of those whose root systems contain a subsystem of type $\tilde{A}_n$ — see [CM, Remark 4.4]). In rank 2, the previous examples of cocompact lattices in G non-affine that are known to us are the free Schottky groups constructed by Carbone–Garland [CG, Section 11], some of the lattices constructed by Rémy–Ronan [RR, Section 4.B], which in rank 2 are free products of finite cyclic groups, and some of the lattices obtained as centralisers of certain involutions by Gramlich–Horn–Mühlherr [GHM, Section 7.3].

As we recall in Section 2.4 below, the Kac–Moody groups G that we consider have Bruhat–Tits building a regular tree X, and the action of G on X induces an edge of groups

$$\mathbb{G} = \underset{B}{\overset{P_1 \qquad\qquad P_2}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet}}$$

where P_1 and P_2 are the standard parahoric subgroups of G and $B = P_1 \cap P_2$ is the standard Iwahori subgroup. The kernel of the G–action on X is the finite group $Z(G)$, the centre of G (see [CR]). Now let Γ be a cocompact lattice in G which acts transitively on the edges of X. Then as we recall in Section 2.3, Γ is the fundamental group $A_1 *_{A_0} A_2$ of an edge of groups

$$\mathbb{A} = \quad \overset{\textstyle A_1 \qquad\qquad\qquad A_2}{\underset{\textstyle A_0}{\bullet\rule{3cm}{0.4pt}\bullet}}$$

with A_0, A_1 and A_2 finite groups (see Section 2.2 for background on graphs of groups).

We now state Theorem 1.1, which constructs new cocompact lattices in G, most of them edge-transitive. We discuss our assumption that G has symmetric generalised Cartan matrix A after the statement of Theorem 1.1. There are some exceptional constructions for small values of q which we then record in Theorem 1.2. Apart from the affine case $G = SL_2(\mathbb{F}_q((t^{-1})))$, there are no known linear representations of the groups G in Theorems 1.1 and 1.2.

Our notation is as follows. We write C_n for the cyclic group of order n and S_n for the symmetric group on n letters. Since for a finite field $\mathbb{F}_q$ and the root system A_1 there exist at most two corresponding finite groups of Lie type, namely $SL_2(\mathbb{F}_q)$ and $PSL_2(\mathbb{F}_q)$, to avoid complications we use Lie-theoretic notation and write $A_1(q)$ in both cases. (We will discuss this ambiguity whenever necessary.) We denote by T a fixed maximal split torus of G with $T \leq P_1 \cap P_2$. Then $Z(G) \leq T$, and T is isomorphic to a quotient of $\mathbb{F}_q^* \times \mathbb{F}_q^*$ (the particular quotient depending upon G). Finally, the parahoric subgroups P_1 and P_2 of G admit Levi decompositions (see Section 2.4.2 and, in particular, Proposition 2.3 below), and for $i = 1, 2$ we denote by L_i a Levi complement of P_i. The group L_i factors as $L_i = M_i T$, where $M_i \cong A_1(q)$ is normalised by T, and we denote by H_i a non-split torus of M_i such that $N_T(H_i)$ is as big as possible. We say that two edge-transitive cocompact lattices $\Gamma = A_1 *_{A_0} A_2$ and $\Gamma' = A_1' *_{A_0'} A_2'$ in G are *isomorphic* if $A_i \cong A_i'$ for $i = 0, 1, 2$ and the obvious diagram commutes.

Theorem 1.1. *Let G be a complete Kac–Moody group of rank 2 with symmetric generalised Cartan matrix* $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$, *defined over the finite field $\mathbb{F}_q$ of order $q = p^a$ where p is prime. Then in the following cases, the group G admits edge-transitive cocompact lattices of each of the following isomorphism types.*

(1) *If $p = 2$ then $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0 \times H_i$ and $H_i \cong C_{q+1}$, and A_0 is any subgroup of $Z(G)$.*

(2) *If p is odd and $L_i/Z(L_i) \cong PSL_2(q)$, assume also that $q \equiv 3 \pmod 4$. Then $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0 N_{M_i}(H_i)$, and A_0 is a subgroup of $Z(G)$ with $A_0 \cap N_{M_i}(H_i) = Z(M_i)$, $i = 1, 2$.*

(3) *If p is odd and $L_i/Z(L_i) \cong PGL_2(q)$, let*

$$C_i := C_{L_i}(H_i), \quad Q_i' \in \mathcal{S}yl_2(C_i), \quad Q_i := (Q_i' \cap Z(L_i)) \in \mathcal{S}yl_2(Z(L_i))$$

and let Q_i^2 be the unique subgroup of Q_i of index 2.

   (a) *If $q \equiv 1 \pmod 4$ and $Q_i^2 \leq Z(G)$ then:*

      (i) *$\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = H_i Q_i' \langle t_i \rangle Z_0$ where $t_i \in N_T(H_i) - C_T(H_i)$ is of order 2, $Z_0 \leq Z(G)$ and $A_0 = Q_i \langle t_i \rangle Z_0$.*

      (ii) *Moreover, if $Q_i \leq Z(G)$, then also $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = H_i Q_i' Z_0$ where $Z_0 \leq Z(G)$ and $A_0 = Q_i Z_0 \leq Z(G)$.*

   (b) *If $q \equiv 3 \pmod 4$:*

      (i) *$\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = H_i Q_i' T_0 Z_0$ with $T_0 \in \mathcal{S}yl_2(T)$, $Q_i' \cap T_0 = Q_i$, $Z_0 \leq Z(G)$ and $A_0 = T_0 Z_0$.*

      (ii) *If $Z(M_i) \leq Z(G)$, then also $\Gamma = A_1 *_{A_0} A_2$ where either for $i = 1, 2$, $A_i = H_i Q_i' A_0$ with $Q_i' \cap A_0 = Z(M_i)$ and $A_0 \leq Z(G)$, or the description of Γ in (2) holds.*

In all other cases, G admits a cocompact lattice Γ' which acts on the tree X inducing a graph of groups of the form

$$\mathbb{A}' = \qquad N_1 \overset{\displaystyle S}{\underset{\displaystyle S}{\bullet\,\,\,\,\,\bullet}} N_2$$
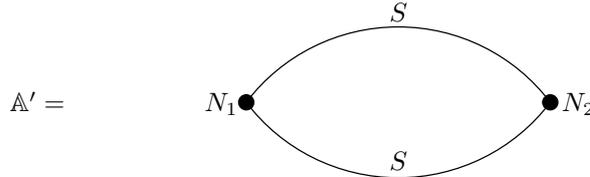
where the finite groups S, N_1 and N_2 will be defined in Section 3.3.1 below.

Remark. If G has generalised Cartan matrix $A = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$ which is not symmetric but satisfies $m, n \geq 2$, then the constructions in Theorem 1.1(1) and (2) above still hold. In case (3) of Theorem 1.1, there are analogous edge-transitive lattices with an even more involved description, which we omit.

Our exceptional constructions for small values of q are as follows.

Theorem 1.2. *Let G be as in Theorem 1.1 above. Then in the following cases, G admits cocompact edge-transitive lattices of the following isomorphism types.*

When p is odd and $q \equiv 1 \pmod 4$:

  (1) *If $q = 5$, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0 N_i$ where $N_i \cong A_1(3)$, $A_0 \leq N_T(N_i)$ and $|N_i : N_i \cap A_0| = 6$; and*

  (2) *If $q = 29$, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0 N_i$ where $N_i \cong A_1(5)$, $A_0 \leq N_T(N_i)$ and $|N_i : N_i \cap A_0| = 30$.*

When p is odd and $q \equiv 3 \pmod 4$:

  (1) *If $q = 7$ or 23, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0 N_i$, $N_i \cong S_4$ or $2S_4$, $A_0 \leq N_T(N_i)$ and $|N_i : N_i \cap A_0| = q + 1$ where $N_i \cap A_0$ is cyclic.*

  (2) *If $q = 11$, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0 N_i$, and $A_0 \leq N_T(N_i)$ with $|N_i : N_i \cap A_0| = 12$, $N_i \cap A_0$ being cyclic, and one of the following holds:*
    (a) *$N_1 \cong N_2 \cong A_1(3)$; or*
    (b) *$N_1 \cong N_2 \cong A_1(5)$.*

  (3) *If $q = 19$ or 59, $\Gamma = A_1 *_{A_0} A_2$ where for $i = 1, 2$, $A_i = A_0 N_i$, $N_i \cong A_1(5)$, $A_0 \leq N_T(N_i)$ and $|N_i : N_i \cap A_0| = q + 1$ with $N_i \cap A_0$ being cyclic.*

We prove Theorems 1.1 and 1.2 in Section 3. By the general theory of tree lattices (see Section 2.3), each Γ or Γ' appearing in Theorems 1.1 and 1.2 yields a cocompact lattice in the full automorphism group $\text{Aut}(X)$ of the tree X. Since $\text{Aut}(X)$ is much larger than the Kac–Moody group G, the key point in proving Theorems 1.1 and 1.2 is to show that each such Γ or Γ' embeds into G as a cocompact lattice.

For this, we develop an embedding criterion, Proposition 3.1 in Section 3.1 below, which may be applied to construct lattices in any closed locally compact group acting transitively on the edges of a locally finite biregular tree. Our embedding criterion generalises [L, Lemma 3.1], which was used in [L] to construct edge-transitive lattices in the affine case $G = SL_2(\mathbb{F}_q((t^{-1})))$. We are able to provide somewhat simpler proofs even in that case by using Bridson and Haefliger's covering theory for complexes of groups [BH] instead of Bass' covering theory for graphs of groups [B], since the theory for complexes of groups has a less complicated notion of morphism.

Our construction of the lattice Γ' in Theorem 1.1 generalises the construction in Example (6.2) of Lubotzky–Weigel [LW] of a cocompact lattice in $G = SL_2(\mathbb{F}_q((t^{-1})))$ when $q \equiv 1 \pmod 4$. Example (6.2) of [LW] uses the embedding criterion [LW, Theorem 5.4], which relies upon Bass' covering theory [B] and is specific to the affine case, while we apply our criterion Proposition 3.1 instead.

1.2. Conjectures.

Let G be as in Theorem 1.1. Our further results, Theorems 1.3, 1.4 and 1.5 below, depend upon conjectures about the behaviour of p–elements in G.

In the affine case $G = SL_2(\mathbb{F}_q((t^{-1})))$, an element of G has order p if and only if it is unipotent. Hence by Godement's Criterion, no p–element is contained in a cocompact lattice. Moreover, any unipotent element of $G = SL_2(\mathbb{F}_q((t^{-1})))$ is contained in a conjugate of the upper unitriangular subgroup of G, which is an infinite group isomorphic to $\mathbb{F}_q((t^{-1}))$. On the other hand, Lubotzky [L] showed that no cocompact lattice of $SL_2(\mathbb{F}_q((t^{-1})))$ contains non-trivial p–elements (analogous to the classical Godement's cocompactness criterion). For general G, as we explain in Section 4 below, there are many p–elements which cannot be contained in any cocompact lattice $\Gamma < G$, since they belong to a conjugate of a canonical subgroup $\mathcal{U}$ of G (see Section 2.4.4 for the definition of $\mathcal{U}$). We thus make the following conjectures in the general case.

Conjecture 1. *Cocompact lattices in G do not contain p–elements.*

Conjecture 2. *Any p–element in G is contained in a conjugate of the subgroup $\mathcal{U}$ of G.*

In Section 4, we also explain why Conjecture 2 implies Conjecture 1.

Theorems 1.3 and 1.4 below only require Conjecture 1, while Conjecture 2 is needed for Theorem 1.5. As in the remark following Theorem 1.1 above, similar results to Theorems 1.3, 1.4 and 1.5 will hold with Cartan matrix $A = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$, $m, n \geq 2$, but we omit these more cumbersome statements.

1.3. Classification of edge-transitive lattices.
Assuming Conjecture 1, we are able to classify the edge-transitive lattices in G up to isomorphism, as follows.

Theorem 1.3. *Let G be as in Theorem 1.1, and assume that Conjecture 1 holds. Then every edge-transitive lattice in G is isomorphic to one of the amalgamated free products of finite groups $A_1 *_{A_0} A_2$ in Theorems 1.1 and 1.2.*

In other words, every edge-transitive lattice in G which does not contain p–elements appears in Theorems 1.1 or 1.2.

The question of classifying amalgams of finite groups is, in general, difficult. An (m, n)–*amalgam* is an amalgamated free product $A_1 *_{A_0} A_2$ where A_0 has index m in A_1 and index n in A_2. To avoid trivial examples, such an amalgam is said to be *faithful* if A_0, A_1 and A_2 have no common normal subgroup. A deep theorem of Goldschmidt [G] established that there are only 15 faithful $(3, 3)$–amalgams of finite groups, and classified such amalgams. Goldschmidt and Sims conjectured that when both m and n are prime, there are only finitely many faithful (m, n)–amalgams of finite groups (see [BK, F, G]). This conjecture remains open, except for the case $(m, n) = (2, 3)$, which was established by Djoković–Miller [DM], and the work of Fan [F], who proved the conjecture when the edge group A_0 is a p–group, with p a prime distinct from both m and n.

On the other hand, Bass–Kulkarni [BK] showed that if either m or n is composite, there are infinitely many faithful (m, n)–amalgams of finite groups. The constructions in [BK] may be viewed as giving infinitely many non-isomorphic edge-transitive lattices in the automorphism group of an (m, n)–biregular tree. Thus if Conjecture 1 holds, there are in contrast only finitely many edge-transitive lattices in Kac–Moody groups G as in Theorem 1.1. We note that, since the action of such G on X is not in general faithful, an amalgam Γ may embed as an edge-transitive lattice in G even though it is not faithful (but its kernel will be at most the finite group $Z(G)$).

We prove Theorem 1.3 in Section 5, making careful use of classical results on the subgroups of $SL_2(q)$, $PSL_2(q)$ and $PGL_2(q)$ of order coprime to p and their actions on the projective line $\mathbb{P}^1(q)$.

1.4. Covolumes of cocompact lattices.
If Conjecture 1 holds, there are also important consequences for the covolumes of lattices in G. We recall in Section 2.3 below that the Haar measure μ on G may be normalised so that the covolume $\mu(\Gamma\backslash G)$ of a cocompact lattice Γ with quotient graph of groups having two vertex groups A_1 and A_2 is equal to $|A_1|^{-1} + |A_2|^{-1}$. Using this normalisation, we obtain the following.

Theorem 1.4. *Let G be as in Theorem 1.1 above and assume that Conjecture 1 holds. Then for $q \geq 514$*

$$\min\{\mu(\Gamma\backslash G) \mid \Gamma \text{ a cocompact lattice in } G\} = \frac{2}{(q + 1)|Z(G)|\delta}$$

where $\delta \in \{1, 2, 4\}$ (depending upon the particular group G). Moreover, we construct a cocompact lattice which realises this minimum, and this lattice appears in Theorem 1.1.

In other words, for q large enough, among cocompact lattices in G which do not contain p–elements the minimal covolume is $2/(q + 1)|Z(G)|\delta$, and this minimum is achieved by one of the lattices in Theorem 1.1.

Theorem 1.4 generalises Theorem 2 of Lubotzky [L] and part of Proposition C of Lubotzky–Weigel [LW], which together determine the cocompact lattices of minimal covolume in the affine case $G = SL_2(\mathbb{F}_q((t^{-1})))$.

We prove Theorem 1.4 in Section 6. The proof is very delicate and has many cases, involving consideration of how various subgroups of $SL_2(q)$, $PSL_2(q)$ and $PGL_2(q)$ of order coprime to p can act on X.

1.5. **Covolumes of all lattices.** Finally, assuming the stronger Conjecture 2, we obtain a lower bound on the covolume of all lattices in G. As we recall in Section 2.4, the minimal Kac–Moody group Λ has twin buildings $X_+ \cong X_-$, and there are two completions $G_+ \cong G_-$ into which Λ embeds, with $G_\pm$ acting on $X_\pm$. As shown independently in Carbone–Garland [CG] and Rémy [R1], a negative maximal parabolic subgroup P_- of Λ is a non-cocompact lattice in $G = G_+$. We have:

Theorem 1.5. *Let G be as in Theorem 1.1 above and assume that Conjecture 2 above holds. Then*

$$\min\{\,\mu(\Gamma\backslash G) \mid \Gamma \ \text{a lattice in } G\,\} = \frac{2}{(q+1)(q-1)|T|}.$$

Moreover, this minimum is realised by the non-cocompact lattice P_-.

Theorem 1.5 generalises Theorem 1 of Lubotzky [L], which establishes that the lattice of minimal covolume in $G = SL_2(\mathbb{F}_q((t^{-1})))$ is the maximal parabolic subgroup $SL_2(\mathbb{F}_q[t])$, a non-cocompact lattice. More recently, Golsefidy [G] has shown that for many Chevalley groups $\mathcal{G}$, the lattice of minimal covolume in $\mathcal{G}(\mathbb{F}_q((t^{-1})))$ is the non-cocompact lattice $\mathcal{G}(\mathbb{F}_q[t])$. Theorem 1.5 and the results of [L] and [G] thus contrast with Siegel's original result [Si] that the lattice of minimal covolume in $SL_2(\mathbb{R})$ is cocompact.

The proof of Theorem 1.5 is similar to that of [L, Theorem 1], after replacing the term "unipotent element" with "p–element". We thus omit the proof.

2. Preliminaries

In Section 2.1 we recall basic definitions and establish notation for graphs and trees. Section 2.2 sketches the theory of graphs of groups, including covering theory. In Section 2.3 we recall some theory of tree lattices. Section 2.4 then presents background on the Kac–Moody groups G that we consider, and in Section 2.5 we recall some classical theorems concerning the finite subgroups of $SL_2(q)$, $PSL_2(q)$ and $PGL_2(q)$.

2.1. **Graphs and trees.** Let A be a connected graph with sets VA of vertices and EA of oriented edges. The initial and terminal vertices of $e \in EA$ are denoted by $\partial_0 e$ and $\partial_1 e$ respectively. The map $e \mapsto \overline{e}$ is orientation reversal, with $\overline{\overline{e}} = e$ and $\partial_{1-j}\overline{e} = \partial_j e$ for $j = 0, 1$. Given a vertex $a \in VA$, we denote by $E_A(a)$ the set of edges

$$E_A(a) := \{e \in EA \mid \partial_0 e = a\}$$

with initial vertex a.

Let A and B be graphs. A *morphism of graphs* is a function $\theta : A \to B$ taking vertices to vertices and edges to edges, such that for every edge $e \in EA$, $\theta(\overline{e}) = \overline{\theta(e)}$ and $\theta(\partial_j(e)) = \partial_j(\theta(e))$ for $j = 0, 1$.

Let X be a simplicial tree with vertex set VX and edge set EX. A group G is said to act *without inversions* on X if for all $g \in G$ and all edges $e \in EX$, if g preserves the edge e then g fixes e pointwise.

Proposition 2.1 (Serre, Proposition 19, Section I.4.3 [S2]). *Let A be a finite group acting without inversions on a simplicial tree X. Then there is a vertex of X which is fixed by A.*

Now equip X with a length metric d by declaring each edge of X to have length 1. Given an edge e of X and an integer $n \geq 0$, we define $\mathrm{Ball}(e, n)$ to be the union of the closed edges in X all of whose points are distance at most n from a point in the closed edge e. By abuse of notation, we then define the distance $d(e, e')$ between edges e and e' of X to be 0 if $e = e'$, and to be $n \geq 1$ if e' is in $\mathrm{Ball}(e, n)$ but not $\mathrm{Ball}(e, n-1)$.

A *geodesic line* in X is an isometry $l : (-\infty, \infty) \to X$, and a *geodesic ray* in X is an isometry $r : [t, \infty) \to X$, $t \in \mathbb{R}$, such that $r(t)$ is a vertex of X; in this case we say that the ray *begins at* the vertex $r(t)$. We will often identify a geodesic line or ray with its image in X. We say that two geodesic rays r and r' in X are *equivalent* if their intersection is infinite. The *boundary* ∂X of X, which is the same thing as the *set of ends* of X, is the collection of equivalence classes of geodesic rays. Given a geodesic ray r, we say that an end ε is *determined by* r if r belongs to the equivalence class ε.

If G is a group of isometries of X which acts without inversions, then an element $g \in G$ is either *elliptic*, meaning that g fixes at least one vertex of X, or *hyperbolic*, meaning that g does not fix any vertex and acts as a translation along its *axis*, a geodesic line in X (see, for example, [BH, Chapter II.6]). If g is hyperbolic then g generates an infinite cyclic subgroup of G.

Let g be a hyperbolic isometry of X, with axis l. Then g has exactly two fixed points in ∂X, which we denote by $l(-\infty)$ and $l(\infty)$. One of these fixed points is repelling and the other is attracting.

2.2. **Bass–Serre theory.** A *graph of groups* $\mathbb{A} = (A, \mathcal{A})$ over a connected graph A consists of an assignment of vertex groups $\mathcal{A}_a$ for each $a \in VA$ and edge groups $\mathcal{A}_e = \mathcal{A}_{\bar{e}}$ for each $e \in EA$, together with monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{\partial_0 e}$ for each $e \in EA$.

Any action of a group Γ on a tree X without inversions induces a graph of groups over the quotient graph $A = \Gamma \backslash X$. See for example [B] for the definitions of the *fundamental group* $\pi_1(\mathbb{A}, a_0)$ and the *universal cover* $X = \widetilde{(A, a_0)}$ of a graph of groups $\mathbb{A} = (A, \mathcal{A})$, with respect to a basepoint $a_0 \in VA$. The universal cover X is a tree, on which $\pi_1(\mathbb{A}, a_0)$ acts by isometries inducing a graph of groups isomorphic to $\mathbb{A}$.

In the special case that $\mathbb{A}$ is a graph of groups over an underlying graph A which is a single edge e, we say that $\mathbb{A}$ is an *edge of groups*. Suppose $\partial_0 e = a_1$ and $\partial_1 e = a_2$. Write A_0 for the edge group $\mathcal{A}_e$, and for $i = 1, 2$ let A_i be the vertex group $\mathcal{A}_{a_i}$. The fundamental group $\pi_1(\mathbb{A}, a_1)$ is then isomorphic to the free product with amalgamation $A_1 *_{A_0} A_2$, and the universal cover $X = \widetilde{(A, a_1)}$ is an (m, n)–biregular tree, where $m = [A_1 : A_0]$ and $n = [A_2 : A_0]$.

We now adapt definitions from covering theory for complexes of groups (see [BH, Chapter III.$\mathcal{C}$]) to graphs of groups, and recall a necessary result from covering theory. For the precise relationship between the category of graphs of groups and the category of complexes of groups over 1–dimensional spaces, see [Th, Proposition 2.1].

Definition 1 (Morphism of graphs of groups)**.** Let $\mathbb{A} = (A, \mathcal{A})$ and $\mathbb{B} = (B, \mathcal{B})$ be graphs of groups, with monomorphisms from edge groups to vertex groups respectively $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{\partial_0 e}$ for $e \in EA$ and $\beta_e : \mathcal{B}_f \to \mathcal{B}_{\partial_0 f}$ for $f \in EB$. Let $\theta : A \to B$ be a morphism of graphs. A *morphism of graphs of groups* $\Phi : \mathbb{A} \to \mathbb{B}$ over θ is given by:

(1) a homomorphism $\phi_x : \mathcal{A}_x \to \mathcal{B}_{\theta(x)}$ of groups, for every $x \in VA \cup EA$; and
(2) an element $\phi(e) \in \mathcal{B}_{\partial_0(\theta(e))}$ for each $e \in EA$ such that the following diagram commutes, where $a = \partial_0 e$:

$$
\begin{array}{ccc}
\mathcal{A}_e & \xrightarrow{\phi_e} & \mathcal{B}_{\theta(e)} \\
\downarrow{\alpha_e} & & \downarrow{\mathrm{ad}(\phi(e)) \circ \beta_{\theta(e)}} \\
\mathcal{A}_a & \xrightarrow{\phi_a} & \mathcal{B}_{\theta(a)}
\end{array}
$$

Definition 2 (Covering of graphs of groups)**.** With notation as in Definition 1 above, $\Phi : \mathbb{A} \to \mathbb{B}$ is a *covering of graphs of groups* if in addition:

(1) for each $x \in VA \cup EA$ the homomorphism $\phi_x : \mathcal{A}_x \to \mathcal{B}_{\theta(x)}$ is injective; and
(2) for each edge $f \in EB$ and each vertex $a \in VA$ with $\partial_0 f = b = \theta(a)$, the map

$$
\Phi_{a/f} : \coprod_{e \in E_A(a) \cap \theta^{-1}(f)} \mathcal{A}_a / \alpha_e(\mathcal{A}_e) \to \mathcal{B}_b / \beta_f(\mathcal{B}_f)
$$

induced by $g \mapsto \phi_a(g)\phi(e)$ is bijective.

The result from covering theory that we will need is:

Proposition 2.2 (Bass, Proposition 2.7 of [B]). *Let $\mathbb{A} = (A, \mathcal{A})$ and $\mathbb{B} = (B, \mathcal{B})$ be graphs of groups. Choose basepoints $a_0 \in A$ and $b_0 \in B$. If there is a covering of graphs of groups $\Phi : \mathbb{A} \to \mathbb{B}$ over $\theta : A \to B$ with $\theta(a_0) = b_0$, then $\pi_1(\mathbb{A}, a_0)$ injects into $\pi_1(\mathbb{B}, b_0)$.*

2.3. **Lattices in groups acting on trees.** Let G be a locally compact topological group with left-invariant Haar measure μ. Recall that a discrete subgroup $\Gamma \leq G$ is a *lattice* if $\Gamma \backslash G$ carries a finite G–invariant measure, and is *cocompact* if $\Gamma \backslash G$ is compact.

Now let X be a locally finite tree and let G be a closed, cocompact group of automorphisms of X, which acts without inversions and with compact open vertex stabilisers. Then a subgroup $\Gamma < G$ is discrete if and only if it acts on X with finite vertex stabilisers, and the Haar measure μ on G may be normalised so that the covolume of a discrete $\Gamma < G$ is given by $\mu(\Gamma \backslash G) = \sum |\Gamma_y|^{-1}$, where the sum is over a set of representatives y of the vertices of the quotient graph $Y = \Gamma \backslash X$ (Bass–Lubotzky [BL]). Hence a discrete subgroup $\Gamma < G$ is a lattice if and only if the series $\sum |\Gamma_y|^{-1}$ converges. Moreover, a discrete subgroup $\Gamma < G$ is a cocompact lattice in G if and only if Y is finite.

2.4. **Kac–Moody groups.** We first in Section 2.4.1 explain how one may associate, to a generalised Cartan matrix A and an arbitrary field, a Kac–Moody group Λ, the so-called minimal or incomplete Kac–Moody group. In Section 2.4.2 we specialise to rank 2 Kac–Moody groups over finite fields. Section 2.4.3 describes the completion G of Λ that appears in the statement of Theorem 1.1 above and Section 2.4.4 recalls some results from [CG] concerning the action of G on the set of ends of its tree X. Our treatment of Kac–Moody groups is brief and combinatorial, and partly follows Dymara–Januszkiewicz [DJ, Appendix TKM]. For a more sophisticated and general approach, using the notion of a "twin root datum", we refer the reader to, for example, Caprace–Rémy [CR].

2.4.1. *Incomplete Kac–Moody groups.* Let I be a finite set. A *generalised Cartan matrix* $A = (A_{ij})_{i,j \in I}$ is a matrix with integer entries, such that $A_{ii} = 2$, $A_{ij} \leq 0$ if $i \neq j$ and $A_{ij} = 0$ if and only if $A_{ji} = 0$. (If A is positive definite, then A is the Cartan matrix of some finite-dimensional semisimple Lie algebra.) A *Kac–Moody datum* is a 5-tuple $(I, \mathfrak{h}, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I}, A)$ where $\mathfrak{h}$ is a finitely generated free abelian group, $\alpha_i \in \mathfrak{h}$, $h_i \in \mathrm{Hom}(\mathfrak{h}, \mathbb{Z})$ and $A_{ij} = h_j(\alpha_i)$. The set $\Pi = \{\alpha_i\}_{i \in I}$ is called the set of *simple roots*.

Given a generalised Cartan matrix A as above, we define a *Coxeter matrix* $M = (m_{ij})_{i,j \in I}$ as follows: $m_{ii} = 1$, and if $i \neq j$ then $m_{ij} = 2, 3, 4, 6$ or ∞ as $A_{ij}A_{ji} = 0, 1, 2, 3$ or is ≥ 4. The associated *Weyl group* W is then the Coxeter group with presentation determined by M:

$$W = \langle \{w_i\}_{i \in I} \mid (w_i w_j)^{m_{ij}} \text{for } m_{ij} \neq \infty \rangle.$$

The Weyl group acts on $\mathfrak{h}$ via $w_i : \beta \mapsto \beta - h_i(\beta)\alpha_i$ for each $\beta \in \mathfrak{h}$ and each $i \in I$. In particular, $w_i(\alpha_i) = -\alpha_i$ for each simple root α_i. The set Φ of *real roots* is defined by $\Phi = W \cdot \Pi$. In general, the set of real roots is infinite.

We will, not by coincidence, use the same terminology and notation for simple roots and real roots which are defined in the following combinatorial fashion. Let ℓ be the word length on the Weyl group W, that is, $\ell(w)$ is the minimal length of a word in the letters $\{w_i\}_{i \in I}$ representing w. The *simple roots* $\Pi = \{\alpha_i\}_{i \in I}$ are then defined by

$$\alpha_i = \{w \in W \mid \ell(w_i w) > \ell(w)\}.$$

The set Φ of *real roots* is $\Phi = W \cdot \Pi = \{w\alpha_i \mid w \in W, \alpha_i \in \Pi\}$, and W acts naturally on Φ. The set Φ_+ of *positive roots* is $\Phi_+ = \{\alpha \in \Phi \mid 1_W \in \alpha\}$, and the set of *negative roots* Φ_- is $\Phi \backslash \Phi_+$. The complement of a root α in W, denoted $-\alpha$, is also a root. As before, $w_i(\alpha_i) = -\alpha_i$ for each simple root α_i.

We now define the split Kac–Moody group Λ associated to a Kac–Moody datum as above, over an arbitrary field k. The group Λ may be given by a presentation, which is essentially due to Tits (see [Ti]), and which appears in Carter [Car]. For simplicity, we state this presentation only for the simply-connected group Λ_u and then discuss the general case. Let $(I, \mathfrak{h}, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I}, A)$ be a Kac–Moody datum and k a field. The associated *simply-connected Kac–Moody group* Λ_u over k is generated by *root subgroups* $U_\alpha = U_\alpha(k) = \langle x_\alpha(t) \mid t \in k \rangle$, one for each real root $\alpha \in \Phi$. We write $x_i(u) = x_{\alpha_i}(u)$ and $x_{-i}(u) = x_{-\alpha_i}(u)$

for each $u \in k$ and $i \in I$, and put $\tilde{w}_i(u) = x_i(u)x_{-i}(u^{-1})x_i(u)$, $\tilde{w}_i = \tilde{w}_i(1)$ and $h_i(u) = \tilde{w}_i(u)\tilde{w}_i^{-1}$ for each $u \in k^*$ and $i \in I$. A set of defining relations for the simply-connected Kac-Moody group Λ_u is then:

(1) $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$, for all roots $\alpha \in \Phi$ and all $t, u \in k$.

(2) If $\alpha, \beta \in \Phi$ is a prenilpotent pair of roots, that is, there exist $w, w' \in W$ such that $w(\alpha) \in \Phi_+$, $w(\beta) \in \Phi_+$, $w'(\alpha) \in \Phi_-$ and $w'(\beta) \in \Phi_-$, then for all $t, u \in k$:

$$[x_\alpha(t), x_\beta(u)] = \prod_{\substack{i, j \in \mathbb{N} \\ i\alpha + j\beta \in \Phi}} x_{i\alpha+j\beta}(C_{ij\alpha\beta}t^i u^j)$$

where the integers $C_{ij\alpha\beta}$ are uniquely determined by $i, j, \alpha, \beta, \Phi$, and the ordering of the terms on the right-hand side.

(3) $h_i(t)h_i(u) = h_i(tu)$ for all $t, u \in k^*$ and all $i \in I$.

(4) $[h_i(t), h_j(u)] = 1$ for all $t, u \in k^*$ and $i, j \in I$.

(5) $h_j(u)x_i(t)h_j(u)^{-1} = x_i(u^{A_{ij}}t)$ for all $t \in k$, $u \in k^*$ and $i, j \in I$.

(6) $\tilde{w}_i h_j(u)\tilde{w}_i^{-1} = h_j(u)h_i(u^{-A_{ij}})$ for all $u \in k^*$ and $i, j \in I$.

(7) $\tilde{w}_i x_\alpha(u)\tilde{w}_i^{-1} = x_{w_i(\alpha)}(\epsilon u)$ where $\epsilon \in \{\pm 1\}$, for all $u \in k$.

By a result of P.-E. Caprace (cf. [C, 3.5(2)]), any two split Kac–Moody groups of the same type defined over the same field are isogenic. That is, if Λ is any split Kac–Moody group associated to the same generalised Cartan matrix A as Λ_u, and defined over the same field k, then there exists a surjective homomorphism $i : \Lambda_u \to \Lambda$ with $\ker(i) \leq Z(\Lambda_u)$. The Kac–Moody group Λ so constructed is sometimes called the *incomplete* or *minimal* Kac–Moody group (for completions of Λ, see Section 2.4.3 below).

A first example of an incomplete Kac–Moody group Λ over a finite field is $\Lambda = SL_n(\mathbb{F}_q[t, t^{-1}])$, which is over the field $\mathbb{F}_q$, and is not simply-connected.

We now discuss several important subgroups of the Kac–Moody group Λ. For any version (simply-connected or not), the *unipotent* subgroup of Λ is

$$U = U_+ = \langle U_\alpha \mid \alpha \in \Phi_+ \rangle.$$

For Λ_u simply-connected, the *torus*

$$T = \langle h_i(u) \mid i \in I, u \in k^* \rangle$$

is isomorphic to the direct product of $|I|$ copies of k^*. In general, the torus T of Λ is a homomorphic image of the direct product of $|I|$ copies of k^*. For all Λ, we define N to be the subgroup of Λ generated by the torus T and by the elements $\{\tilde{w}_i\}_{i \in I}$ (where, in general as in the simply-connected case, $\tilde{w}_i = x_{\alpha_i}(1)x_{-\alpha_i}(1)x_{\alpha_i}(1)$ for all $i \in I$). The *standard Borel subgroup* $B = B_+$ of Λ is defined by

$$B = \langle T, U_+ \rangle = \langle T, U \rangle.$$

The group B has decomposition $B = T \ltimes U_+ = T \ltimes U$ (see [R2]).

The subgroups B and N of Λ form a BN–pair (also known as a Tits system) with Weyl group W, and hence Λ has a Bruhat–Tits building X. (In fact, the group Λ has isomorphic twin buildings $X_+ \cong X_-$, associated to twin BN–pairs (B_+, N) and (B_-, N) respectively.) The chambers of X correspond to the cosets of B in Λ, hence Λ acts naturally on X with quotient a single chamber. For each apartment Σ of X, the chambers in Σ are in bijection with the elements of the Weyl group W. Each root $\alpha \subset W$ corresponds to a "half-apartment". The construction of the building X for Λ of rank 2 is explained further in Section 2.4.2 below.

2.4.2. *Rank* 2. We now specialise to the cases considered in Theorem 1.1 above. Let A be a generalised Cartan matrix of the form $A = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, with $m \geq 2$. If $m = 2$, then A is *affine*, meaning that A is positive semidefinite but not positive definite. For all such A (affine and non-affine) the associated Weyl group W is

$$W = \langle w_1, w_2 \mid w_1^2, w_2^2 \rangle.$$

That is, W is the infinite dihedral group. Let ℓ be the word length on W. The simple roots $\Pi = \{\alpha_1, \alpha_2\}$ are then given by, for $i = 1, 2$,

$$\alpha_i = \{w \in W \mid \ell(w_i w) > \ell(w)\} = \{1, w_{3-i}, w_{3-i} w_i, w_{3-i} w_i w_{3-i}, \ldots\}.$$

The set Φ of real roots is $\Phi = \{w \alpha_i \mid w \in W, i = 1, 2\}$.

Now let Λ be an incomplete Kac–Moody group with generalised Cartan matrix A, defined over a finite field $\mathbb{F}_q$, where $q = p^a$ with p prime. As Λ is a group with BN–pair, as described above, for $i = 1, 2$, the *parabolic subgroup* P_i of Λ is defined by

$$P_i = B \sqcup B \tilde{w}_i B.$$

Since $J_i = \{\alpha_i\}$ is a root system of type A_1, and thus is of finite type, now [R2, 6.2] applies. Hence, the group P_i has a *Levi decomposition* $P_i = L_i \ltimes U_i$. Here $U_i = U \cap U^{w_i}$ is called a *unipotent radical* of P_i, and the group L_i is called a *Levi complement* of P_i. The Levi complement factors as $L_i = T M_i$, where T is the torus of Λ, and $M_i = \langle U_{\alpha_i}, U_{-\alpha_i} \rangle$, that is, $A_1(q) \cong M_i \lhd L_i$.

To describe the building X for Λ, we first describe its apartments. Let Σ be the *Coxeter complex* for the Weyl group W (the infinite dihedral group). Then Σ is the one-dimensional simplicial complex homeomorphic to the line. The set of real roots Φ, described above, is in bijection with the set of half-lines in Σ. Thus we may regard each real root as a geodesic ray in X. So each real root determines an end of X.

In particular, the set Φ^+ of positive real roots is the disjoint union of the sets

$$\Phi_+^1 := \{\alpha_1, w_1 \alpha_2, w_1 w_2 \alpha_1, w_1 w_2 w_1 \alpha_2, \ldots, (w_1 w_2)^n \alpha_1, (w_1 w_2)^n w_1 \alpha_2, \ldots\}$$

and

$$\Phi_+^2 := \{\alpha_2, w_2 \alpha_1, w_2 w_1 \alpha_2, w_2 w_1 w_2 \alpha_1, \ldots, (w_2 w_1)^n \alpha_2, (w_2 w_1)^n w_2 \alpha_1, \ldots\}$$

and the roots in Φ^+ are in bijection with the set of half-lines in Σ which contain the base chamber B. See Figure 1 below.
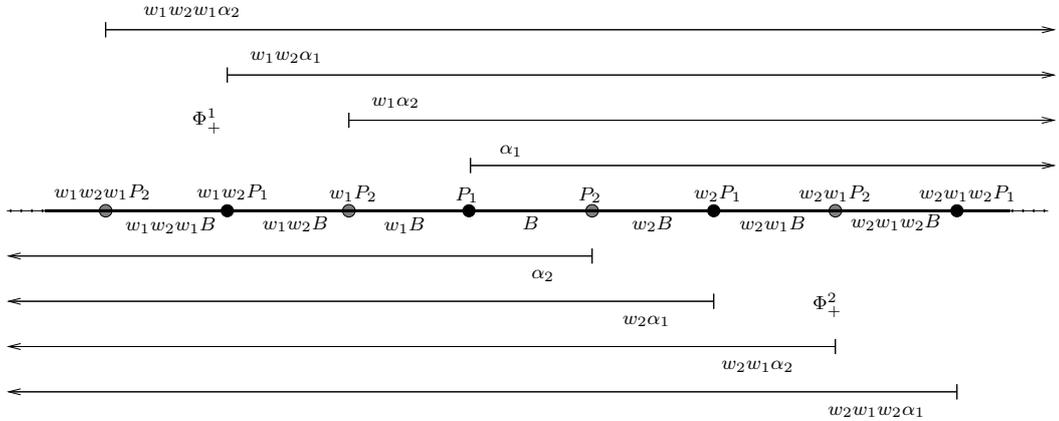

FIGURE 1. The sets of positive real roots Φ_+^1 and Φ_+^2, with each such root identified with a half-apartment of the standard apartment Σ containing B.

The apartments of the building X are copies of the Coxeter complex Σ for W and X is a $(q+1)$–regular simplicial tree, with chambers the edges of this tree.

2.4.3. *Completions of Λ.* We are finally ready to describe the main object of our study: the locally compact topological Kac–Moody groups. In order to do this we will have to define a topological completion of the incomplete Kac–Moody group Λ. There are several completions appearing in the literature. For example, Carbone–Garland [CG] defined a representation-theoretic completion of Λ using the "weight topology". A different approach by Rémy and Ronan, appearing for instance in [RR], is to use the action of Λ on the building X, as follows. The kernel K of the Λ–action on X is the centre $Z(\Lambda)$, which is a finite group when Λ

is over a finite field (Rémy [R2]). The closure of Λ/K in the automorphism group of X is then a completion of Λ. For example, when $\Lambda = SL_n(\mathbb{F}_q[t, t^{-1}])$, the centre $Z(\Lambda)$ is the finite group $\mu_n(\mathbb{F}_q)$ of nth roots of unity in $\mathbb{F}_q$, and the completion in this topology is $SL_n(\mathbb{F}_q((t^{-1})))/\mu_n(\mathbb{F}_q) \cong PSL_n(\mathbb{F}_q((t^{-1})))$. To avoid dealing with representation-theoretic constructions or with quotients, we are going to follow instead the completion in the building topology, defined by Caprace and Rémy in [CR].

So, let Λ be an incomplete Kac–Moody group over a finite field, as defined in Section 2.4.1 above. We now describe the completion G of Λ which appears in Theorem 1.1 (for Λ with generalised Cartan matrix A as in Section 2.4.2 above). Let $c_+ = B_+$ be the chamber of the Bruhat–Tits building X for Λ which is fixed by $B = B_+$. For each $n \in \mathbb{N}$, let

$$U_{+,n} = \{ g \in U_+ \mid g \cdot c = c \text{ for each chamber } c \text{ such that } d(c, c_+) \leq n \ \}.$$

That is, $U_{+,n}$ is the kernel of the action of $U_+ = U$ on $\mathrm{Ball}(c_+, n)$. Now define a function $\mathrm{dist}_+ : \Lambda \times \Lambda \to \mathbb{R}_+$ by $\mathrm{dist}_+(g, h) = 2$ if $h^{-1}g \notin U_+$, and $\mathrm{dist}_+(g, h) = 2^{-n}$ if $g^{-1}h \in U_+$ and $n = \max\{k \in \mathbb{N} \mid g^{-1}h \in U_{+,k}\}$. Then dist_+ is a left-invariant metric on Λ (see [CR]). Let G be the completion of Λ with respect to this metric. The group G is called the *completion of Λ in the building topology*, and we will refer to G as a *topological Kac–Moody group*. For example, when $\Lambda = SL_n(\mathbb{F}_q[t, t^{-1}])$, the topological Kac–Moody group G is $G = SL_n(\mathbb{F}_q((t^{-1})))$.

Some properties of topological Kac–Moody groups that we will need are gathered in Proposition 2.3 below. We state these results only for G as in Theorem 1.1 above, although they hold more generally.

Proposition 2.3. *Let G be a topological Kac–Moody group as in Theorem 1.1 above, with G being the completion in the building topology of an incomplete Kac–Moody group Λ.*

   (1) *G is a locally compact, totally disconnected topological group.*
   (2) *Let $\hat{B}$, $\hat{U}$, $\hat{P}_i$ and $\hat{U}_i$ be the closures in G of the subgroups $B = B_+$, $U = U_+$, P_i and U_i respectively of Λ. Then $\hat{B} \cong T \ltimes \hat{U}$ and $\hat{P}_i \cong L_i \ltimes \hat{U}_i$.*
   (3) *$(\hat{B}, N)$ is a BN–pair of G. The corresponding building is canonically isomorphic to X, and so by abuse of notation we will denote it by X as well. The kernel of the action of G on X is the centre $Z(G)$, and $Z(G) = Z(\Lambda)$.*

Items (1) and (3) are established by Caprace–Rémy in [CR], and item (2) in [CR] and [C].

Following the terminology used by Rémy and Ronan [RR, p. 196] for topological Kac–Moody groups, we will refer to $\hat{B}$ as the *(standard) Iwahori subgroup* of G, and to $\hat{P}_1$ and $\hat{P}_2$ as the *(maximal or standard) parahoric subgroups* of G. To simplify notation, when the context is clear we will write B, P_1 and P_2 for the Iwahori and maximal parahoric subgroups of the topological Kac–Moody group G, rather than respectively $\hat{B}$, $\hat{P}_1$ and $\hat{P}_2$.

2.4.4. *Action of G on ends.* In this section we recall definitions and results from Carbone–Garland [CG] concerning the action of G on the set of ends of X. These will be used in Section 4 below where we discuss our conjectures about p–elements in G.

For $i = 1, 2$, let ε_i be the unique end of X determined by the simple root α_i (see Figure 1 above). We partition ∂X into two sets $\mathcal{E}_1$ and $\mathcal{E}_2$, so that $\varepsilon_i \in \mathcal{E}_i$ for $i = 1, 2$, as follows. For $i = 1, 2$, $\mathcal{E}_i$ consists of those ends represented by a geodesic ray which starts at the vertex P_{3-i} and does not contain the edge B.

Lemma 2.4. *For any end $\varepsilon \in \mathcal{E}_1$, there is a $u_1 \in \hat{U}_+$ such that $u_1 \cdot \varepsilon = \varepsilon_1$. For any end $\varepsilon' \in \mathcal{E}_2$, there is a $u_2 \in \hat{U}_+$ such that $u_2 \cdot \varepsilon' = \varepsilon_2$. Moreover, u_2 may be chosen to fix ε_1.*

Proof. This follows from [CG, Lemma 14.1]. In [CG], a different completion of minimal Kac–Moody groups was used, but the same proof goes through in the completion that we are using. □

The following corollary of Lemma 2.4 is the same statement as [CG, Corollary 14.1].

Corollary 2.5. *The group G acts 2–transitively on the set of ends of X.*

Now let V_1 be the elementary abelian p–subgroup of U_+ defined by

$$V_1 := \langle U_\alpha \mid \alpha \in \Phi_+^1 \rangle$$

and let $-V_2$ be the elementary abelian p–subgroup of U_- defined by

$$-V_2 := \langle U_\alpha \mid -\alpha \in \Phi_+^2 \rangle.$$

Then the groups V_1 and $-V_2$ both fix the end ε_1. Notice that $[V_1, -V_2] = 1$ and $V_1 \cap -V_2 = 1$, and so $\langle V_1, -V_2 \rangle = V_1 \times -V_2$. Let $\mathcal{U}$ be the closed abelian group

$$\mathcal{U} := \widehat{V_1 \times -V_2}$$

and let

$$\mathcal{M} := \bigcap_{w \in W} w\hat{B}w^{-1}.$$

Note that $\mathcal{M}$ is precisely the subgroup of G which fixes the standard apartment Σ pointwise. In particular, the torus T is a (finite) subgroup of $\mathcal{M}$, of exponent $(q - 1)$.

Take $g_\tau \in G$ such that g_τ induces the element $\tau := w_1 w_2 \in W$. The element g_τ is hyperbolic, with axis the standard apartment Σ, translation length two edges, and with attracting fixed point ε_2 and repelling fixed point ε_1. (In Figure 1 above, g_τ moves the standard apartment two edges to the left.) Let R be the infinite cyclic subgroup of G generated by g_τ.

The following decomposition of the stabiliser G_{ε_1} is from [CG], where the group $\mathcal{M}$ is denoted by $\mathcal{B}_\mathcal{I}$ and R is denoted by T. As with Lemma 2.4 above, the same proof applies even though we are using a different completion to [CG].

Theorem 2.6 (Theorem 14.1, [CG]). *The group G_{ε_1} can be expressed as $G_{\varepsilon_1} = \mathcal{U}\mathcal{M}R$.*

2.5. **Finite groups.** In our constructions of cocompact lattices in Kac–Moody groups and the proofs of Theorems 1.3 and 1.4, we will need to look carefully at the finite subgroups of G. The following celebrated result of L.E. Dickson and its corollary will be especially useful for us.

Theorem 2.7 (Dickson, 6.5.1 of [GLS3]). *Let $K = PSL_2(q)$, where $q = p^a \geq 5$ and p is a prime. Set $d = (2, q - 1)$. Then K has subgroups of the following isomorphism types in the indicated cases, and every subgroup of K is isomorphic to a subgroup of one of the following groups:*

   (1) *Borel subgroups of K, which are Frobenius groups of order $q(q - 1)/d$;*
   (2) *Dihedral groups of orders $2(q - 1)/d$ and $2(q + 1)/d$;*
   (3) *The groups $PGL_2(p^b)$ (if $2b \mid a$) and $PSL_2(p^b)$ (if b is a proper divisor of a);*
   (4) *The alternating group A_5, if 5 divides $|K|$;*
   (5) *The symmetric group S_4, if 8 divides $|K|$; and*
   (6) *The alternating group A_4.*

Corollary 2.8. *Let $K = SL_2(q)$, where $q = p^a$ with p prime, and suppose A is a proper subgroup of K.*

If $p = 2$ and $q + 1$ divides $|A|$, then either $A \cong C_{q+1}$, a cyclic group of order $q + 1$, or $A \cong D_{2(q+1)}$, a dihedral group of order $2(q + 1)$.

If p is odd and the image of A in $K/Z(K) \cong PSL_2(q)$ has order divisible by $q+1$, then $Z(K) = \langle -I \rangle \leq A$. Moreover, either A is a subgroup of K of order $2(q+1)$ such that $A/Z(K) \cong D_{q+1}$, a dihedral group of order $q + 1$, or one of the following conditions hold:

   (1) *$q = 5$, $A \cong SL_2(3)$;*
   (2) *$q = 7$, $A \cong 2S_4$;*
   (3) *$q = 9$, $A \cong SL_2(5)$;*
   (4) *$q = 11$, $A \cong SL_2(3)$ or $A \cong SL_2(5)$;*
   (5) *$q = 19$, $A \cong SL_2(5)$;*
   (6) *$q = 23$, $A \cong 2S_4$;*
   (7) *$q = 29$, $A \cong SL_2(5)$; or*
   (8) *$q = 59$, $A \cong SL_2(5)$.*

Proof. Suppose that $p = 2$. Then $d = 1$ and $SL_2(q) = PSL_2(q)$. Assume first that $q \geq 5$. Then if $q + 1$ divides $|A|$, Theorem 2.7 above asserts that both C_{q+1} and $D_{2(q+1)}$ are the obvious candidates for the role of A. If not, A would be one of the following groups: A_4, S_4 or A_5. Then $q + 1$ would divide 12, 24 or 60.

Since q is a power of 2 and $q \geq 5$, this is not possible, proving the result. Otherwise $q \in \{2, 4\}$, and the result follows immediately from the structures of $K = SL_2(2) \cong S_3$ and $K = SL_2(4) \cong A_5$.

Suppose now that p is odd. This time $d = 2$ and the image of A in $PSL_2(q)$ is a group of order divisible by $q + 1$. Since $|A|$ is even while K contains a unique involution $-I$, $\langle -I \rangle = Z(K) \leq A$. If $q \geq 5$, using the same argument as above, we obtain the desired conclusion. Otherwise $q = 3$ and $K = SL_2(3) \cong Q_8 C_3$, and the result follows immediately. $\qquad\square$

We will also need another result of L.E. Dickson about the subgroups of $PGL_2(q)$, stated in the following form in [COTR].

Theorem 2.9 (Theorem 2 of [COTR]). *Let $q \equiv \epsilon \mod 4$ where $\epsilon = \pm 1$. The subgroups of $PGL_2(q)$ are as follows.*

(1) *Two conjugacy classes of cyclic subgroups C_2.*
(2) *One conjugacy class of cyclic subgroups of C_d where $d \mid (q \pm \epsilon)$ and $d > 2$.*
(3) *Two conjugacy classes of dihedral subgroups D_4 (only one class contained in the subgroup $PSL_2(q)$).*
(4) *Two conjugacy classes of dihedral subgroups D_{2d}, where $d \mid \frac{q \pm \epsilon}{2}$ and $d > 2$ (only one class contained in the subgroup $PSL_2(q)$).*
(5) *One conjugacy class of dihedral subgroups D_{2d}, where $(q \pm \epsilon)/d$ is an odd integer and $d > 2$.*
(6) *Subgroups A_4, S_4 and A_5 when $q \equiv \pm 1 \mod 10$. There is only one conjugacy class of any of these types of subgroups and all lie in the subgroup $PSL_2(q)$, except for S_4 when $q \equiv \pm 3 \mod 8$.*
(7) *Subgroups $PSL(2, p^m)$, $PGL(2, p^m)$, the elementary Abelian group of order p^m and a semidirect product of the elementary Abelian group of order p^m and the cyclic group of order d, where $m \leq \log_p q$, $d \mid (q - 1)$ and $d \mid (p^m - 1)$.*

3. CONSTRUCTION OF COCOMPACT LATTICES

In this section we prove Theorems 1.1 and 1.2, stated in the introduction. We first establish our embedding criterion for cocompact lattices, Proposition 3.1, in Section 3.1 below. We then apply this criterion in Section 3.2 to construct the edge-transitive lattices Γ in Theorems 1.1 and 1.2, and in Section 3.3 to construct the lattices Γ' in Theorem 1.1.

3.1. **Embedding criterion.** Our embedding criterion applies not only to G as in Theorem 1.1, but to more general locally compact groups acting on trees, as follows.

Let q_1 and q_2 be positive integers and let X be the $(q_1 + 1, q_2 + 1)$–biregular tree. Let G be any closed locally compact group of automorphisms of X which acts on X without inversions, with compact open vertex stabilisers G_x for $x \in VX$ and with fundamental domain an edge $[x_1, x_2]$, where for $i = 1, 2$ the vertex x_i has valence $q_i + 1$. Denote by P_i the stabiliser G_{x_i} for $i = 1, 2$, and let $B = P_1 \cap P_2$. For notational convenience, we denote by C the subgraph of X with vertex set $\{x_1, x_2\}$ and edge set $\{f, \overline{f}\}$, such that $\partial_0(f) = \partial_1(\overline{f}) = x_1$ and $\partial_1(f) = \partial_0(\overline{f}) = x_2$. Then G is the fundamental group of an edge of groups $\mathbb{G}$ over C, as sketched in the introduction.

For some integer $n \geq 1$ dividing both $q_1 + 1$ and $q_2 + 1$, let $A = A_n$ be the graph with two vertices a_1 and a_2 and edge set $\{e_1, \ldots, e_n, \overline{e_1}, \ldots, \overline{e_n}\}$, so that $\partial_0(e_j) = \partial_1(\overline{e_j}) = a_1$ and $\partial_1(e_j) = \partial_0(\overline{e_j}) = a_2$ for $1 \leq j \leq n$. The case $n = 2$ is sketched in the statement of Theorem 1.1. We now state and prove a sufficient criterion for the fundamental group of a graph of groups over A to embed in G as a cocompact lattice.

Proposition 3.1. *Suppose that there are finite groups $A_1 \leq P_1$ and $A_2 \leq P_2$ such that:*

(1) *for $i = 1, 2$, the group A_i has n orbits of equal size on $E_X(x_i)$;*
(2) *there are:*
   - *representatives $f_1 = f$, f_2, $\ldots$, f_n of the orbits of A_1 on $E_X(x_1)$ and $\hat{f}_1 = \overline{f}$, $\hat{f}_2$, $\ldots$, $\hat{f}_n$ of the orbits of A_2 on $E_X(x_2)$; and*
   - *elements $g_1 = 1$, g_2, $\ldots$, $g_n \in P_1$ and $\hat{g}_1 = 1$, $\hat{g}_2$, $\ldots$, $\hat{g}_n \in P_2$;*
   such that for $1 \leq j \leq n$:
   (a) *$g_j \cdot f_1 = f_j$ and $\hat{g}_j \cdot \hat{f}_1 = \hat{f}_j$;*

(b) $A_1 \cap B^{g_j} = A_1 \cap A_2 = A_2 \cap B^{\hat{g}_j}$; and
(c) $(A_1 \cap A_2)^{g_j} = (A_1 \cap A_2)^{\hat{g}_j}$.

Let $\mathbb{A}$ be the graph of groups over A with:

- *vertex groups $\mathcal{A}_{a_i} = A_i$ for $i = 1, 2$;*
- *edge groups $\mathcal{A}_{e_j} = \mathcal{A}_{\overline{e_j}} = A_1 \cap A_2$ for $1 \leq j \leq n$;*
- *each monomorphism α_{e_j} from an edge group $A_1 \cap A_2$ into A_1 inclusion composed with $\mathrm{ad}(g_j \hat{g}_j^{-1})$; and*
- *the monomorphisms $\alpha_{\overline{e_j}}$ from edge groups $A_1 \cap A_2$ into A_2 inclusions.*

Then the fundamental group of the graph of groups $\mathbb{A}$ is a cocompact lattice in G, with quotient A.

Note that in the special case $n = 1$, where for $i = 1, 2$ the group A_i acts transitively on $E_X(x_i)$, condition (2) reduces to the requirement that $\mathrm{Stab}_{A_i}(x_{3-i}) = A_1 \cap A_2$ for $i = 1, 2$.

Proof. We construct a covering of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$. Since A is a finite graph and the vertex groups A_1 and A_2 are finite, it then follows from our discussion of lattices in Section 2.3 and Proposition 2.2 above that the fundamental group of $\mathbb{A}$ is a cocompact lattice in G with quotient the graph A.

Let $\theta : A \to C$ be the graph morphism given by $\theta(a_i) = x_i$ for $i = 1, 2$, and $\theta(e_j) = f$ and $\theta(\overline{e_j}) = \overline{f}$ for $1 \leq j \leq n$. We construct a morphism of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$ over θ as follows. For $i = 1, 2$ let $\phi_{a_i} : \mathcal{A}_{a_i} \to P_i$ be the natural inclusion $A_i \hookrightarrow P_i$. For $1 \leq j \leq n$ let $\phi_{e_j} : \mathcal{A}_{e_j} \to B$ be the composition of the natural inclusion $A_1 \cap A_2 \hookrightarrow B^{\hat{g}_j}$ with the map $\mathrm{ad}(\hat{g}_j^{-1}) : B^{\hat{g}_j} \to B$. Define $\phi(e_j) = g_j$ and $\phi(\overline{e_j}) = \hat{g}_j$. Then it may be checked that Φ is indeed a morphism of graphs of groups.

To show that Φ is a covering, we first show that the map

$$\Phi_{a_1/f} : \coprod_{j=1}^{n} \mathcal{A}_{a_1}/\alpha_{e_j}(\mathcal{A}_{e_j}) \to P_1/B$$

induced by $g \mapsto \phi_{a_1}(g)\phi(e_j) = g g_j$ for g representing a coset of $\alpha_{e_j}(\mathcal{A}_{e_j}) = (A_1 \cap A_2)^{g_j \hat{g}_j^{-1}} = A_1 \cap A_2$ in $\mathcal{A}_{a_1} = A_1$ is a bijection. For this, we note that since the edges $f_j = g_j \cdot f_1 = g_j \cdot f$ represent pairwise distinct A_1–orbits on $E_X(x_1)$, for all $g, h \in A_1$ and all $1 \leq j \neq j' \leq n$ the cosets $g g_j B$ and $h g_{j'} B$ are pairwise distinct. The conclusion that $\Phi_{a_1/f}$ is a bijection then follows from the assumption that $A_1 \cap B^{g_j} = A_1 \cap A_2$.

The proof that the map

$$\Phi_{a_2/\overline{f}} : \coprod_{j=1}^{n} \mathcal{A}_{a_2}/\alpha_{\overline{e_j}}(\mathcal{A}_{\overline{e_j}}) \to P_2/B$$

is a bijection is similar. We conclude that $\Phi : \mathbb{A} \to \mathbb{G}$ is a covering of graphs of groups, as desired. $\square$

3.2. **Construction of edge-transitive lattices.** Let G be as in Theorem 1.1. We now apply Proposition 3.1 in the case $n = 1$ to construct the edge-transitive amalgams $\Gamma = A_1 *_{A_0} A_2$ described in Theorems 1.1 and 1.2. Throughout this section, P_1 and P_2 are the maximal parahoric subgroups of G, and for $i = 1, 2$ the group P_i is the stabiliser in G of the vertex x_i of X. Recall that each P_i has Levi decomposition $P_i = L_i \ltimes U_i$, and $L_i = TM_i$ where T is a fixed maximal split torus of G and $A_1(q) \cong M_i \triangleleft L_i$.

We now prove parts (1), (2) and (3) of Theorem 1.1 in Sections 3.2.1, 3.2.2 and 3.2.3 respectively, and then prove Theorem 1.2 in Section 3.2.4. In all cases below, H_i is a non-split torus of M_i such that $|N_T(H_i)|$ is as large as possible.

3.2.1. *Part (1) of Theorem 1.1.* We begin with the case $p = 2$. Since q is even, $SL_2(q) \cong PSL_2(q)$ and so $M_i \cong SL_2(q)$. Thus $H_i \cong C_{q+1}$.

Lemma 3.2. *For $i = 1, 2$, the group H_i acts simply transitively on $E_X(x_i)$.*

Proof. The action of M_i on $E_X(x_i)$ can be identified with the natural action of $SL_2(q)$ on the projective line $\mathbb{P}^1(q)$, in which any one-point stabiliser has trivial intersection with H_i. $\square$

Now let A_0 be any subgroup of $Z(G)$ and for $i = 1, 2$ let $A_i = A_0 \times H_i$. Then by Lemma 3.2, condition (1) of Proposition 3.1 is satisfied with $n = 1$, and $\operatorname{Stab}_{A_1}(x_2) = A_1 \cap A_2 = A_0 = \operatorname{Stab}_{A_2}(x_1)$, so condition (2) of Proposition 3.1 also holds. Proposition 3.1 then implies that $\Gamma = A_1 *_{A_0} A_2$ is an edge-transitive lattice in G. This completes the proof of Theorem 1.1(1).

3.2.2. *Part* (2) *of Theorem 1.1.* If p is odd, the two possibilities for $L_i/Z(L_i)$ are $L_i/Z(L_i) \cong PSL_2(q)$ or $L_i/Z(L_i) \cong PGL_2(q)$. We first consider the case $L_i/Z(L_i) \cong PSL_2(q)$. Assume also that $q \equiv 3 \pmod 4$. Then since either $M_i \cong SL_2(q)$ or $M_i \cong PSL_2(q)$, H_i is cyclic of order $|Z(M_i)|(q+1)/2$ and $N_{M_i}(H_i)$ is a group of order $|Z(M_i)|(q+1)$. Moreover, $N_{M_i}(H_i) \cap T = Z(M_i)$.

Lemma 3.3. *For* $i = 1, 2$, *the group* $N_{M_i}(H_i)$ *acts transitively on* $E_X(x_i)$.

Proof. Again, the action of M_i on $E_X(x_i)$ can be identified with the natural action of $SL_2(q)$ on the projective line. In particular, a stabiliser of an edge $\operatorname{Stab}_{M_i}(e)$ in M_i has order $\frac{q(q-1)}{2}|Z(M_i)|$. Hence $N_{M_i}(H_i) \cap \operatorname{Stab}_{M_i}(e) \leq N_{M_i}(H_i) \cap T = Z(M_i)$. Thus the length of the orbit of $N_{M_i}(H_i)$ on $E_X(x_i)$ is $|Z(M_i)|(q+1)/|Z(M_i)| = q+1$ which proves the result. $\square$

For $i = 1, 2$, consider $Z(M_i)$. Then $Z(M_i) \leq T_0$ where $T_0 \in \mathcal{S}yl_2(T)$. Since $q \equiv 3 \pmod 4$ while $L_i/Z(L_i) \cong PSL_2(q)$, $|T_0|$ divides 4 and $\exp(T) \leq 2$. In fact, $[T_0, M_i] = 1$ for $i = 1, 2$. Thus $T_0 \leq C_G(\langle M_1, M_2 \rangle) = C_G(\Lambda) \leq Z(G)$, and so $Z(M_i) \leq Z(G)$ for $i = 1, 2$. Now take A_0 to be any subgroup of $Z(G)$ with $Z(M_i) \leq A_0$ for $i = 1, 2$. Let $A_i := A_0 N_{M_i}(H_i)$. Then $A_0 \cap N_{M_i}(H_i) = Z(M_i)$. Lemma 3.3 implies that A_i is transitive on $E_X(x_i)$. Moreover, $\operatorname{Stab}_{A_i}(x_{3-i}) = A_0$. Now an easy application of Proposition 3.1 finishes the proof of Theorem 1.1(2).

3.2.3. *Part* (3) *of Theorem 1.1.* To complete the proof of Theorem 1.1, we consider the case $L_i/Z(L_i) \cong PGL_2(q)$ (with p odd). Define
$$N_i := N_{L_i}(H_i) \quad \text{and} \quad C_i := C_{L_i}(H_i).$$
Then $H_i \leq C_i$, C_i is a cyclic group of order dividing $(q^2 - 1)$ (more precisely, $|C_i| = (q+1)|Z(L_i)|$), $Z(L_i) \leq C_i$ and $|N_i : C_i| = 2$ with $N_i = C_i N_T(H_i)$. For any group C such that $Z(L_i) \leq C \leq L_i$, denote by $\overline{C}$ the image of C in $\overline{L}_i := L_i/Z(L_i)$. Then $\overline{C}_i \cong C_{q+1}$.

Assume first that $q \equiv 3 \pmod 4$, so that $\frac{q-1}{2}$ is odd. Then $C_i = C_i' \times Z_i'$ where $(|C_i'|, |Z_i'|) = 1$, $Z_i' \leq Z(L_i)$ is the "odd part" of $Z(L_i)$ (i.e., $|Z_i'|$ divides $\frac{q-1}{2}$ and $Z(L_i) = Z(M_i)Z_i'$), and $\overline{C}_i' = \overline{C}_i$. Moreover, $|C_i' : H_i| = 2$. In fact, if $Q_i' \in \mathcal{S}yl_2(C_i)$, $C_i' = H_i Q_i' = H_i' \times Q_i'$ where $H_i' = O(H_i) \cong C_{\frac{(q+1)|Z(M_i)|}{|Q_i'|}}$ and $Q_i' \cap Z(L_i) = Z(M_i)$.

Lemma 3.4. *For* $i = 1, 2$ *the group* $C_i' = H_i Q_i'$ *acts transitively on* $E_X(x_i)$.

Proof. This time the action of L_i on the set $E_X(x_i)$ is the natural action of $PGL_2(q)$ on the projective line. As we noticed earlier the image $\overline{C}_i'$ of C_i' in $\overline{L}_i$ is isomorphic to C_{q+1} while the image $\overline{H}_i$ of H_i is isomorphic to $C_{\frac{q+1}{2}}$. Moreover, there exists $\overline{c}_i' \in \overline{C}_i'$ such that $\overline{C}_i' = \langle \overline{c}_i' \rangle$ and $1 \neq \overline{c}_i'^2 \in \overline{H}_i$. As $H_i \cap Z(L_i) = Z(M_i)$, $\operatorname{Stab}_{\overline{H}_i}(e) = 1$ for any $e \in E_X(x_i)$, and so $\operatorname{Stab}_{\overline{C}_i'}(e) = 1$ for $e \in E_X(x_i)$. Hence the length of the orbit of $\overline{C}_i'$ on $E_X(x_i)$ is $q+1$. The result follows immediately. $\square$

Now let T_0 be the Sylow 2–subgroup of T. Then as $q \equiv 3 \pmod 4$ while $|T|$ divides $(q-1)^2$, we observe that $|T_0| = 2|Z(M_i)|$, $\exp(T_0) = 2$, $T_0 \cap Q_i' = Z(M_i)$ and $T_0 \leq N_{L_i}(H_i)$. Take Z_0 to be any subgroup of $Z(G)$ and $A_0 = T_0 Z_0$. Let $A_1 = C_1' A_0$ and $A_2 = C_2' A_0$. Lemma 3.4 then implies that condition (1) of Proposition 3.1 holds, with $n = 1$. We also have $A_1 \cap A_2 = A_0$, and so condition (2) of Proposition 3.1 is satisfied. We have proved Theorem 1.1(3(b)i).

To prove Theorem 1.1(3(b)ii), assume that $Z(M_i) \leq Z(G)$ for $i = 1, 2$. Take A_0 to be a subgroup of $Z(G)$ with $Z(M_i) \leq A_0$ for $i = 1, 2$. Then if $A_1 = C_1' A_0$ and $A_2 = C_2' A_0$, we similarly obtain an edge-transitive lattice $\Gamma = A_1 *_{A_0} A_2$. Notice that $C_i' \cap A_0 = Z(M_i)$. We remark also that $Z(M_i) \leq Z(G)$ is crucial in this case, for it allows for the condition $A_1 \cap A_2 = \operatorname{Stab}_{A_i}(x_{3-i})$ to be satisfied. This completes the proof of part (3b) of Theorem 1.1.

Suppose now that $q \equiv 1 \pmod 4$. In particular, $\frac{q+1}{2}$ is odd. Then $C_i = H_i' \times Q_i' \times Z_i'$ where the following holds:

(1) $H_i' \cong C_{\frac{q+1}{2}}$ and $H_i = H_i' \times Z(M_i)$;

(2) $Q_i' = O_2(C_i)$, in particular, as before $Q_i' \in \mathcal{S}yl_2(C_i)$, Q_i' is cyclic and $Q_i := Q_i' \cap Z(L_i)$ is in $\mathcal{S}yl_2(Z(L_i))$; and

(3) $Z_i' \leq Z(L_i)$ with $|Z_i'|$ dividing $\frac{q-1}{2}$, i.e., Z_i' is a cyclic subgroup of odd order and whose order is the part of $|C_i|$ coprime to $(q+1)$.

Notice that $H_i \cap Q_i' = Z(M_i)$ and so $H_i' \times Q_i' = H_i Q_i'$.

Lemma 3.5. *For $i = 1, 2$ the group $H_i Q_i'$ acts transitively on $E_X(x_i)$.*

Proof. As in the previous case, the action of L_i on the set $E_X(x_i)$ is the natural action of $PGL_2(q)$ on the projective line. Now the image $\overline{H_i Q_i'}$ of $H_i Q_i'$ in $L_i/Z(L_i)$ is isomorphic to C_{q+1} while the image $\overline{H_i}$ of H_i is isomorphic to $C_{\frac{q+1}{2}}$. Moreover, $\overline{Q_i'} \cap \overline{T} = 1$, hence $\mathrm{Stab}_{\overline{H_i Q_i'}}(e) = 1$ for any $e \in E_X(x_i)$. Another application of the Orbit-Stabiliser Theorem immediately gives the result. $\square$

Suppose now that $Q_i \leq Z(G)$. Take $A_i := H_i Q_i' Z_0$ with $Z_0 \leq Z(G)$. Then A_i acts transitively on $E_X(x_i)$ by Lemma 3.5, and $A_1 \cap A_2 = Q_i Z_0 \leq Z(G)$ with $Q_i Z_0 = \mathrm{Stab}_{A_i}(x_{3-i})$. Hence, all conditions of Proposition 3.1 are satisfied, which proves part (3)(a)ii of Theorem 1.1.

Now let us impose a slightly weaker condition: instead of assuming $Q_i \leq Z(G)$, let us suppose that $Q_i^2 \leq Z(G)$ where Q_i^2 is the unique subgroup of Q_i of index 2 (i.e., $Q_i^2 = \langle x_i^2 \mid \langle x_i \rangle = Q_i \rangle$). Now take $A_i := O_{\left(\frac{q-1}{|Q_i'|}\right)'}(N_i) = H_i Q_i' Z_0 \langle t_i \rangle$ where $t_i \in N_T(H_i) - C_T(H_i)$ is of order 2 and $Z_0 \leq Z(G)$. Then by Lemma 3.5, A_i acts transitively on $E_X(x_i)$, as it contains $H_i Q_i'$.

Finally, let $T_0 \in \mathcal{S}yl_2(T)$. If $Q_i \leq Z(G)$, $A_1 \cap A_2 = Q_i \langle t_i \rangle Z_0 = (Z(G) \cap A_i) \langle t_i \rangle = \mathrm{Stab}_{A_i}(x_{3-i})$ as $Q_i \langle t_i \rangle$ contains all elements of order 2 in T_0, and so $t_1 \in Q_2 \langle t_2 \rangle$ while $t_2 \in Q_1 \langle t_1 \rangle$. Assume though that $Q_i \not\leq Z(G)$. Since $Q_i^2 \leq Z(G)$, $Q_i \langle t_i \rangle = \{x \in T_0 \mid x^2 \in Z(G)\}$ for $i = 1, 2$, and so $Q_1 \langle t_1 \rangle = Q_2 \langle t_2 \rangle$. Therefore $A_1 \cap A_2 = Q_i \langle t_i \rangle Z_0 = \mathrm{Stab}_{A_i}(x_{3-i})$ which finishes the proof of Theorem 1.1.

3.2.4. *Proof of Theorem 1.2.* We finally construct the exceptional edge-transitive lattices in Theorem 1.2. We provide only a brief discussion, since our proof consists of carrying out for Kac–Moody groups G as in Theorem 1.1 the constructions used by Lubotzky to prove [L, Theorem 3.3] for $SL_2(\mathbb{F}_q((t^{-1})))$.

The reason for the existence of these "exceptional" lattices is that when q is small enough, L_i occasionally contains a subgroup N_i distinct from the ones listed in the conclusions of Theorem 1.1, yet still acting transitively on the set $E_X(x_i)$. In those cases we then take $A_i = N_i A_0$ with $A_0 \leq N_T(N_i)$, $i = 1, 2$, and check that both conditions of Proposition 3.1 are satisfied with $n = 1$.

To determine N_i, we use the fact that L_i acts on $E_X(x_i)$ as $\overline{L_i} = L_i/Z(L_i)$ (which is either $PSL_2(q)$ or $PGL_2(q)$) naturally acts on $\mathbb{P}^1(q)$. Thus we may as well look for subgroups $\overline{N_i}$ of $\overline{L_i}$ that act transitively on the points of $\mathbb{P}^1(q)$ and are different from $N_{\overline{L_i}}(\overline{H_i})$. Since we are interested in constructing lattices without p–elements, we will also assume that $(q, |N_i|) = 1$. The results of Dickson together with the fact that the normaliser of a split torus of $\overline{L_i}$ is not transitive on the points of $\mathbb{P}^1(q)$, tell us immediately that q is odd and $\overline{N_i}$ can only be isomorphic to one of the following groups: A_4, S_4 or A_5. On the other hand, as $\overline{N_i}$ acts transitively on $E_X(x_i)$, the Orbit-Stabiliser Theorem implies that $|E_X(x_i)| = q+1$ divides $|\overline{N_i}|$. Combining these two easy arguments gives us $q + 1 \leq 60$. In fact, when $(5, |L_i|) = 1$, we have $q + 1 \leq 24$.

Thus we first list all the odd prime powers q that are less or equal to 59. If $(5, q^2 - 1) = 1$, then $q \leq 23$. This leaves us with $5, 7, 9, 11, 13, 17, 19, 23, 25, 29, 31, 41, 49$ and 59. Furthermore, $q + 1$ divides $|\overline{N_i}|$ and as $|\overline{N_i}|$ divides $2^3 \cdot 3 \cdot 5$, we may exclude $13, 17, 25, 31, 41$ and 49. Finally, if $q = 9$, $q + 1 = 10$, and so $\overline{N_i} \cong A_5$, which contradicts the fact that $(q, |N_i|) = 1$. We are now down to the same list of exceptional q as in [L]. Hence, all that remains to do is for a given prime q take a candidate for $\overline{N_i}$ for $i = 1, 2$ as in Theorem 1.2, and check that the conditions of Proposition 3.1 are satisfied with $n = 1$. This follows immediately as in [L] by simply looking at the action of the appropriately chosen N_i on the points of $\mathbb{P}^1(q)$.

3.3. **Construction of the lattice Γ'.** Let G be as in Theorem 1.1 and assume that we are *not* in any of the cases in Theorem 1.1 where G admits an edge-transitive lattice. We now construct a cocompact lattice $\Gamma' < G$ which is the fundamental group of the graph of groups $\mathbb{A}'$ sketched in the statement of Theorem 1.1. For this, we first in Section 3.3.1 define certain finite subgroups S, N_1 and N_2 of G and discuss their structure, then in Section 3.3.2 construct Γ' by verifying that our embedding criterion, Proposition 3.1 above, may be applied with $n = 2$, $A_1 = N_1$ and $A_2 = N_2$.

3.3.1. *The groups S, N_1 and N_2.* Let P_1 and P_2 be the standard maximal parahoric subgroups of G. Then for $i = 1, 2$, P_i is the stabiliser in G of a vertex x_i of X with $[x_1, x_2]$ an edge of X. Recall that since G is rank 2 and has symmetric Cartan matrix, $P_1 \cong P_2$. Moreover, if L_i is a Levi complement of P_i, then $L_i = M_i T$ where $T \leq B \leq P_1 \cap P_2$ is a torus of G and $A_1(q) \cong M_i \triangleleft L_i$, where $A_1(q)$ is isomorphic to either $SL_2(q)$ or $PSL_2(q)$, depending upon G. By assumption, $q \equiv 1 \pmod 4$, and either $L_i/Z(L_i) \cong PSL_2(q)$, or if $L_i/Z(L_i) \cong PGL_2(q)$, $Q_i^2 \not\leq Z(G)$ where Q_i^2 is the unique subgroup of index 2 of the Sylow 2–subgroup of $Z(L_i)$.

For $i = 1, 2$ let H_i be a fixed non-split torus of M_i such that $N_T(H_i)$ is as big as possible. Then either $H_i \cong C_{\frac{q+1}{2}}$ or C_{q+1}, depending on whether $M_i \cong PSL_2(q)$ or $SL_2(q)$ respectively. Also, $N_T(H_i)/C_T(H_i) \cong C_2$ and $H_i \cap N_T(H_i) = Z(M_i)$. Define

$$S := N_T(H_1) \cap N_T(H_2).$$

Let us try to describe S in more definite terms. Let Q be the Sylow 2–subgroup of T (it is unique since T is abelian). First, notice that if $z \in N_T(H_i)$, $i = 1, 2$, is of odd order, then $[z, H_i] = 1$ and $[z, M_i] = 1$. Hence if $z \in S$ and z is of odd order, $z \in C_G(\langle M_1, M_2 \rangle)$ thus $z \in Z(G)$. It follows immediately that $Z(G) \leq S \leq Z(G)Q$. Let us now investigate what happens when $z \in N_T(H_i) \cap Q$, $i = 1, 2$.

Take $x \in Q$ such that x normalises but not centralises H_i for some $i \in \{1, 2\}$. Then x acts on H_i as an element of order 2, and so x^2 centralises H_i. It follows that x^2 centralises M_i. Now consider $R_i := \{x \in Q \mid x^2 \in C_T(M_i)\}$. Then $R_i \leq Q$ and $R_i \leq N_T(H_i)$. Define

$$Q_0 := R_1 \cap R_2 = \{x \in Q \mid x^2 \in C_T(M_1) \cap C_T(M_2)\} = \{x \in Q \mid x^2 \in Z(G)\}.$$

Clearly, $Q_0 \leq S$. On the other hand, take $s \in S \cap Q$. If $[s, H_i] = 1$ for both $i = 1, 2$, then $[s, M_i] = 1$ for $i = 1, 2$ implying $s \in Z(G) \cap Q \leq Q_0$. Let $s \in S \cap Q$ be such that $[s, H_i] \neq 1$ for some $i \in \{1, 2\}$. As noticed above, $s^2 \in C_T(H_i) \leq C_T(M_i)$. Hence, $s^2 \in C_T(M_j)$, $\{i, j\} = \{1, 2\}$. Therefore, $s^2 \in Z(G)$. Thus $S \cap Q \leq Q_0$. It follows that:

Lemma 3.6. $S = Z(G)Q_0$.

Notice that $|N_S(H_i) : C_S(H_i)| = 2$. We also define

$$N_1 := SH_1 \quad \text{and} \quad N_2 := SH_2.$$

3.3.2. *Application of embedding criterion.* By construction, for $i = 1, 2$, N_i is a finite subgroup of P_i, and $S = N_1 \cap N_2$. We now verify that our embedding criterion, Proposition 3.1 above, may be applied with $n = 2$, $A_1 = N_1$ and $A_2 = N_2$.

Notice first that for $i = 1, 2$, the intersection of N_i with an edge stabiliser in L_i is of index $\frac{q+1}{2}$. The Orbit-Stabiliser Theorem yields immediately that N_i has 2 orbits of equal size $\frac{1}{2}(q+1)$ on $E_X(x_i)$. That is, with $n = 2$, condition (1) in the statement of Proposition 3.1 above holds.

Denote by f_1 the edge $[x_1, x_2]$ of X and by $\hat{f}_1$ the edge $[x_2, x_1]$. Choose an edge $f_2 \in E_X(x_1)$ so that the edges f_1 and f_2 represent the two N_1–orbits on $E_X(x_1)$, and choose an edge $\hat{f}_2 \in E_X(x_2)$ so that the edges $\hat{f}_1$ and $\hat{f}_2$ represent the two N_2–orbits on $E_X(x_2)$.

The edges f_1 and $\hat{f}_1$ are fixed by S, since $S \leq T \leq B = P_1 \cap P_2$. We claim that the edges f_2 and $\hat{f}_2$ may be chosen so that S fixes both f_2 and $\hat{f}_2$. To see this, consider first the action of N_1 on the edges $E_X(x_1)$. Now $N_1 \leq L_1$, and L_1 acts on the set $E_X(x_1)$ as on the points of projective line, that is, we observe this action via a homomorphism $\phi : L_1 \to PGL_2(q)$. The kernel of this action is $\ker(\phi) = Z(L_1) = C_T(M_1)$. We know that N_1 has 2 orbits, say θ_1 and θ_2, in this action, each of length $\frac{q+1}{2}$ which is odd. Assume that the fixed points of S all lie inside the same orbit of N_1, say θ_1. Then S would act fixed-point free on θ_2.

Now, $S \ker(\phi)/\ker(\phi) \cong S/S \cap \ker(\phi)$ and as $|S \ker(\phi)/\ker(\phi)| = 2$, S would have a fixed point on θ_2, a contradiction. Hence we may choose the edge $f_2 \in E_X(x_1)$ so that f_2 is fixed by S. Similarly, we may choose $\hat{f}_2 \in E_X(x_2)$ to be fixed by S.

Now let $g_1 = \hat{g}_1 = 1_G$. Consider the fixed points of S on $E_X(x_i)$, $i = 1, 2$. Since $|S \cap M_i : Z(M_i)| = 2$, they are the two points fixed by the whole of T. Choose $g_2 \in N_{P_1}(T)$ that represents $w_1 \in W$. Then $g_2 \cdot f_1 = f_2$. Similarly, we may choose $\hat{g}_2 \in N_{P_2}(T)$ that represents w_2 and such that $\hat{g}_2 \cdot \hat{f}_1 = \hat{f}_2$. Then (2a) in Proposition 3.1 above holds. Let $\tau := g_2 \hat{g}_2^{-1}$. We observe that $S^\tau = S$, since by Lemma 3.6 above $S = Z(G)Q_0$, a characteristic subgroup of T which is therefore $N_G(T)$–invariant. Hence $(N_1 \cap N_2)^{g_2} = (N_1 \cap N_2)^{\hat{g}_2}$, and so (2c) in Proposition 3.1 above is satisfied.

To show that (2b) in Proposition 3.1 above holds, we must show that $N_1 \cap B = N_1 \cap B^{g_2} = N_2 \cap B = N_2 \cap B^{\hat{g}_2} = S$. Since $N_1 \leq L_1 \leq P_1$, we have that $N_1 \cap B = N_1 \cap (B \cap L_1)$. Now, $B \cap L_1$ is isomorphic to a Borel subgroup TU_0 of L_1, where $U_0 \cong E_{p^a}$, the elementary abelian group of exponent p and order $q = p^a$, is normalised by T. On the other hand N_1 is a finite subgroup of L_1. The order of N_1 is $|S|\frac{q+1}{2}$ and it divides $|T|\frac{q+1}{2}$. Moreover, $(|S|, \frac{q+1}{2}) = 1$. Therefore, numerical reasons imply that $N_1 \cap B$ is a finite group whose order divides $|T|$ and is actually at most $|S|$. But $S \leq N_1$ and $S \leq T \leq B$. Hence $N_1 \cap B = S$ as required. The argument that $N_2 \cap B = S$ is similar.

Since S fixes the edge $f_2 = g_2 \cdot f_1$, we have $S \leq B^{g_2}$. The argument that $N_1 \cap B^{g_2} = S$ is then similar to the previous paragraph. Finally, S also fixes the edge $\hat{f}_2 = \hat{g}_2 \cdot \hat{f}_1$, and again by similar arguments we conclude that $N_2 \cap B^{\hat{g}_2} = S$. Therefore all hypotheses of Proposition 3.1 above are satisfied with $n = 2$, $A_1 = N_1$ and $A_2 = N_2$, and so the fundamental group Γ' of the graph of groups $\mathbb{A}$ as sketched in the statement of Theorem 1.1 is a cocompact lattice in G with quotient the graph A.

4. Discussion of conjectures

Let G be as in Theorem 1.1. In this section we motivate Conjectures 1 and 2, stated in the introduction, and explain why Conjecture 2 implies Conjecture 1.

We will use the following well-known property of cocompact lattices.

Proposition 4.1 (see p. 10 of Gelfand–Graev–Piatetsky-Shapiro [GGPS]). *Let G be any locally compact topological group, and Γ a cocompact lattice in G. If $u \in \Gamma$, then*

$$u^G := \{gug^{-1} \mid g \in G\}$$

is a closed subset of G.

We will also need the next result, which follows from the reference Ronan [R, Theorem 6.15].

Lemma 4.2. *Let α be a positive real root whose corresponding geodesic ray in Σ begins at a vertex distance at least $n + 1$ from both P_1 and P_2. Then U_α fixes the ball of radius n about the edge B.*

Recall the definition of the closed abelian group $\mathcal{U}$ from Section 2.4.4 above. Observe that every torsion element of $\mathcal{U}$ has order p. We now show that nontrivial elements of conjugates of $\mathcal{U}$ cannot be contained in cocompact lattices. It is then immediate that Conjecture 2 implies Conjecture 1.

Corollary 4.3. *Let u be a nontrivial element of any G–conjugate of $\mathcal{U}$. Then there is a $g \in G$ such that*

$$\lim_{n \to \infty} g^n u g^{-n} = 1_G.$$

Hence by Proposition 4.1 above, u is not contained in any cocompact lattice $\Gamma < G$.

Proof. It suffices to prove the result for $u \in \mathcal{U}$. Let $g_\tau \in G$ induce the translation $\tau = w_1 w_2$, as defined in Section 2.4.4 above, and for each positive integer n let $f_n : G \to G$ be given by $f_n(x) = g_\tau^n x g_\tau^{-n}$. We will show that $f_n(u) \to 1_G$.

By definition, u is the limit in the building topology of a sequence of elements u_k in the elementary abelian p–group $V_1 \times -V_2$. Since u fixes the end ε_1, there is a root $\alpha \in \Phi_+^1 \cup \Phi_-^2$ such that u fixes pointwise the half-line α. Now there are at most finitely many root groups in $-V_2$ which fix such an α, namely those root groups U_β with $\alpha \subseteq \beta \subseteq -\alpha_2$. We may thus assume without loss of generality that for all u_k, all of the

root groups U_β in $-V_2$ on which u_k has a nontrivial projection satisfy $\alpha \subseteq \beta \subseteq -\alpha_2$. Hence for all n large enough, $f_n(u) \in \hat{V}_1$. So we reduce to the case that $u \in \hat{V}_1$.

For each root $\alpha \in \Phi^1_+$, consider the group

$$f_n(U_\alpha) = g_\tau^n U_\alpha g_\tau^{-n} = U_{\tau^n \alpha}.$$

Since $\alpha \in \Phi^1_+$ and τ acts by translation by two edges with repelling fixed point ε_1, the distance from the initial vertex of the ray $\tau^n \alpha$ to the edge B is at least $2n \geq n + 1$. Lemma 4.2 then implies that $f_n(U_\alpha)$ fixes pointwise the ball of radius n about B. Therefore $f_n(V_1)$ fixes this ball pointwise. By definition of the building topology, it follows that $f_n(\hat{V}_1) = g_\tau^n \hat{V}_1 g_\tau^{-n}$ fixes pointwise the ball of radius n about B. Hence for all $u \in \hat{V}_1$, $f_n(u) \to 1_G$, as required. $\qquad\square$

5. Classification of edge-transitive lattices

Let G be as in Theorem 1.1. In this section we assume Conjecture 1, that is, that cocompact lattices in G do not contain p–elements, and under this assumption prove Theorem 1.3, which says that the edge-transitive lattices in G are exactly those described in Theorems 1.1 and 1.2.

To fix notation throughout this section, for $i = 1, 2$ let x_i be the vertex of the tree X which is fixed by the standard parahoric P_i. We have that P_i has Levi decomposition $P_i = L_i \ltimes U_i$ (see Proposition 2.3) where $L_i = T_i M_i$.

In order to classify the edge-transitive lattices in G, we will rely on the following converse to the case $n = 1$ of Proposition 3.1, which follows from Bass–Serre theory and the discussion of lattices in Section 2.3. Our notation is the same as in Proposition 3.1.

Lemma 5.1. *Suppose Γ is a cocompact lattice in G with fundamental domain the edge $[x_1, x_2]$. For $i = 1, 2$, let $A_i = \operatorname{Stab}_\Gamma(x_i)$. Then $\Gamma \cong A_1 *_{A_1 \cap A_2} A_2$ and A_1 and A_2 are finite subgroups of G such that for $i = 1, 2$,*

(1) A_i *acts transitively on $E_X(x_i)$; and*
(2) $\operatorname{Stab}_{A_i}(x_{3-i}) = A_1 \cap A_2$.

Now suppose Γ is a cocompact edge-transitive lattice in G. Our classification of edge-transitive lattices is up to isomorphism, so by Lemma 5.1 we may assume without loss of generality that $\Gamma \cong A_1 *_{A_0} A_2$, where for $i = 1, 2$ the group A_i is a finite subgroup of the standard parahoric P_i, and $A_0 = A_1 \cap A_2$. By Conjecture 1, the groups A_1 and A_2 do not contain any p–elements. Also, since the finite group A_0 has index $(q+1)$ in both A_1 and A_2, for $i = 1, 2$ the group A_i has order divisible by $(q+1)$.

We continue by listing the groups A_1 and A_2 which satisfy the restrictions stated so far. Recall from Section 2.4 that if $L = L_i$ is the Levi factor of a standard maximal parahoric subgroup $P = P_i$ of G, then $L = TM$ where T is the torus of G and $A_1(q) \cong M \lhd L$.

Proposition 5.2. *Let $A = A_i$ be a finite subgroup of $P = P_i$ such that $|A|$ is coprime to p and is divisible by $(q+1)$. We may assume without loss of generality that A is a subgroup of the Levi factor L. Then one of the following conditions hold:*

(1) *If $p = 2$, then $H \leq A \leq C_T(M)H$ where $H \cong C_{q+1}$ is a non-split torus of $M \cong SL_2(q)$.*
(2) *If p is odd and $A \leq MZ(L)$, then $A \cong T_0 H$ where $T_0 \leq N_T(H)$ and $H \leq M$. More precisely, if $A_1(q)$ is universal (that is, if $M \cong SL_2(q)$), then H is isomorphic to a subgroup listed in the conclusions to Corollary 2.8 above. Otherwise, $M \cong PSL_2(q)$ and $H \cong H'/\langle -I \rangle$ where H' is a conclusion to Corollary 2.8 above. But if $p = 3$ and $q = 9$, $H/Z(H) \not\cong A_5$.*
(3) *If p is odd, $L/Z(L) \cong PGL_2(q)$ and $A \not\leq MZ(L)$, then either $AZ(L)/Z(L) \leq H_0$ where $H_0 \cong D_{2(q+1)}$, or $q \in \{5, 11\}$ and $AZ(L)/Z(L) \cong S_4$.*

Proof. Since $P = P_i$ has Levi decomposition $P = L \ltimes U$ where U is pro–p (see Proposition 2.3), we may assume without loss of generality that $A \leq L$. Parts (1) and (2) then follow immediately from Corollary 2.8 above. Part (3) follows from Theorem 2.9 above and the fact that subgroups isomorphic to D_{q+1} from Part (4) of Theorem 2.9 are either contained in subgroups isomorphic to $D_{2(q+1)}$ from Part (5) of the same result, or possibly $q \equiv \pm 3 \pmod 8$ and $A \cong S_4$. Now [ATLAS] confirms the result. $\qquad\square$

The remaining proof of Theorem 1.3 is divided into the same cases as in the statement of Theorem 1.1.

5.1. **Case** (1) **of Theorem 1.1.** We first prove Theorem 1.3 in the case $p = 2$, by showing that every edge-transitive lattice in G has the form described in Theorem 1.1(1). By Lemma 5.1 and the discussion before Proposition 5.2, any edge-transitive lattice $\Gamma < G$ satisfies $\Gamma \cong A_1 *_{A_1 \cap A_2} A_2$ where for $i = 1, 2$, A_i is a finite subgroup of P_i of order divisible by $(q+1)$ and coprime to p. But then by Proposition 5.2(1), for $i = 1, 2$ we have $H_i \leq A_i \leq C_T(M_i)H_i$ where $H_i \cong C_{q+1}$ is a non-split torus of M_i. (Note that since H_i acts transitively on $E_X(x_i)$, Lemma 5.1(1) automatically holds.) Now observe that

$$A_0 := A_1 \cap A_2 \leq C_T(M_1) \cap C_T(M_2) \leq C_T(\langle M_1, M_2 \rangle) \leq Z(G),$$

as required. By Lemma 5.1(2), it follows that $A_i = A_0 H_i$, which completes the proof of Theorem 1.3 in this case.

5.2. **Case** (2) **of Theorem 1.1.** We are now in the case that p is odd and $L_i/Z(L_i) \cong PSL_2(q)$. To prove Theorem 1.3 in this case, assume first that $q \equiv 3 \pmod 4$. As in the proof of Theorem 1.1, we may conclude that in this case $Z(M_i) \leq Z(G)$ for $i = 1, 2$. Then for $i = 1, 2$, we are considering subgroups A_i of M_i whose order in $M_i/Z(M_i) \cong PSL_2(q)$ is divisible by $(q+1)$. By Proposition 5.2(2), either $A_i = A_0 N_{M_i}(H_i)$ where H_i is a non-split torus of M_i and $A_0 \leq N_T(H_i)$, or $A_i = A_0 N_i$ where $N_i \cong H$ above and $A_0 \leq N_T(N_i)$. Then in all cases $A_0 \leq C_T(M_i)$ for $i = 1, 2$, and so $A_0 \leq C_G(\langle M_1, M_2 \rangle) \leq Z(G)$ and $A_0 \cap N_{M_i} = Z(M_i)$ as required.

We now assume that $q \equiv 1 \pmod 4$.

Lemma 5.3. *The group $A_i = A_0 N_{M_i}(H_i)$ does not act transitively on $E_X(x_i)$.*

Proof. To see this, in its action on $E_X(x_i)$, $A_i = A_0 N_{M_i}(H_i)$ intersects a one-point stabiliser B_i of L_i in a subgroup of index $4/d$, that is,

$$|A_i : A_i \cap B_i| = |A_0 N_{M_i}(H_i) : A_0 N_{M_i}(H_i) \cap T \operatorname{Stab}_{M_i}(x_{3-i})| = |N_{M_i}(H_i) : N_{M_i}(H_i) \cap T| = 4/d,$$

where $d = 2$ if $M_i \cong SL_2(q)$ and $d = 1$ if $M_i \cong PSL_2(q)$. Hence the length of the orbit of A_i in its action on $E_X(x_i)$ is at most $\frac{2(q+1)/d}{4/d} = \frac{q+1}{2}$. And so A_i is not transitive on $E_X(x_i)$. $\square$

Now Lemma 5.1(2) implies that G does not contain edge-transitive cocompact lattices unless possibly one of the following holds: $q = 5$ and $B_1 \cong B_2 \cong A_1(3)$, or $q = 29$ and $B_1 \cong B_2 \cong A_1(5)$. These exceptional cases are constructed in Theorem 1.2.

5.3. **Case** (3) **of Theorem 1.1.** We are now in the case that p is odd and $L_i/Z(L_i) \cong PGL_2(q)$. We will use the notation introduced in the proof of Theorem 1.1. Recall that $C_i = C_{L_i}(H_i)$, $H_i \leq C_i$ and C_i is a cyclic group of order $(q+1)|Z(L_i)|$, $Z(L_i) \leq C_i$ and $|N_i : C_i| = 2$. Moreover, if $\overline{C}_i$ is the image of C_i in $\overline{L}_i = L_i/Z(L_i)$, $\overline{C}_i \cong C_{q+1}$ and $N_i = C_i N_T(H_i)$. Using part (3) of Proposition 5.2 we obtain that if A_i is a subgroup we are looking for, $\overline{A}_i = A_i Z(L_i)/Z(L_i)$ must be isomorphic to a subgroup of $D_{2(q+1)}$ whose order is divisible by $(q+1)$ or if $q \in \{5, 11\}$, $\overline{A}_i \cong S_4$. Assume now that Γ is not an exceptional lattice, i.e., we are not in the latter case. Then A_i is a subgroup of $N_i := N_{L_i}(H_i)$. This is so because N_i is the full pre-image of $D_{2(q+1)}$ in L_i.

Suppose first that $q \equiv 3 \pmod 4$. In terms of notation from the proof of Theorem 1.1, $C_i = C_i' \times Z_i'$. Suppose first that $A_i \leq N_i$ is such that $\overline{A}_i \cong D_{2(q+1)}$. Then $A_i \geq C_i'$ and $A_i \geq T_0$. This is true as $T_0 \leq N_T(H_i)$, $\overline{C_i' T_0} = \overline{N}_i$ and if a subgroup of N_i does not contain $C_i' T_0$, its image in L_i would be properly contained in $D_{2(q+1)}$. On the other hand, $A_i \leq N_i = C_i' T_0 Z_i'$. Now, A_0 is a subgroup of A_i of index $(q+1)$, and as $A_0 = A_1 \cap A_2 \leq P_1 \cap P_2 = B$ while $(|A_0|, p) = 1$, $A_0 \leq T$. It follows that $A_0 \leq T_0 Z_i'$. Notice that $g \in A_0$ is of odd order if and only if $g \in Z_i'$. Hence, $g \in O(A_0)$ implies $g \in Z(L_1) \cap Z(L_2) \leq C_G(\langle M_1, M_2 \rangle) \leq Z(G)$. We may now conclude that $A_0 \leq T_0 Z(G)$ which gives us the lattices from (3)(b)i of Theorem 1.1.

Let us now investigate which subgroups of N_i will give us other edge-transitive lattices in G. We are looking for $A_i \leq N_i$ that acts edge-transitively on $E_X(x_i)$ and is such that $(q+1)$ dividing $|A_i|$. Suppose first that A_i is a subgroup of C_i such that $\overline{A}_i \cong C_{q+1}$. Then $C_i' \leq A_i \leq C_i' Z_i'$. Recall that A_0 is a subgroup of A_i of index $(q+1)$, and as before one can easily see that $A_0 \leq T$. Now, $C_i' Z_i' \cap T = Z(M_i) Z_i'$ and $Z(M_i) \leq C_i'$. Again, $g \in A_0$ is of odd order if and only if $g \in Z_i'$, implying $g \in Z(G)$. Assume now that $Z(M_i) \neq 1$. It

follows that $A_0 = Z(M_1)(A_0 \cap Z_1') = Z(M_2)(A_0 \cap Z_2')$. Because the Sylow 2–subgroup of A_0 is normal in A_0, it follows that $Z(M_1) = Z(M_2)$, and so $Z(M_i) \leq Z(G)$ for $i = 1, 2$. If $Z(M_i) \nleq Z(G)$, we will obtain that condition (2) of Lemma 5.1 is violated. Therefore, if $Z(M_i) \leq Z(G)$, we obtain some lattices from Theorem 1.1(3(b)ii). Finally, the only other subgroups of N_i that act transitively on $E_X(x_i)$ are subgroups that map onto $\overline{N_{M_i}(H_i)} \cong D_{q+1}$. These are the subgroups that map into $\overline{M_i}$ and we already described them in (2) of Theorem 1.1: $A_i = N_{M_i}(H_i)Z_0$ where $Z_0 \leq Z(G)$. However, $N_{M_1}(H_1)Z_0 \cap N_{M_2}(H_2)Z_0 = Z(M_i)Z_0$ for $i = 1, 2$, and it follows immediately that $Z(M_1) = Z(M_2) \leq Z(G)$, giving the remaining lattices from (3b) of Theorem 1.1.

Suppose now that $q \equiv 1 \pmod 4$. This time $C_i = H_iQ_i' \times Z_i'$. Suppose first that $A_i \leq N_i$ is such that $\overline{A}_i \cong D_{2(q+1)}$. Then $A_i \geq H_iQ_i'\langle t_i \rangle$ where t_i is an involution in $N_T(H_i) - C_T(H_i)$. This holds because if a subgroup of N_i does not contain $H_iQ_i'\langle t_i \rangle$, its image in L_i would be properly contained in $D_{2(q+1)}$. It follows that $A_i \leq H_iQ_i'\langle t_i \rangle O(Z_i')$. As in the previous case, A_0 is a subgroup of A_i of index $(q+1)$ such that $(|A_0|, p) = 1$ and $A_0 \leq T$. It follows that $A_0 \leq (Q_i' \cap T)\langle t_i \rangle O(Z_i')$. As before $g \in A_0$ is of odd order if and only if $g \in O(Z_i')$, implying $g \in Z(G)$. We may now conclude that $A_0 \leq (Q_i' \cap T)\langle t_i \rangle Z(G)$. Now, $Q_i \leq A_i$ and in fact $Q_i \leq A_0$. Hence, the Sylow 2–subgroup of A_0 is $Q_i\langle t_i \rangle$. It follows that $Q_1\langle t_1 \rangle = Q_2\langle t_2 \rangle$. As before let Q_i^2 be the subgroup of Q_i generated by the squares of its elements. It follows that $Q_1^2 = Q_2^2$, and so $Q_i^2 \leq Z(L_1) \cap Z(L_2) \leq Z(G)$. If this condition fails, it is clearly impossible to have an edge-transitive lattice, while if it holds, the conditions of Proposition 3.1 are satisfied. We therefore have obtained the lattices from (3(a)i) of Theorem 1.1.

Let us now investigate which subgroups of N_i will give us other edge-transitive lattices in G. Again we are in quest of $A_i \leq N_i$ that acts edge-transitively on $E_X(x_i)$ (in particular, with $(q+1)$ dividing $|A_i|$). Suppose first that A_i is a subgroup of C_i such that $\overline{A}_i \cong C_{q+1}$. Then $Q_i'H_i \leq A_i \leq C_iH_i$. Now A_0 is a subgroup of A_i of index $(q+1)$, and as before it follows that $A_0 \leq T$. Thus $|Q_i' : Q_i' \cap A_0| = 2$. Again, $g \in A_0$ is of odd order if and only if $g \in Z_i'$, implying $g \in Z(G)$. Hence, $A_0 = (Q_i' \cap A_0)(A_0 \cap Z_i') = Q_i(A_0 \cap Z_i')$. Because the Sylow 2–subgroup of A_0 is normal in A_0, $Q_1 = Q_2$. In particular, the usual argument implies $Q_i \leq Z(G)$ for $i = 1, 2$. If $Q_i \nleq Z(G)$, we will obtain that condition (2) of Lemma 5.1 is violated. Therefore we obtain the remaining conclusions of Theorem 1.1. Since there are no other subgroups of N_i that act transitively on $E_X(x_i)$, we are done with the proof of Theorem 1.3.

6. Minimality of covolumes among cocompact lattices

Let G be as in Theorem 1.1. In this section we assume Conjecture 1, that is, that cocompact lattices in G do not contain p–elements, and we prove Theorem 1.4, stated in the introduction, on the minimal covolume of cocompact lattices in G.

We will subdivide our proof of Theorem 1.4 between the cases where G admits an edge-transitive lattice, discussed in Sections 6.1–6.3, and the remaining cases, discussed in Sections 6.4–6.6. Many arguments become rather tedious for small values of odd q. We thus assume that q is large enough (in particular, our arguments work for $q \geq 300$ for the edge-transitive case, and $q \geq 514$ for the remaining cases). Our discussion can be carried out in similar fashion when p is odd and $q \leq 514$, but we decided to skip it in order to have "cleaner" statements. Hence our further assumption is that $q \geq 514$.

6.1. **Edge-transitive case.** The main results are Lemmas 6.2 and 6.3 below, which show that, in the cases where G admits an edge-transitive lattice, an edge-transitive lattice is a cocompact lattice of minimal covolume in G. Thus by Theorem 1.3, a cocompact lattice of minimal covolume is listed in the statement of Theorem 1.1, and so the minimal covolume among cocompact lattices may be easily computed.

In the proofs below, we assume by contradiction that there is a cocompact lattice Γ of covolume strictly less than that of the edge-transitive lattice Γ_0 of smallest covolume, and then show by careful case analysis that in fact $\mu(\Gamma \backslash G) \geq \mu(\Gamma_0 \backslash G)$. Extending these arguments would, we believe, show that any cocompact lattice of minimal covolume in G must be edge-transitive. Since we did not need this further statement to determine the strict lower bound on covolumes given by Theorem 1.4, and the proofs are already lengthy, we have not carried out this extension.

As before, for $i = 1, 2$, let P_i be a maximal parahoric subgroup of G, with P_i the stabiliser in G of a vertex x_i of X. Recall that $P_1 \cong P_2$, and that if L_i is a Levi factor of P_i, then $L_i = TM_i$ where $T \leq B \leq P_1 \cap P_2$ is a torus of G, and $A_1(q) \cong M_i \lhd L_i$. Now M_i is normalised by T, and $T \cap M_i$ induces what are called inner-diagonal automorphisms on $M_i \cong A_1(q)$. On the other hand there are various possibilities for the action of elements of $T - T \cap M_i$ on M_i. In particular, either "none of them" or "some of them" induce non-trivial outer-diagonal automorphisms on M_i. This results in the following subcases:

Subcase 1 ("none of them"). *For $i = 1, 2$, $L_i/Z(L_i) \cong PSL_2(q)$.*

Subcase 2 ("some of them"). *For $i = 1, 2$, $L_i/Z(L_i) \cong PGL_2(q)$.*

We further subdivide our discussion of the edge-transitive case based on this observation, with Subcase 1 being considered in Section 6.2 and Subcase 2 in Section 6.3.

6.2. **Edge-transitive case, Subcase 1.** In this case $L_i = M_i \circ T_i$, that is, L_i is a central (commuting) product of M_i and $T_i = C_T(M_i)$. It is possible but not necessary that $T_i \cap M_i = 1$.

Example 1. Let G have generalised Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

(1) Let $p = 2$ and $G = G_u$, the universal version of the group. Then G is a central extension of $SL_2(\mathbb{F}_q((t^{-1})))$ by $\mathbb{F}_q^\times$, and so $L_i \cong C_{q-1} \times PSL_2(q)$ with $T_i \cap M_i = 1$ and $|T_i| = q - 1$.
(2) Let p be an odd prime, and $G \cong SL_2(\mathbb{F}_q((t^{-1})))$. Then $L_i \cong SL_2(q)$ with $T_i = T \cap M_i = \langle -I \rangle \cong C_2$.

An interesting and unusual consequence is the following observation that we already used in the proofs of Theorems 1.1 and 1.3, but now will state more explicitly.

Corollary 6.1. *Let G be as in Theorem 1.1. Suppose further that $q \equiv 3 \mod 4$ and $L_i/Z(L_i) \cong PSL_2(q)$ for $i = 1, 2$. If $M_i \cong SL_2(q)$, then $Z(M_i) \leq Z(G)$, and in particular, $Z(G) \neq 1$.*

Proof. We are now in the conclusions of Theorem 1.1(1) and (2). If $M_i \cong SL_2(q)$, then because of the structure of H_i, we have $A_0 \cap H_i = A_0 \cap M_i = Z(M_i)$. Hence $A_0 \geq Z(M_i) \cong C_2$. But $A_0 \leq Z(G)$, proving the result. $\square$

Let us now discuss the question of covolumes. Let $\Gamma = A_1 *_{A_0} A_2$ be an edge-transitive lattice in G. Then by Theorem 1.3, Γ is one of the conclusions listed in Theorem 1.1(1) or (2) above. As recalled in Section 2.3 above, the covolume of Γ in G may be calculated as follows:

$$\mu(\Gamma \backslash G) = \frac{1}{|A_1|} + \frac{1}{|A_2|} = \frac{1}{(q+1)|A_0|} + \frac{1}{(q+1)|A_0|} = \frac{2}{(q+1)|A_0|}.$$

In all conclusions to Theorem 1.1(1) and (2), the edge group A_0 satisfies $A_0 \leq Z(G)$. It follows that among all the *edge-transitive* cocompact lattices in G, the lattice Γ_0 with edge group $A_0 = Z(G)$ has the *smallest possible covolume*.

Now take Γ to be a cocompact, not necessarily edge-transitive, lattice in G. What happens then?

Lemma 6.2. *Let G be as in Theorem 1.1(1) or (2). In fact, if p is odd, suppose that $q > 300$. Assume that Conjecture 1 holds. Then an edge-transitive lattice is a cocompact lattice in G of minimal covolume.*

Combined with the discussion above, Lemma 6.2 proves Theorem 1.4 for this subcase. Note that in the statement of Theorem 1.4 in the introduction, $\delta = 1$ in this case.

Proof of Lemma 6.2. Since $Z(G)$ is finite, without loss of generality we may assume that $Z(G) = 1$, hence all lattices act faithfully. We have already constructed an edge-transitive lattice Γ_0 of covolume $\mu(\Gamma_0 \backslash G) = \frac{2}{|Z(G)|(q+1)} = \frac{2}{q+1}$. We will show that this is a lattice of minimal covolume among the cocompact lattices of G. In order to do so, assume that there exists a cocompact lattice Γ in G whose covolume is strictly smaller than that of Γ_0.

The quotient graph $S := \Gamma \backslash X$ is a bipartite graph containing at least two adjacent vertices. Let us call them x_1 and x_2. Then

$$\mu(\Gamma \backslash G) = \sum_{s \in S} \frac{1}{|\Gamma_s|} \geq \frac{1}{|\Gamma_{x_1}|} + \frac{1}{|\Gamma_{x_2}|}.$$

Moreover, as Γ_{x_i} is finite, by Proposition 2.1 above, we may assume without loss of generality that $\Gamma_{x_i} \leq P_i$. Hence by abuse of notation, we will for $i = 1, 2$ denote by x_i the vertex of X stabilised by P_i, so that the edge $[x_1, x_2]$ of X is the edge stabilised by B. By Conjecture 1, we have $(|\Gamma_{x_i}|, p) = 1$. Thus in fact we may suppose that $\Gamma_{x_i} \leq L_i$. Notice that as $T \leq P_1 \cap P_2$, we have $\Gamma \cap T \leq \Gamma \cap P_i = \Gamma_{x_i}$. It follows that $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$ and since $(p, |\Gamma_{x_i}|) = 1$, $\Gamma_{x_1} \cap \Gamma_{x_2} \leq T$.

We now consider two further subcases depending on the value of p. The case $p = 2$ is considered in Section 6.2.1 and the case p odd in Section 6.2.2.

6.2.1. *Proof of Lemma 6.2 when $p = 2$.* In this case we have that $M_i \cong SL_2(q) \cong PSL_2(q)$. Thus $L_i = M_i \times T_i$ with $T_i \leq T$ and T_i isomorphic to a subgroup of C_{q-1}. Since $q = 2^a$ for some $a \in \mathbb{N}$, $(q - 1)$ is odd. Moreover, as we already remarked, since we are assuming that Conjecture 1 holds, Γ_{x_i} is a subgroup of L_i of odd order.

We first observe that if $T_i \neq 1$ then $[T_i, M_j] \neq 1$, where $\{i, j\} = \{1, 2\}$. Otherwise, $T_i \leq C_G(\langle M_1, M_2 \rangle) \leq Z(G) = 1$, a contradiction.

Next we note that $\Gamma_{x_i} \cap T_i \neq 1$ for at least one of $i = 1, 2$. Suppose that $\Gamma_{x_i} \cap T_i = 1$. Then

$$\Gamma_{x_i} \cong \Gamma_{x_i} T_i / T_i \leq M_i T_i / T_i \cong M_i.$$

Since by Dickson's Theorem, the only subgroups of odd order of M_i are isomorphic to subgroups of either C_{q-1} or C_{q+1}, it follows that $|\Gamma_{x_i}| \leq (q + 1)$. Therefore if $\Gamma_{x_i} \cap T_i = 1$ for both $i = 1, 2$, we would have that $\mu(\Gamma \backslash G) \geq \frac{2}{(q+1)} = \mu(\Gamma_0 \backslash G)$, contradicting our assumption that the covolume of Γ is strictly smaller than the covolume of Γ_0.

Thus without loss of generality we may assume that there exists $1 \neq y_1 \in \Gamma_{x_1} \cap T_1$. By the observations above, $o(y_1) \mid (q - 1)$ and $[y_1, M_2] \neq 1$. In fact, $\langle y_1 \rangle$ acts faithfully on M_2 inducing inner automorphisms. Note also that as $y_1 \in \Gamma_{x_1} \cap T$, in fact, $y_1 \in \Gamma_{x_2}$.

If there exists $1 \neq y \in \Gamma_{x_2}$ with $o(y) \mid (q + 1)$, then as $\langle y, y_1 \rangle$ acts faithfully on $M_2 \cong PSL_2(q)$, Dickson's Theorem implies that $\langle y, y_2 \rangle$ contains an element of order 2, a contradiction with the cocompactness of Γ. Hence, for every $g \in \Gamma_{x_2}$, $o(g) \mid (q - 1)$, and in fact $g \in T$. Thus $\Gamma_{x_2} \leq T$, and so $\Gamma_{x_2} \leq \Gamma_{x_1}$. Now, if $\Gamma_{x_2} \cap T_2 \neq 1$, then going through the same discussion, but switching the roles of x_1 and x_2, we will obtain that $\Gamma_{x_1} \leq \Gamma_{x_2}$, implying that $\Gamma_{x_1} = \Gamma_{x_2}$. Thus if all the vertices s in the quotient graph $S = \Gamma \backslash X$ have this property (i.e., $\Gamma_s \cap T_s \neq 1$), we will be getting the same group as a vertex group at every step, and so Γ will be finite, a contradiction. Hence, we may assume that $\Gamma_{x_2} \cap T_2 = 1$, and it follows that Γ_{x_2} is a subgroup of T isomorphic to a subgroup of C_{q-1}, with $[\Gamma_{x_2}, M_2] \neq 1$. In particular, $\Gamma_{x_2} \leq \Gamma_{x_1}$.

Let us look at the possibilities for Γ_{x_1}. If $\Gamma_{x_1} \leq T$, it follows that $\Gamma_{x_1} \leq \Gamma_{x_2}$ implying $\Gamma_{x_1} = \Gamma_{x_2}$. In particular, $|\Gamma_{x_i}| \leq (q - 1)$ for $i = 1, 2$, which violates the minimality of covolume of Γ. Therefore, $\Gamma_{x_1} \not\leq T$. Now Dickson's Theorem implies that the only other choice is for $\Gamma_{x_1} \leq H_1 \times T_1$ where H_1 is a non-split torus of M_1. If there are more than two vertices adjacent to x_1 in S, then $\mu(\Gamma \backslash G) \geq \frac{2}{q-1} > \mu(\Gamma_0 \backslash G)$, a contradiction. Hence on the quotient graph, x_2 is the only vertex neighbouring x_1. Moreover, all other vertices of S are neighbours of x_2 and their stabilisers are of the same type as that of x_1. In particular, it follows that Γ_{x_2} is a normal subgroup of all other vertex groups, which contradicts the fact that Γ is a faithful lattice, since $Z(G) = 1$.

6.2.2. *Proof of Lemma 6.2 when p is odd.* In this case, $q = p^a$ for some $a \in \mathbb{N}$, $q \equiv 3 \pmod 4$ and we also assume $q > 300$.

Since we are working in the situation where $Z(G) = 1$, Corollary 6.1 implies that $Z(M_i) = 1$. Assume there exists an involution $t_i \in T_i$. If $[t_i, M_j] \neq 1$, then $t_i \in T$ would induce a non-trivial automorphism on M_j. But $q \equiv 3 \mod 4$, and so all the non-trivial involutory inner automorphisms of M_j come from the elements of a non-split torus, a contradiction. Hence $t_i \in C_G(\langle M_i, M_j \rangle \leq Z(G) = 1$, contradicting our assumption. Therefore $L_i \cong M_i \times T_i$ with $M_i \cong PSL_2(q)$ and T_i isomorphic to a subgroup of $C_{\frac{q-1}{2}}$.

Moreover, as in the previous case, if $T_i \neq 1$, it must act faithfully on M_j inducing inner automorphisms, for otherwise $Z(G) \neq 1$.

If $\Gamma_{x_i} \cap T_i = 1$, then

$$\Gamma_{x_i} \cong \Gamma_{x_i} T_i / T_i \leq M_i T_i / T_i \cong M_i.$$

In particular, $|\Gamma_{x_i}| \leq (q+1)$. Therefore if $\Gamma_{x_i} \cap T_i = 1$ for both $i = 1, 2$, it follows that $\mu(\Gamma\backslash G) \geq \mu(\Gamma_0\backslash G)$, a contradiction. Hence, without loss of generality we may assume that there exists $1 \neq y_1 \in \Gamma_{x_1} \cap T_1$ such that $\langle y_1 \rangle = \Gamma_{x_1} \cap T_1$. Notice that $o(y_1) \mid \frac{q-1}{2}$ and $\langle y_1 \rangle$ acts faithfully on M_2 inducing non-trivial inner automorphisms. Recall that as $y_1 \in T \cap \Gamma_{x_1}$, $y_1 \in \Gamma_{x_2}$. And so Γ_{x_2} acts on M_2 either as a subgroup of $N_{M_2}(M_2 \cap T)$, or as a subgroup of K_2 where $K_2 \in \{S_4, A_5\}$. Notice that in the latter case $o(y_1) \in \{3, 5\}$.

Assume first that $\Gamma_{x_2} \cap T_2 = 1$. Then Γ_{x_2} is isomorphic to a subgroup of M_2. If Γ_{x_2} is isomorphic to a subgroup of K, then using the previous paragraph we obtain that $\mu(\Gamma\backslash G) \geq \frac{1}{60} + \frac{1}{5 \cdot 2 \cdot (q+1)/2} > \frac{2}{q+1}$ for $q > 47$. Since this obviously contradicts the minimality of covolume of Γ, Γ_{x_2} must be isomorphic to a subgroup of $N_{M_2}(M_2 \cap T)$. Since we have full information about the action of L_i on the set of edges $E_X(x_i)$ coming out of x_i, let us first consider the action of Γ_{x_1} on $E_X(x_1)$. Since T_1 fixes $E_X(x_1)$ pointwise, we are interested in the action of the projection of Γ_{x_1} on M_1. Assume first that Γ_{x_1} acts transitively on $E_X(x_1)$. Then in the quotient graph x_1 has a unique neighbour x_2. Consider the set of neighbouring vertices of x_2 in S. If x_1 is its unique neighbour, $VS = \{x_1, x_2\}$ and since Γ_{x_1} acts transitively on $E_X(x_1)$, the lattice Γ is edge-transitive and we are done. Hence there are at least two neighbouring vertices of x_2 in S (one already represented by x_1). Let $z_1, \ldots, z_k$ be representatives of the other neighbouring vertices of x_2 in S. If each Γ_{z_i} acts transitively on the edges $E_X(z_i)$ coming out of z_i, we are looking at the whole $VS = \{z_1, \ldots, z_k, x_2, x_1\}$. If for some i, $C_{\Gamma_{z_i}}(M_{z_i}) = 1$, Γ_{z_i} is isomorphic to a subgroup of M_{z_i} and so $|\Gamma_{z_i}| \leq (q+1)$. Hence $\mu(\Gamma\backslash G) \geq \mu(\Gamma_0\backslash G)$, a contradiction. Therefore for all i, $C_{\Gamma_{z_i}}(M_{z_i}) \neq 1$.

Now denote by y_{z_i} an element of Γ_{z_i} such that $\langle y_{z_i} \rangle = C_{\Gamma_{z_i}}(M_{z_i})$. Then just like y_1, y_{z_i} acts faithfully on M_2 and $y_{z_i} \in \Gamma_{x_2}$. But Γ_{x_2} is a subgroup of the normaliser of the split torus of M_2, and so $[y_1, y_{z_i}] = 1$. Again using the fact that we know how L_2 acts on $E_X(x_2)$, we observe that y_1 then must fix the edge (x_2, z_i). It follows that y_1 fixes all vertices of S and so is in the kernel of the action of Γ, which since Γ acts faithfully is a contradiction. Therefore, either Γ_{x_1} or some Γ_{z_i} does not act transitively on the corresponding set of edges, and so by taking $x_1 = z_i$ if necessary, we are in the following case: Γ_{x_1} has at least two orbits on $E_X(x_1)$.

Therefore, either x_1 has a unique neighbouring vertex x_2 in S but the number of edges between them in S is greater than one, or there are at least two neighbouring vertices of x_1 in the quotient graph S. Let us discuss these two cases. In the former one, if $VS = \{x_1, x_2\}$, then $\langle y_1 \rangle \triangleleft \Gamma$, which is a contradiction as Γ is a faithful lattice since $Z(G) = 1$. It follows that $|VS| > 2$ and x_2 has at least two neighbouring vertices $z_1, \ldots, z_k$. Not to contradict the minimality of Γ we may now assume that $C_{\Gamma_{z_i}}(M_{z_i}) \neq 1$. Using the same argument as above, we obtain that $y_1 \in \Gamma_{z_i}$. Assume that y_1 acts on M_{z_i} as an element of $K_{z_i} \in \{S_4, A_5\}$. Suppose first that $o(y_1) \leq 15$. Let us evaluate the covolume of Γ. Let $a_2 \in \Gamma_{x_2}$ be such that $\langle a_2 \rangle \leq \Gamma_{x_2} = \langle a_2 \rangle \rtimes \langle s_2 \rangle \cong D_{2 \cdot o(a_2)}$ and $\langle a_2 \rangle = T \cap \Gamma_2$. Then $a_2 \in \Gamma_{x_1}$ and $y_1 \in \langle a_2 \rangle$. If $o(y_1) = o(a_2)$, then $|\Gamma_{x_2}| \leq 30$ which immediately contradicts the minimality of covolume of Γ ($\frac{1}{30} \geq \frac{2}{q+1}$ for $q > 59$). Hence, $o(a_2) > o(y_1)$ and so Γ_{x_1} acts on M_1 as either a subgroup of $K_1 \in \{S_4, A_5\}$, or as a subgroup of $N_{M_1}(M_1 \cap T)$.

In the former case $|\Gamma_{x_1}| \leq 15 \cdot 60$ and in particular, $|\Gamma_{x_1} \cap T| \leq 15 \cdot 5 = 75$. As we noticed earlier, $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$, and so $|\Gamma_{x_2} \cap T| \leq 75$. Since Γ_{x_2} is isomorphic to a subgroup of a normaliser of split torus of M_2, $|\Gamma_{x_2}| \leq 2 \cdot 75 = 150$. If follows that $\mu(\Gamma\backslash G) > \frac{2}{q+1}$ for $q \geq 257$, again a clear contradiction. In the latter case $|\Gamma_{x_1}| \leq o(y_1) 2 \frac{o(a_2)}{o(y_1)} \leq (q-1)$. Since $|\Gamma_{x_2}| \leq (q-1)$, we again get a contradiction with the minimality of covolume of Γ. Hence $o(y_1) > 15$ and so there exists $1 \neq y_1' \in \langle y_1 \rangle$ such that $[y_1', M_{z_i}] = 1$ for all i. We then use y_1' in the place of y_1 in all the previous subgroups. Therefore y_1 either centralises M_{z_i} or acts on it as a subgroup of a normaliser of a split torus of M_{z_i}. In both situations, $\langle y_1 \rangle$ is normal in Γ_{z_i} for $i = 1, \ldots, k$. Continuing with this argument, we obtain that y_1 is in the kernel of the action of Γ, a contradiction.

Therefore we are in the case that x_1 has more than one neighbouring vertex in S. One is x_2 and let z be among the other neighbouring vertices of x_1. If $\Gamma_z \cap T_z = 1$, then using $|\Gamma_{x_2}|$ and $|\Gamma_z|$, we obtain a contradiction with the minimality of covolume of Γ. Therefore $\Gamma_z \cap T_z \neq 1$ and we may take $x_2 = z$.

Thus we may assume that $\Gamma_{x_2} \cap T_2 \neq 1$. Hence there exists $y_2 \in \Gamma_{x_2} \cap T_2$ with $\langle y_2 \rangle = \Gamma_{x_2} \cap T_2$. As before we notice that $o(y_2) \mid \frac{q-1}{2}$, $y_2 \in \Gamma_{x_1}$ and $\langle y_2 \rangle$ acts faithfully on M_1 via inner automorphisms. As before now Dickson's Theorem allows us to conclude that either Γ_{x_1} acts on M_1 as a subgroup of K_1 with $K_1 \in \{S_4, A_5\}$, or Γ_{x_1} acts on M_1 as a subgroup of $N_{M_1}(M_1 \cap T)$. In the former case $o(y_2) \in \{3, 5\}$.

Let us begin with the case when Γ_{x_1} acts on M_1 as a subgroup of K_1. If Γ_{x_2} acts on M_2 as a subgroup of K_2, then $\mu(\Gamma \backslash G) \geq \frac{2}{60 \cdot 5} > \frac{2}{q+1} = \mu(\Gamma_0 \backslash G)$ for $q \geq 300$, a contradiction. Hence Γ_{x_2} acts on M_2 as a subgroup $N_{M_2}(M_2 \cap T)$. Recall that $|VS| > 2$. Let $v_1, \dots, v_k$ be the neighbours of x_1 in $VS - \{x_2\}$. Since y_1 fixes every edge in $E_X(x_1)$, it follows that y_1 acts faithfully on M_{v_i} and holding a discussion similar to the above one with v_i in place of x_2, we may assume that Γ_{v_i} acts on M_{v_i} as a subgroup of a normaliser of a split torus of M_{v_i}. It follows that $\langle y_1 \rangle$ is normal in each Γ_{v_i}. Now let $z_1, \dots, z_m$ be the neighbours of x_2 in $S - \{x_1\}$. Let us consider $\Gamma_{z_i} = P_{z_i} \cap \Gamma$. If $\Gamma_{z_i} \cap C_G(M_{z_i}) = 1$, then there is at most one such vertex, otherwise we would contradict the minimality of covolume of Γ. Hence we may assume that if it happens, $i = 1$, that is, $\Gamma_{z_1} \cap C_G(M_{z_1}) = 1$. Then we may further assume that $T \leq P_{z_1}$. Thus $y_1, y_2 \in \Gamma_{z_1}$. If Γ_{z_1} acts on M_{z_1} as a subgroup of $K_{z_1} \in \{S_4, A_5\}$, then $|\Gamma_{z_1}| \leq 60$, which is a contradiction, as always ($\frac{1}{60} \geq \frac{2}{q+1}$ for $q > 120$). Hence Γ_{z_1} acts on M_{z_1} as a subgroup of a normaliser of a split torus of M_{z_1} (it is not possible to have a normaliser of a non-split torus, as $o(y_j)$, $j = 1, 2$, is co-prime to $(q+1)$). It follows that $\langle y_1 \rangle$ is a normal subgroup of Γ_{z_1}. Hence, for $i > 1$, there exists $y_{z_i} \in C_G(M_{z_i})$ of order dividing $\frac{q-1}{2}$. But this element sits in the kernel of action of L_{z_i} on $E_X(z_i)$ and therefore, $y_{z_i} \in \Gamma_{x_2}$. On the other hand by the usual argument, y_2 acts faithfully on M_{z_i} and so $[y_2, y_{z_i}] = 1$. It follows that $\langle y_{z_i} \rangle$ is normal in Γ_{x_2}.

Finally, as $C_{\Gamma_{x_2}}(M_2)$ stabilises (x_2, z_i), it follows that $C_{\Gamma_{x_2}}(y_{z_i}) \leq \Gamma_{z_i}$. It follows that $y_1 \in \Gamma_{z_i}$ and so $\langle y_1, y_2 \rangle \leq \Gamma_{z_i}$. Assume that y_1 acts on M_{z_i} as a subgroup of $K_{z_i} \in \{S_4, A_5\}$. If $o(y_1) \leq 15$, then $|\Gamma_{x_1}| \leq 15 \cdot 60$ and in particular, $|\Gamma_{x_1} \cap T| \leq 15 \cdot 5 = 75$. As we remarked earlier, $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$, and so $|\Gamma_{x_2} \cap T| \leq 75$. Since Γ_{x_2} acts on M_2 as a subgroup of a normaliser of split torus, $|\Gamma_{x_2}| \leq 2 \cdot 75 = 150$, which leads to the usual contradiction as $\frac{1}{15 \cdot 60} + \frac{1}{150} \geq \frac{2}{q+1}$ for $q \geq 257$. Hence, $o(y_1) > 15$ and so there exists $y_1' \in \langle y_1 \rangle$ with $[y_1', M_{z_i}] = 1$ for all i. In this case we will use y_1' in the place of y_1 in all the previous subgroups. By iterating this argument we may show that $\langle y_1 \rangle \lhd \Gamma$ which is a contradiction.

We are now reduced to the last possible situation: Γ_{x_1} acts on M_1 as a subgroup of $N_{M_1}(M_1 \cap T)$. Notice that because of the symmetry between x_1 and x_2, to finish the analysis it remains to consider the case when Γ_{x_2} acts on M_2 as a subgroup of $N_{M_2}(M_2 \cap T)$.

But in this case $\langle \Gamma_{x_1}, \Gamma_{x_2} \rangle \leq N$. Hence we may move to the next vertex y on our graph. Using the previous argument we obtain again that the only possibility will be $\Gamma_y \leq N$, and so on and so forth. Therefore, in the end of this case, the only possible conclusion will be $\Gamma \leq N$. This is a contradiction, as N is not a cocompact lattice of G, nor does it contain any cocompact lattice. $\square$

6.3. Edge-transitive case, Subcase 2.

We are now in the case that G admits edge-transitive lattices and T induces non-trivial outer-diagonal automorphisms on M_i for $i = 1, 2$. As $M_i \cong A_1(q)$, p is **odd**, for if $p = 2$, $A_1(q) = SL_2(q) = PSL_2(q)$ does not admit non-trivial outer-diagonal automorphisms. It follows that L_i is isomorphic to a homomorphic image of $GL_2(q)$. In particular, $L_i = T_i M_i \langle t_i \rangle$ where $T_i = C_T(M_i)$, $t_i \in T - T_i$ and $L_i/T_i \cong PGL_2(q)$. Moreover, if $q \equiv 3 \pmod 4$, $T_i/T_i \cap M_i$ is a cyclic group of odd order and $t_i \in T$ is an involution, while if $q \equiv 1 \pmod 4$, $1 \neq t_i^2 \in M_i T_i$.

As in Subcase 1 in Section 6.2 above, we investigate minimality of covolumes. Assume that $q \geq 300$ and let $\Gamma = A_1 *_{A_0} A_2$ be an edge-transitive lattice in G. Reading carefully through the statement of Theorem 1.1(3), we observe that $|A_0| = \delta |Z_0|$ where $Z_0 \leq Z(G)$ and $\delta \in \{1, 2, 4\}$ depends on the structure of G and Γ.

Now, among all the edge-transitive cocompact lattices in G, choose $\Gamma_0 = A_1 *_{A_0} A_2$ such that $|A_0|$ is as large as possible. It follows that if $q \equiv 1 \pmod 4$, Γ_0 is described in (3(a)i) of Theorem 1.1, while if $q \equiv 3 \pmod 4$, Γ_0 is described in (3(b)i) of Theorem 1.1, and in both cases $Z_0 = Z(G)$. In particular,

$|A_0| = \delta |Z(G)|$ with $\delta \in \{2, 4\}$. Therefore for any other edge-transitive lattice $\Gamma = A_1 *_{A_0} A_2$ in G, we have

$$\mu(\Gamma \backslash G) \geq \mu(\Gamma_0 \backslash G) = \frac{2}{(q+1)|Z(G)|\delta}$$

where $\delta \in \{2, 4\}$ as described above. And so among all the *edge-transitive* cocompact lattices in G, the lattice Γ_0 with edge group A_0 of order $|Z(G)|\delta$ has the *smallest possible covolume*.

Now take Γ to be a cocompact, not necessarily edge-transitive, lattice in G. What happens then?

Lemma 6.3. *Let G be as in Theorem 1.1(3), $q \geq 300$, and assume that G admits an edge-transitive lattice. Assume further that Conjecture 1 holds. Then an edge-transitive lattice is a cocompact lattice in G of minimal covolume.*

Again, the discussion above together with Lemma 6.3 proves Theorem 1.4 for this case.

Proof of Lemma 6.3. We consider the case $q \equiv 3 \pmod 4$ in Section 6.3.1 and then the case $q \equiv 1 \pmod 4$ in Section 6.3.2.

6.3.1. *Proof of Lemma 6.3 when $q \equiv 3 \pmod 4$.* Consider $L_i = M_i T$. Then $M_i \triangleleft L_i$, $T_i = C_T(M_i)$ is a homomorphic image of C_{q-1}, $T_i / Z(M_i)$ is a cyclic group of order dividing $\frac{q-1}{2}$ and there exists an involution $t_i \in T$ that induces an outer-diagonal (PGL_2-) automorphism on M_i. Notice that even if $Z(G) = 1$, this time it is possible to have $Z(M_i) \neq 1$. However, if $Z(G) = 1$, T_i acts faithfully on M_j where $\{i, j\} = \{1, 2\}$ for $C_{T_i}(M_j) \leq C_T(\langle M_1, M_2 \rangle) \leq Z(G)$. Again we assume without loss of generality that $Z(G) = 1$, and we suppose that Γ is a cocompact lattice in G whose covolume is strictly less than that of Γ_0 (discussed above).

A few more comments: if $Z(M_i) = 1$, $|T|_2 = 2$ and so there exists a unique involution $t \in T$ that induces a PGL_2–automorphism on both M_1 and M_2. As above, if $Z(M_i) = 1$, $\delta = 2$, while if $|Z(M_i)| = 2$, $\delta = 4$. The covolume of Γ_0 is then $\frac{1}{(q+1)\delta}$. Put $\delta_0 := \frac{\delta}{2}$.

If $\Gamma_{x_i} \cap T_i \leq Z(M_i)$, then

$$\Gamma_{x_i} Z(M_i) / Z(M_i) \cong \Gamma_{x_i} T_i / T_i \leq M_i T / T_i \cong PGL_2(q).$$

In particular, $|\Gamma_{x_i}| \leq (q+1)\delta$. Therefore if $\Gamma_{x_i} \cap T_i \leq Z(M_i)$ for both $i = 1, 2$, it follows that $\mu(\Gamma \backslash G) \geq \mu(\Gamma_0 \backslash G)$, a contradiction. Hence without loss of generality we may assume that there exists $1 \neq y_1 \in \Gamma_{x_1} \cap T_1 - Z(M_1)$ such that $\langle y_1 \rangle = O(\Gamma_{x_1} \cap T_1)$. (Recall that for a finite group H, $O(H)$ denotes the largest normal subgroup of H of odd order.) Notice that $o(y_1) \mid \frac{q-1}{2}$ and $\langle y_1 \rangle$ acts faithfully on M_2 via inner automorphisms. Recall that as $y_1 \in T \cap \Gamma_{x_1}$, $y_1 \in \Gamma_{x_2}$. And so Γ_{x_2} acts on M_2 either as a subgroup of a normaliser of a split torus of $M_2 \langle t_2 \rangle$, or as a subgroup of K_2 where $K_2 / Z(K_2) \in \{S_4, A_5\}$. Note that in the latter case $o(y_1) \in \{3, 5\}$.

Assume first that $\Gamma_{x_2} \cap T_2 \leq Z(M_2)$. Then Γ_{x_2} is isomorphic to a subgroup of $M_2 \langle t_2 \rangle$. If Γ_{x_2} is isomorphic to a subgroup of K_2, then using the previous paragraph we obtain that $\mu(\Gamma \backslash G) \geq \frac{1}{60\delta_0} + \frac{1}{5 \cdot 2 \cdot (q+1)\delta_0} > \frac{2}{2(q+1)\delta_0} = \frac{2}{(q+1)\delta}$ for $q > 53$. Since this obviously contradicts the minimality of covolume of Γ, Γ_{x_2} must be isomorphic to a subgroup of a normaliser of a split torus in $M_2 \langle t_2 \rangle$.

Since we have full information about the action of L_i on the set of edges $E_X(x_i)$ coming out of x_i, $i = 1, 2$, let us first consider the action of Γ_{x_1} on $E_X(x_1)$. Since T_1 fixes $E_X(x_1)$ pointwise, we are interested in the action of the projection of Γ_{x_1} on $M_1 \langle t_1 \rangle$. Assume first that Γ_{x_1} acts transitively on $E_X(x_1)$. Then in the quotient graph S, x_1 has a unique neighbour vertex x_2. Consider the set of neighbours of x_2 in S. If $VS = \{x_1, x_2\}$, the quotient graph corresponds to the edge-transitive lattice, and we are done. Therefore x_2 has at least two neighbours vertices in S. Let $z_1, \ldots, z_k$ be representatives of the neighbouring vertices of x_2 in S other than x_1. If each Γ_{z_i} acts transitively on the set of edges $E_X(z_i)$ coming out of z_i, then we are looking at the whole $VS = \{z_1, \ldots, z_k, x_2, x_1\}$. If for some i, $C_{\Gamma_{z_i}}(M_{z_i}) \leq Z(M_i)$, Γ_{z_i} is isomorphic to a subgroup of $PGL_2(q)$ and so $|\Gamma_{z_i}| \leq (q+1)\delta$. Hence $\mu(\Gamma \backslash G) \geq \mu(\Gamma_0 \backslash G)$, a contradiction. Therefore for all i, $C_{\Gamma_{z_i}}(M_{z_i}) \nleq Z(M_i)$. Denote by y_{z_i} an element of Γ_{z_i} such that $\langle y_{z_i} \rangle = O(C_{\Gamma_{z_i}}(M_{z_i}))$. Then just like y_1, y_{z_i} acts faithfully on M_2 and $y_{z_i} \in \Gamma_{x_2}$. But Γ_{x_2} is a subgroup of the normaliser of the split torus of M_2, and so $[y_1, y_{z_i}] = 1$. Again using the fact that we know how L_2 acts on the set $E_X(x_2)$, we observe that y_1 must fix the edge (x_2, z_i). It follows that y_1 fixes all vertices of S, and so is in the kernel of the action of

Γ, a contradiction. Therefore, either Γ_{x_1}, or some Γ_{z_i} does not act transitively on the corresponding set of edges. Thus by taking $x_1 = z_i$ if necessary, we reduce to the following case: Γ_{x_1} has at least two orbits on $E_X(x_1)$.

Therefore, either x_1 has a unique neighbouring vertex x_2 in S but the number of edges between x_1 and x_2 in S is greater than one, or there are at least two neighbouring vertices of x_1 in the quotient graph. Let us discuss these two cases. In the former one, if $|VS| = 2$, $\langle y_1 \rangle \lhd \Gamma$ which is a contradiction since Γ is a faithful lattice. It follows that in the former case $|VS| > 2$ and so x_2 has more than one neighbouring vertex in S: $z_1, \ldots, z_k$. Again not to contradict the minimality of covolume of Γ, we may assume that $C_{\Gamma_{z_i}}(M_{z_i}) \not\leq Z(M_i)$. Using the same argument as above we obtain that $y_1 \in \Gamma_{z_i}$. Assume that y_1 acts on M_{z_i} as an element of K_{z_i} where $K_{z_i}/Z(K_{z_i}) \in \{S_4, A_5\}$.

Suppose first that $o(y_1) \leq 15$. Let us evaluate the covolume of Γ. Let $a_2 \in \Gamma_{x_2}$ be such that $\langle a_2 \rangle = O(\Gamma_{x_2})$. Then $y_1 \in \langle a_2 \rangle$. If $o(y_1) = o(a_2)$, $|\Gamma_{x_2}| \leq 30\delta$ which immediately contradicts the minimality of covolume of Γ $\left(\frac{1}{30\delta} > \frac{2}{(q+1)\delta} \text{ as long as } q > 59\right)$. Hence, $o(a_2) > o(y_1)$. Now, consider an action of Γ_{x_1} on M_1. If it acts as a subgroup of K_1 with $K_1/Z(K_1) \in \{S_4, A_5\}$, then $|\Gamma_{x_1}| \leq 15 \cdot 60\delta_0$ and in particular, $|\Gamma_{x_1} \cap T| \leq 15 \cdot 5\delta = 75\delta$. But $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$, and so $|\Gamma_{x_2} \cap T| \leq 75\delta$. As Γ_{x_2} acts on M_2 as a subgroup of a normaliser of split torus, $|\Gamma_{x_2}| \leq 2 \cdot 75\delta = 150\delta$. Therefore, $\mu(\Gamma \backslash G) \geq \frac{1}{900\delta_0} + \frac{1}{150\delta} \geq \frac{2}{(q+1)\delta}$ for $q \geq 225$, a contradiction with the minimality of covolume of Γ. If it acts as a subgroup of a normaliser of a non-split torus, then together with the action a_2, we will get that Γ_{x_1} will act as a subgroup containing p–elements, a contradiction. Thus it can only act on M_1 as a subgroup of a normaliser of a split torus, and so $|\Gamma_{x_1}| \leq o(y_1) 2 \frac{o(a_2)}{o(y_1)} \delta \leq (q-1)\delta$. Since $|\Gamma_{x_2}| \leq (q-1)\delta$, we again get a contradiction with the minimality of the covolume of Γ. Hence, $o(y_1) > 15$ and so there exists $y_1' \in \langle y_1 \rangle$ such that $[y_1', M_{z_i}] = 1$ for all i. In this case we will use y_1' instead of y_1 in all the previous discussions. Therefore for all i, either y_1 centralises M_{z_i} or acts on it as a normal subgroup of a normaliser of a split torus. In both situations, $\langle y_1 \rangle$ is normal in Γ_{z_i}, for $i = 1, \ldots, k$. Continuing with this argument we obtain that $\langle y_1 \rangle$ is a normal subgroup of Γ which implies that y_1 is in the kernel of the action of Γ on X, a contradiction.

Therefore we may suppose that x_1 has more than one neighbouring vertex in S. One is x_2, and let z be among the other neighbouring vertices of x_1. If $C_{\Gamma_z}(M_z) \leq Z(M_z)$, then using $|\Gamma_{x_2}|$ and $|\Gamma_z|$, we obtain a contradiction with the minimality of $\mu(\Gamma \backslash G)$. Therefore $C_{\Gamma_z}(M_z) \not\leq Z(M_z)$ and we may take $x_2 = z$. Therefore, we may assume that $\Gamma_{x_2} \cap T_2 \not\leq Z(M_2)$. It follows that there exists $y_2 \in \Gamma_2 \cap T_2$ with $\langle y_2 \rangle = O(\Gamma_2 \cap T_2)$. As before notice that $y_2 \in \Gamma_{x_1}$ and $\langle y_2 \rangle$ acts faithfully on M_1 via inner automorphisms. Also, the subgroup structure of L_1 implies that either Γ_{x_1} acts on M_1 as a subgroup of K_1 where $K_1/Z(K_1) \in \{S_4, A_5\}$, or as a subgroup of a normaliser of a split torus of $M_1 \langle t_1 \rangle$. Assume the former, i.e., let Γ_{x_1} act on M_1 as a subgroup of K_1. If Γ_{x_2} acts on M_2 as a subgroup of K_2 where $K_2/Z(K_2) \in \{S_4, A_5\}$, then $\mu(\Gamma \backslash G) \geq \frac{2}{5 \cdot 60\delta_0} > \frac{2}{(q+1)\delta} = \mu(\Gamma_0 \backslash G)$ for $q \geq 300$, a contradiction. Hence, Γ_{x_2} acts on M_2 as a subgroup of a normaliser of a split torus. Recall that $|VS| > 2$ and x_1 has neighbouring vertices in S other than x_2. Let us call them $v_1, \ldots, v_k$. Arguing for v_i as we did for x_2, we obtain that $\langle y_1 \rangle$ acts faithfully on M_{v_i} and that Γ_{v_i} acts on M_{v_i} as a normaliser of a non-split torus. It follows that $\langle y_1 \rangle$ is normal in each Γ_{v_i}.

Now let $z_1, \ldots, z_m$ be the neighbouring vertices of x_2 in $S - \{x_1\}$. Let us consider $\Gamma_{z_i} = \Gamma \cap P_{z_i}$. If $C_{\Gamma_{z_i}}(M_{z_i}) \leq Z(M_{z_i})$ for some i, then there is at most one such vertex, for otherwise we would contradict the minimality of covolume of Γ. Assume there exists such a vertex z_i, then without loss of generality assume that $i = 1$, i.e., $C_{\Gamma_{z_1}}(M_{z_1}) \leq Z(M_{z_1})$ and for $i > 1$, $C_{\Gamma_{z_i}}(M_{z_i}) \not\leq Z(M_{z_i})$. Further, we may assume that $T \leq P_{z_1}$. Then $y_1, y_2 \in \Gamma_{z_1}$. If Γ_{z_1} acts on M_{z_1} as a subgroup of K_{z_1} where $K_{z_1}/Z(K_{z_1}) \in \{S_4, A_5\}$, then $|\Gamma_{z_1}| \leq 60\delta_0$, which leads to the usual contradiction. Thus Γ_{z_1} acts on M_{z_1} as a subgroup of a normaliser of a split torus, and so $\langle y_1 \rangle \lhd \Gamma_{z_1}$. Now, let us look at Γ_{z_i} for $i \geq 2$ (or it is possible that the previous case does not happen, then we are looking at Γ_{z_i} for $i \geq 1$). It follows that for each i there exists y_{z_i} such that $\langle y_{z_i} \rangle = O(C_{\Gamma_{z_i}}(M_{z_i}))$. Since y_{z_i} sits in the kernel of L_{z_i} in its action on $E_X(z_i)$, y_{z_i} fixes every vertex neighbouring z_i, and in particular fixes x_2. It follows that $y_{z_i} \in \Gamma_{x_2}$. On the other hand, by the usual argument, y_2 acts faithfully on M_{z_i} implying $[y_2, y_{z_i}] = 1$. Therefore, $\langle y_{z_i} \rangle \lhd \Gamma_{x_2}$. Finally, as $C_{\Gamma_{x_2}}(M_2)$ stabilises every (x_2, z_i), it follows that $C_{\Gamma_{x_2}}(y_{z_i}) \leq \Gamma_{z_i}$. It follows that $y_1 \in \Gamma_{z_i}$ and so $\langle y_1, y_2 \rangle \leq \Gamma_{z_i}$. Suppose y_1 acts on M_{z_i} as an element of a subgroup of K_{z_i} with $K_{z_i}/Z(K_{z_i}) \in \{S_4, A_5\}$. If $o(y_1) \leq 15$,

$|\Gamma_{x_1}| \leq 15 \cdot 60\delta_0$ and in particular, $|\Gamma_{x_1} \cap T| \leq 15 \cdot 5\delta = 75\delta$. Now, as we noticed earlier, $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$, and so $|\Gamma_{x_2} \cap T| \leq 75\delta$. As Γ_{x_2} acts on M_2 as a subgroup of a normaliser of split torus, $|\Gamma_{x_2}| \leq 2 \cdot 75\delta = 150\delta$, leading to the usual contradiction for $q \geq 300$. Hence, $o(y_1) > 15$ and so there exists $y_1' \in \langle y_1 \rangle$ such that $[y_1', M_{z_i}] = 1$ for all is. In this case we will use y_1' instead of y_1 in all the previous discussions. By carefully iterating this argument, we obtain that there exists a nontrivial element y of odd order such that $\langle y \rangle \triangleleft \Gamma$, a contradiction.

We are now reduced to the case when Γ_{x_1} acts on $M_1\langle t_1 \rangle$ as a subgroup of $N_{M_1\langle t_1 \rangle}(T)$. Notice, that because of symmetry between x_1 and x_2, to finish the analysis it remains to consider the case when Γ_{x_2} acts on $M_2\langle t_2 \rangle$ as a subgroup of $N_{M_2\langle t_2 \rangle}(T)$. But in this case $\langle \Gamma_{x_1}, \Gamma_{x_2} \rangle \leq N$. Hence, we may move to the next vertex y on our graph. However, again, the only possible case will be $\Gamma_y \leq N$, and so on and so forth. Therefore, in the end of this case, the only possible conclusion will be $\Gamma \leq N$, which is a contradiction as N is not a cocompact lattice of G, nor does it contain one. This completes the proof of Lemma 6.3 when $q \equiv 1$ (mod 4).

6.3.2. *Proof of Lemma 6.3 when $q \equiv 1$ (mod 4).* Again consider $L_i = M_i T$, $i = 1, 2$. As in the case $q \equiv 3$ (mod 4), $M_i \triangleleft L_i$, T_i is a homomorphic image of C_{q-1} and $T_i/Z(M_i)$ is a cyclic group of order dividing $\frac{q-1}{2}$. But this time if x is an involution in $L_i \cap T$, then $x \in M_i T_i$. As usual we may suppose that $Z(G) = 1$. As before, it follows that T_i must act faithfully on M_j, $\{i, j\} = \{1, 2\}$.

Recall that $Q_i \in \mathcal{S}yl_2(Z(L_i))$ and Q_i^2 is the unique subgroup of Q_i of index 2. Since we assume that G admits an edge-transitive lattice as in the conclusion (3(a)i) of Theorem 1.1 and that $Z(G) = 1$, $T_i = Z_i \times Q_i$ where $|Q_i| \leq 2$, $|Z_i|$ is odd and $M_i \cong PSL_2(q)$. If $|Q_i| = 2$, $\delta = 4$, while if $|Q_i| = 1$, $\delta = 2$. The covolume of Γ_0 is then $\frac{2}{(q+1)\delta}$. Notice that if $|Q_i| = 1$, the lattice in the conclusion (3(a)ii) of Theorem 1.1 also exists, but its covolume is twice $\mu(\Gamma_0 \backslash G)$. Put $\delta_0 := \frac{\delta}{2}$.

As in the previous case, let us assume that Γ is a cocompact lattice in G whose covolume is strictly less than that of Γ_0. If $\Gamma_{x_i} \cap T_i \leq Q_i$, then

$$\Gamma_{x_i} Q_i / Q_i \cong \Gamma_{x_i} T_i / T_i \leq M_i T / T_i \cong PGL_2(q).$$

In particular, $|\Gamma_{x_i}| \leq (q+1)\delta_0$. Therefore if $\Gamma_{x_i} \cap T_i \leq Q_i$ for both $i = 1, 2$, it follows that $\mu(\Gamma \backslash G) \geq \mu(\Gamma_0 \backslash G)$, a contradiction. Hence without loss of generality we may assume that there exists $1 \neq y_1 \in \Gamma_{x_1} \cap T_1$ such that $\langle y_1 \rangle = O(\Gamma_{x_1} \cap T_1)$. Note that $1 \neq o(y_1)$ is odd, $o(y_1) \mid \frac{q-1}{2}\delta_0$ and $\langle y_1 \rangle$ acts faithfully on M_2. Recall that as $y_1 \in T \cap \Gamma_{x_1}$, $y_1 \in \Gamma_{x_2}$. And so there are several possibilities for the action of Γ_{x_2} on M_2: either as a subgroup of a normaliser of a split torus, or as a subgroup of K_2 where $K_2 \in \{S_4, A_5\}$, in which case $o(y_1) \leq 5$.

Assume first that $\Gamma_{x_2} \cap T_2 \leq Q_2$. Then Γ_{x_2} is isomorphic to a subgroup of $2PGL_2(q)$. If Γ_{x_2} is isomorphic to a subgroup of K_2, we obtain the usual contradiction with the minimality of covolume, as $\frac{1}{60} > \frac{2}{(q+1)\delta}$ for $q \geq 60$. Hence Γ_{x_2} acts on M_2 a subgroup of a normaliser of a split torus and $O(\Gamma_{x_2}) \neq 1$.

Since we have full information about the action of L_i on the set of edges $E_X(x_i)$ coming out of x_i, $i = 1, 2$, let us first consider the action of Γ_{x_1} on $E_X(x_1)$. Since T_1 fixes $E_X(x_1)$ pointwise, we are interested in the action of the projection of Γ_{x_1} on $M_1\langle t_1 \rangle$. Assume first that Γ_{x_1} acts transitively on $E_X(x_1)$. Then in the quotient graph S, x_1 has a unique neighbouring vertex x_2. Consider the set of neighbours of x_2 in S. If $VS = \{x_1, x_2\}$, the quotient graph corresponds to the edge-transitive lattice, and we are done. Therefore, there exists $x_2 \in VS$ which has at least two neighbouring vertices in S. Let $z_1, \ldots, z_k$ be representatives of the neighbouring vertices of x_2 in S other than x_1. If each Γ_{z_i} acts transitively on the set of edges $E_X(z_i)$ coming out of z_i, then we are looking at the whole $VS = \{z_1, \ldots, z_k, x_2, x_1\}$. If for some i, $C_{\Gamma_{z_i}}(M_{z_i}) \leq Q_i$, then $|\Gamma_{z_i}| \leq (q+1)\delta$. Looking at $\frac{1}{|\Gamma_{x_2}|} + \frac{1}{|\Gamma_{z_i}|}$, we obtain that $\mu(\Gamma \backslash G) \geq \mu(\Gamma_0 \backslash G)$, a contradiction. Therefore for all i, $C_{\Gamma_{z_i}}(M_{z_i}) \not\leq Q_i$. Hence there exists y_{z_i}, an element of Γ_{z_i}, such that $\langle y_{z_i} \rangle = O(C_{\Gamma_{z_i}}(M_{z_i}))$. Then just like y_1, y_{z_i} acts faithfully on M_2 and $y_{z_i} \in \Gamma_{x_2}$. But Γ_{x_2} is a subgroup of the normaliser of the split torus of M_2, and so $[y_1, y_{z_i}] = 1$. Again using the fact that we know how L_2 acts on the set $E_X(x_2)$, we observe that y_1 must fix the edge $[x_2, z_i]$. It follows that y_1 fixes all vertices of S, and so is in the kernel of the action of Γ, a contradiction. Therefore, either Γ_{x_1}, or some Γ_{z_i} does not act transitively on the corresponding set

of edges. Thus by taking $x_1 = z_i$ if necessary, we reduce to the following case: Γ_{x_1} has at least two orbits on $E_X(x_1)$.

Therefore either x_1 has a unique neighbouring vertex x_2 in S but the number of edges between them in S is greater than one, or there are at least two neighbouring vertices of x_1 in the quotient graph. Let us discuss those two cases. In the former one, if $|VS| = 2$ then $\langle y_1 \rangle \lhd \Gamma$, which is a contradiction since Γ is a faithful lattice. It follows that if x_1 has a unique neighbouring vertex x_2 is S then $|VS| > 2$, and so x_2 has more than one neighbouring vertex in S. Let us denote these neighbours (other than x_1) by $z_1, \ldots, z_k$. Again not to contradict the minimality of covolume of Γ, we may assume that $C_{\Gamma_{z_i}}(M_{z_i}) \nleq Q_i$. Using the same argument as above we obtain that $y_1 \in \Gamma_{z_i}$. Assume that y_1 acts on M_{z_i} as an element of K_{z_i} where $K_{z_i} \in \{S_4, A_5\}$. If $o(y_1) \leq 15$, let us evaluate the covolume of Γ. Let $a_2 \in \Gamma_{x_2}$ be such that $\langle a_2 \rangle = O(\Gamma_{x_2})$. Then $y_1 \in \langle a_2 \rangle$. If $o(y_1) = o(a_2)$, $|\Gamma_{x_2}| \leq 30\delta$ which immediately contradicts the minimality of covolume of Γ ($\frac{1}{30\delta} > \frac{2}{(q+1)\delta}$ for $q > 59$). Hence, $o(a_2) > o(y_1)$. Now, consider the action of Γ_{x_1} on M_1. If it acts as a subgroup of K_1 where $K_1 \in \{S_4, A_5\}$, then $|\Gamma_{x_1}| \leq 15 \cdot 60\delta_0$, and arguing as in the previous case ($q \equiv 3 \pmod 4$), we get $|\Gamma_{x_2}| \leq 150\delta$ leading to the usual contradiction with the minimality of covolume of Γ as long as $q \geq 257$. If it acts as a subgroup of a normaliser of a non-split torus, then together with the action of a_2, we will get that Γ_{x_1} will act as a subgroup containing p–elements, a contradiction. Thus it can only act on M_1 as a subgroup of a normaliser of a split torus, and so $|\Gamma_{x_1}| \leq o(y_1) 2 \frac{o(a_2)}{o(y_1)} \delta \leq (q-1)\delta$. Since $|\Gamma_{x_2}| \leq (q-1)\delta$, we again get a contradiction with the minimality of the covolume of Γ. Now suppose that $o(y_1) > 15$, then there exists y_1' such that $y_1' \in \langle y_1 \rangle$ and y_1' centralises all the M_{z_i}s. We may use it instead of y_1. Therefore for all i, either y_1 centralises M_{z_i} or acts on it as a normal subgroup of a normaliser of a split torus. In both cases, $\langle y_1 \rangle$ is normal in Γ_{z_i}, $i = 1, \ldots, k$. Continuing with this argument we obtain that $\langle y_1 \rangle$ is a normal subgroup of Γ which implies that y_1 is in the kernel of the action of Γ on X, a contradiction.

Therefore we may suppose that x_1 has more than one neighbouring vertex in S. One is x_2, and let z be among the other neighbouring vertices of x_1. If $C_{\Gamma_z}(M_z) \leq Q_z$, then using $|\Gamma_{x_2}|$ and $|\Gamma_z|$, we obtain a contradiction with the minimality of $\mu(\Gamma \backslash G)$: $\mu(\Gamma \backslash G) \geq \frac{1}{|\Gamma_{x_2}|} + \frac{1}{|\Gamma_z|} \geq \frac{2}{(q+1)\delta}$. Therefore $C_{\Gamma_z}(M_z) \nleq Q_z$ and we may take $x_2 = z$. Therefore, we may assume that $\Gamma_{x_2} \cap T_2 \neq 1$. It follows that there exists $y_2 \in O(\Gamma_2 \cap T_2)$ such that $1 \neq \langle y_2 \rangle = O(\Gamma_2 \cap T_2)$.

As before notice that $y_2 \in \Gamma_{x_1}$ and $\langle y_2 \rangle$ acts faithfully on M_1 via inner automorphisms. Also, the subgroup structure of L_1 implies that either Γ_{x_1} acts on M_1 as a subgroup of K_1 where $K_1 \in \{S_4, A_5\}$, or as a subgroup of a normaliser of a split torus of $M_1 \langle t_1 \rangle$.

Assume the former, i.e., let Γ_{x_1} act on M_1 as a subgroup of K_1. If $o(y_1) = 2$, then $|\Gamma_{x_1}| \leq 2 \cdot 60$ which as always implies that $\mu(\Gamma \backslash G)$ is not minimal. Hence, $O(\Gamma_{x_1} \cap T_1) \neq 1$ and we may choose y_1 to be such that $\langle y_1 \rangle = O(\Gamma_{x_1} \cap T_1)$. As always notice that $y_1 \in \Gamma_{x_2}$ and acts faithfully on M_2 via inner automorphisms. If Γ_{x_2} acts on M_2 as a subgroup of K_2 where $K_2 \in \{S_4, A_5\}$, then $\mu(\Gamma \backslash G) \geq \frac{2}{5 \cdot 60\delta_0} > \frac{2}{(q+1)\delta} = \mu(\Gamma_0 \backslash G)$ for $q \geq 300$, a contradiction.

Hence Γ_{x_2} acts on M_2 as a subgroup of a normaliser of a split torus. Recall that $|VS| > 2$ and x_1 has neighbouring vertices other than x_2 in S. Let us call them $v_1, \ldots, v_k$. Arguing for v_i as we did for x_2, we obtain that $\langle y_1 \rangle$ acts faithfully on M_{v_i} and that Γ_{v_i} acts on M_{v_i} as a normaliser of a non-split torus. It follows that $\langle y_1 \rangle$ is normal in each Γ_{v_i}. Now let $z_1, \ldots, z_m$ be the neighbouring vertices of x_2 in $VS - \{x_1\}$. Let us consider $\Gamma_{z_i} = \Gamma \cap P_{z_i}$. If $C_{\Gamma_{z_i}}(M_{z_i}) \leq Q_{z_i}$ for some i, then there is at most one such vertex, for otherwise we would contradict the minimality of covolume of Γ. If there exists such a vertex, then without loss of generality assume that such $i = 1$, i.e., $C_{\Gamma_{z_1}}(M_{z_1}) \leq Q_{z_1}$, and for $i > 1$, $C_{\Gamma_{z_i}}(M_{z_i}) \nleq Q_{z_i}$. Further, we may assume that $T \leq P_{z_1}$. Then $y_1, y_2 \in \Gamma_{z_1}$. If Γ_{z_1} acts on M_{z_1} as a subgroup of K_{z_1} where $K_{z_1} \in \{S_4, A_5\}$, then $|\Gamma_{z_1}| \leq 60\delta_0$, which leads to the usual contradiction as long as $q > 59$. Thus Γ_{z_1} acts on M_{z_1} as a subgroup of a normaliser of a split torus, and so $\langle y_1 \rangle \lhd \Gamma_{z_1}$. Now, let us look at Γ_{z_i} for $i \geq 2$ (or it is possible that the previous case does not happen, then we are looking at Γ_{z_i} for $i \geq 1$). It follows that for each $i \geq 1$ (or strictly greater than 1 if $o(y_{z_1}) \mid 2$) there exists y_{z_i} such that $\langle y_{z_i} \rangle = O(C_{\Gamma_{z_i}}(M_{z_i}))$.

Since y_{z_i} sits in the kernel of L_{z_i} in its action on $E_X(z_i)$, y_{z_i} fixes every vertex neighbouring z_i, and in particular fixes x_2. It follows that $y_{z_i} \in \Gamma_{x_2}$. On the other hand, by the usual argument, y_2 acts faithfully on M_{z_i} implying $[y_2, y_{z_i}] = 1$. Therefore, $\langle y_{z_i} \rangle \lhd \Gamma_{x_2}$. Finally, as $C_{\Gamma_{x_2}}(M_2)$ stabilises every edge $[x_2, z_i]$, it

follows that $C_{\Gamma_{x_2}}(y_{z_i}) \leq \Gamma_{z_i}$. It follows that $y_1 \in \Gamma_{z_i}$ and so $\langle y_1, y_2 \rangle \leq \Gamma_{z_i}$. Suppose y_1 acts on M_{z_i} as an element of a subgroup of K_{z_i} with $K_{z_i} \in \{S_4, A_5\}$. By the argument identical to the one for $q \equiv 3 \pmod 4$, we obtain that it is not possible and that there exists a nontrivial element y of odd order such that $\langle y \rangle \lhd \Gamma$, a contradiction.

We are now reduced to the case when Γ_{x_1} acts on $M_1 \langle t_1 \rangle$ as a subgroup of $N_{M_1 \langle t_1 \rangle}(M_1 \langle t_1 \rangle \cap T)$. Notice that because of symmetry between x_1 and x_2, to finish the analysis it remains to consider the case when Γ_{x_2} acts on $M_2 \langle t_2 \rangle$ as a subgroup of $N_{M_2 \langle t_2 \rangle}(M_2 \langle t_2 \rangle \cap T)$. But in this case $\langle \Gamma_{x_1}, \Gamma_{x_2} \rangle \leq N$. Hence we may move to the next vertex y on our graph. However, again, the only possible case will be $\Gamma_y \leq N$, and so on and so forth. Therefore, in the end of this case, the only possible conclusion will be $\Gamma \leq N$, which is a contradiction as N is not a cocompact lattice of G, nor does it contain one. This completes the proof of Lemma 6.3, and thus the proof of Theorem 1.4 when G admits an edge-transitive lattice. $\qquad\square$

6.4. Non-edge transitive case.

Let G be as in Theorem 1.4 above, and assume that G does not admit any edge-transitive lattices. In this section we compute the covolume of the lattice Γ' constructed in Section 3.3 above, which is not edge-transitive, and then prove that for $q \geq 514$, the lattice Γ' is the cocompact lattice of minimal covolume in G. The important difference with the edge-transitive case is that

$$q \equiv 1 \pmod 4 \quad \text{while} \quad Q_i^2 \nleq Z(G).$$

Hence certain arguments that worked very well in the cases $q = 2^a$ and $q \equiv 3 \pmod 4$ and even for $q \equiv 1 \pmod 4$ above, now fail and/or become considerably longer and more delicate. Thus we produce them carefully and at length.

From the construction of Γ' in Section 3.3 and the discussion of covolumes in Section 2.3, it follows that the covolume of Γ' is given by

$$\mu(\Gamma' \backslash G) = \frac{1}{|\mathcal{A}_{x_1}|} + \frac{1}{|\mathcal{A}_{x_2}|} = \frac{1}{|SH_1|} + \frac{1}{|SH_2|}.$$

Recall that $S \cap H_i = Z(M_i) \leq Q_0$ and $|H_i : Z(M_i)| = \frac{q+1}{2}$. Hence

$$|SH_i| = \frac{|S||H_i|}{|S \cap H_i|} = |S| \frac{q+1}{2} = |Z(G)||Q_0 : (Q_0 \cap Z(G))| \frac{q+1}{2}.$$

Since $|Q_0 : (Q_0 \cap Z(G))| = 2\delta$ where $\delta \in \{1, 2\}$ and its precise value depends on G, we obtain that

$$\mu(\Gamma' \backslash G) = \frac{2}{\delta |Z(G)|(q+1)} \quad \text{with } \delta \in \{1, 2\} \text{ depending on } G. \tag{1}$$

Let us now discuss the issue of minimality of covolume of Γ'. Our goal now is to prove the following statement.

Lemma 6.4. *Let G be as in Theorem 1.1, $q \geq 514$ and assume that G does not admit an edge-transitive lattice. Assume further that Conjecture 1 holds. Then Γ' is a cocompact lattice in G of minimal covolume.*

Assume that there is a cocompact lattice Γ in G whose covolume $\mu(\Gamma \backslash G)$ is strictly smaller than the covolume of Γ' given above.

Remark. Notice that $T \leq P_1 \cap P_2$ together with $\Gamma \cap P_i = \Gamma_{x_i}$ yields $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$.

As in the edge-transitive case, our plan is to subdivide further into the subcases:

Subcase 1 : *For $i = 1, 2$, $L_i / Z(L_i) \cong PSL_2(q)$,* and **Subcase 2** : *For $i = 1, 2$, $L_i / Z(L_i) \cong PGL_2(q)$.*

We consider Subcase 1 in Section 6.5 and Subcase 2 in Section 6.6. Again, to simplify arguments we assume that G has trivial centre, that is, the finite group $Z(G)$ satisfies $|Z(G)| = 1$.

6.5. **Non-edge-transitive case, Subcase 1.** In this case $L_i = M_i \circ T_i$, that is, L_i is a central (commuting) product of M_i and $T_i = C_T(M_i)$. Moreover, if an element of T centralises a non-split torus of M_i, then from the structure of M_i and L_i, it follows immediately that it centralises M_i. Now, $Z(G) = 1$ implies that $T_i \cap T_j = 1$ and T_i acts faithfully on M_j with $\{i, j\} = \{1, 2\}$. Let us make a few more comments about the structure of the L_is. Recall that for a finite group F, $O_2(F)$ denotes the largest normal 2–subgroup of F.

Suppose first that $L_i = M_i \times T_i$. Assume that $Z(M_i) \neq 1$, i.e., $M_i \cong SL_2(q)$. Then $1 \neq Q_0 \leq C_G(M_i)$ for $i = 1, 2$, and so $Q_0 \leq C_G(\langle M_1, M_2 \rangle) \leq Z(G) = 1$, a contradiction. Thus if $L_i = M_i \times T_i$, $M_i \cong PSL_2(q)$. Moreover, as far as the value of our parameter δ is concerned, it follows immediately that $|T_i|$ is odd whenever $\delta = 1$, and $|T_i|$ is even whenever $\delta = 2$. (In particular, in the key example $G = PSL_2(\mathbb{F}_q((t)))$, $|T_i| = 1$ and $\delta = 1$.) Furthermore, T_i must act faithfully on M_j and so T_i must be isomorphic to a subgroup of M_j. It follows that $|T_i|$ divides $\frac{q-1}{2}$. Suppose now that $Z(M_i) \neq 1$, $T_i \cap M_i \neq 1$, $4 \mid |T_i|$ and $L_i = M_i \circ_{\langle -I \rangle} T_i$. In particular, $M_i \cong M_j \cong SL_2(q)$. Choose an element $g_i \in T \cap M_i$ of order $(q-1)$. Then $\langle g_i^{\frac{q-1}{2}} \rangle = Z(M_i)$. Since $g_i \in T$, it follows that $g_i \in L_j$ and $g_i^{\frac{q-1}{2}}$ must act faithfully on M_j. Thus $O_2(\langle g_i \rangle)$ acts faithfully on M_j via inner automorphisms, which is a contradiction since $O_2(\langle g_i \rangle) \cong C_{2-part\ of\ (q-1)}$ while $\mathrm{Inn}(SL_2(q)) = PSL_2(q)$ does not contain such a subgroup. Therefore, this case does not happen. Hence $M_i \cong PSL_2(q)$ and $L_i \cong T_i \times M_i$.

If $|\Gamma_{x_i} \cap T_i| \leq \delta$, then

$$(2) \qquad \Gamma_{x_i}/\Gamma_{x_i} \cap T_i \cong \Gamma_{x_i} T_i/T_i \leq M_i T_i/T_i \cong M_i \cong PSL_2(q).$$

Notice that $|\Gamma_{x_i} \cap T_i| = 2$ implies that $\delta = 2$. By Dickson's Theorem, it follows that $|\Gamma_{x_i}| \leq \delta(q+1)$. Therefore if $|\Gamma_{x_i} \cap T_i| \leq \delta$ for both $i = 1, 2$, it follows that $\mu(\Gamma \backslash G) \geq \mu(\Gamma' \backslash G)$, a contradiction (this is precisely the case in [LW, Example (6.2)] implying the minimality of the lattice constructed there). Hence, without loss of generality we may assume that there exists $1 \neq y_1 \in \Gamma_{x_1} \cap T_1$ such that $\langle y_1 \rangle = \Gamma_{x_1} \cap T_1$ with $o(y_1) > \delta$. Then $o(y_1) \mid \frac{q-1}{2}$ and $\langle y_1 \rangle$ acts faithfully on M_2 via inner automorphisms. Notice that if $\delta = 1$, $o(y_1) \neq 1$ is odd, and so for $\delta \in \{1, 2\}$, $o(y_1) \geq 3$. As noticed in Remark 6.4, since $y_1 \in \Gamma_{x_1} \cap T$, $y_1 \in \Gamma_{x_2}$. Thus Γ_{x_2} acts non-trivially on M_2. Now Dickson's Theorem asserts that Γ_{x_2} must act on M_2 either as a subgroup of a normaliser of a split torus of M_2, or as a subgroup of K_2 with $K_2 \in \{S_4, A_5\}$ (notice that in this case $|o(y_1)| \leq 5$).

Assume first that $|\Gamma_{x_2} \cap T_2| \leq \delta$. Then (2) implies that Γ_{x_2} is actually isomorphic to a subgroup of $PSL_2(q) \times C_\delta$. If Γ_{x_2} acts on M_2 as a subgroup of K_2, then using the previous paragraph we obtain that $\mu(\Gamma \backslash G) \geq \frac{1}{5 \cdot (q+1)} + \frac{1}{60\delta} > \frac{2}{\delta(q+1)} = \mu(\Gamma' \backslash G)$ for $q > 107$. Since this obviously contradicts the minimality of covolume of Γ, Γ_{x_2} must be acting on M_2 as a subgroup of a normaliser of a split torus of M_2. It follows that $\langle y_1 \rangle$ is normal in Γ_{x_2}.

We are now interested in the action of Γ_{x_1} on M_1. By abuse of notation, we identify x_i with its image in the quotient graph $A = \Gamma \backslash X$ for $i = 1, 2$. Then in the graph A, x_1 is a neighbour of x_2. If $|A| = 2$, it follows immediately that $\langle y_1 \rangle \lhd \Gamma$, a contradiction. And so $|A| > 2$. Let $z_1, \ldots, z_k$ be representatives of the other neighbouring vertices of x_2 in A. If for some i, $|C_{\Gamma_{z_i}}(M_{z_i})| \leq \delta$, $|\Gamma_{z_i}| \leq (q+1)\delta$. Hence, $\mu(\Gamma \backslash G) \geq \mu(\Gamma' \backslash G)$, a contradiction. Therefore for all i, $|C_{\Gamma_{z_i}}(M_{z_i})| > \delta$. Denote by y_{z_i} an element of Γ_{z_i} such that $\langle y_{z_i} \rangle = C_{\Gamma_{z_i}}(M_{z_i})$. We may use exactly the same arguments for y_{z_i} as we did for y_1. Then $o(y_{z_i}) \geq 3$, just like y_1, y_{z_i} acts faithfully on M_2 and $y_{z_i} \in \Gamma_{x_2}$. But Γ_{x_2} acts on M_2 as a subgroup of the normaliser of the split torus. Hence, $[y_1, y_{z_i}] = 1$. Using the fact that we know how L_2 acts on $E_X(x_2)$, we observe that y_1 then must fix the edge $[x_2, z_i]$. It follows that y_1 fixes $x_1, x_2, z_1, \ldots, z_k$ and that $y_1 \in \Gamma_{z_i}$.

Assume that y_1 acts on M_{z_i} as a non-trivial element of $K_{z_i} \in \{S_4, A_5\}$. Suppose first that $o(y_1) = 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3}$ where either $\beta_1 \leq 2$ and $\beta_i \leq 1$ for $i = 2, 3$ (this corresponds to $K_{z_i} \cong A_5$), or $\beta_1 \leq 3$, $\beta_2 \leq 1$ while $\beta_3 = 0$ (this is when $K_{z_i} \cong S_4$). Let us evaluate the covolume of Γ. Let $a_2 \in \Gamma_{x_2}$ be such that $\langle a_2 \rangle$ acts faithfully on M_2 and $\langle a_2 \rangle \leq \Gamma_{x_2} \leq (\langle a_2 \rangle \rtimes \langle s_2 \rangle)(\Gamma_{x_2} \cap T_2)$ where $\langle a_2 \rangle \rtimes \langle s_2 \rangle \cong D_{2 \cdot o(a_2)}$, $|\Gamma_{x_2} \cap T_2| \leq \delta$ and $\langle a_2 \rangle \times (\Gamma_{x_2} \cap T_2) = \Gamma_2 \cap T$. Then $a_2 \in \Gamma_{x_1}$ and without loss of generality we may assume that $y_1 \in \langle a_2 \rangle$. If $o(y_1) = o(a_2)$, then $|\Gamma_{x_2}| \leq (2^2 \cdot 3 \cdot 5) \cdot 2\delta$ which immediately contradicts the minimality of covolume of Γ ($\frac{1}{120\delta} \geq \frac{2}{(q+1)\delta}$ for $q \geq 240$). Hence, $o(a_2) > o(y_1)$ and so Γ_{x_1} acts on M_1 as either a subgroup of $K_1 \in \{S_4, A_5\}$, or as a subgroup of $N_{M_1}(M_1 \cap T)$.

In the former case $|\Gamma_{x_1}| \leq 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3} \cdot 60$. In particular, $|\Gamma_{x_1} \cap T| \leq 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3} \cdot 5$. As we noticed earlier, $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$, and so $|\Gamma_{x_2} \cap T| \leq 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3+1}$. Since Γ_{x_2} acts on M_2 as a subgroup of a normaliser of a split torus of M_2, $|\Gamma_{x_2}| \leq (2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3+1}) \cdot 2$. Note that if $\beta_1 = 1$, $\delta = 2$. If follows that $\mu(\Gamma \backslash G) > \frac{1}{2^{\beta_1+2} \cdot 3^{\beta_2+1} \cdot 5^{\beta_3+1}} + \frac{1}{2^{\beta_1+1} \cdot 3^{\beta_2} \cdot 5^{\beta_3+1}} \geq \frac{2}{(q+1)\delta}$ for $q \geq 514$, again a contradiction.

In the latter case, $|\Gamma_{x_1}| \leq o(y_1)2\frac{o(a_2)\delta}{o(y_1)} \leq (q-1)\delta$. Since $|\Gamma_{x_2}| \leq (q-1)\delta$, we again get a contradiction with the minimality of covolume of Γ. Therefore either $o(y_1)$ is divisible by $2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3}$ with either $\beta_1 \geq 3$ or $\beta_i \geq 2$ for some $i = 2, 3$, or by $2^{\beta_1} \cdot 3^{\beta_2}$ with $\beta_1 \geq 4$ or $\beta_2 \geq 2$, or $o(y_1)$ is divisible by $\alpha_1 \neq 1$ with $(\alpha_1, 30) = 1$. In all the cases there exists $1 \neq y_1' \in \langle y_1 \rangle$ such that $[y_1', M_{z_i}] = 1$ for all i's. Moreover, $o(y_1') \geq 3$. We may now replace y_1 by y_1' if necessary in all the previous subgroups to obtain the following conclusion: either y_1 centralises M_{z_i} or acts on it as a subgroup of a normaliser of a split torus of M_{z_i} where $i = 1, \ldots, k$. In both situations, $\langle y_1 \rangle$ is normal in Γ_{z_i} for $i = 1, \ldots, k$. If x_1 has no other neighbouring vertices than x_2 in A, we continue with the argument (i.e., next look at y_1 in the Γ–stabilisers of the neighbouring vertices of the y_{z_i}s in A and so on) only to conclude that $\langle y_1 \rangle \triangleleft \Gamma$, a contradiction.

Therefore, it is possible that x_1 has more than one neighbouring vertex in A. One is x_2 and let z be among the other neighbouring vertices of x_1. If $|\Gamma_z \cap T_z| \leq \delta$, then using $|\Gamma_{x_2}|$ and $|\Gamma_z|$, we obtain a contradiction with the minimality of covolume of Γ. Therefore $|\Gamma_z \cap T_z| > \delta$ and we may take $x_2 = z$. Thus whether x_1 has one or more neighbouring vertices in A, we may assume that $|\Gamma_{x_2} \cap T_2| > \delta$. Hence, there exists $y_2 \in \Gamma_{x_2} \cap T_2$ with $o(y_2) > \delta$ and $\langle y_2 \rangle = \Gamma_{x_2} \cap T_2$. As for y_1, we notice that $o(y_2) \mid \frac{q-1}{2}$, $o(y_2) \geq 3$, $y_2 \in \Gamma_{x_1}$ and $\langle y_2 \rangle$ acts faithfully on M_1 via inner automorphisms. Now Dickson's Theorem allows us to conclude that either Γ_{x_1} acts on M_1 as a subgroup of K_1 where $K_1 \in \{S_4, A_5\}$ (in which case $o(y_2) \leq 5$), or Γ_{x_1} acts on M_1 as a subgroup of $N_{M_1}(M_1 \cap T)$.

Let us begin with the case when Γ_{x_1} acts on M_1 as a subgroup of K_1. If Γ_{x_2} acts on M_2 as a subgroup of K_2, then $\mu(\Gamma \backslash G) \geq \frac{2}{60 \cdot 5} > \frac{2}{(q+1)\delta} = \mu(\Gamma' \backslash G)$ for $q \geq 514$, a contradiction. Hence, Γ_{x_2} acts on M_2 as a subgroup $N_{M_2}(M_2 \cap T)$ and in particular, $\langle y_1 \rangle \triangleleft \Gamma_{x_2}$. Again, if $|VA| = 2$, $\langle y_1 \rangle \triangleleft \Gamma$, a clear contradiction. Thus $|VA| > 2$ and let $v_1, \ldots, v_k$ be the neighbours of x_1 in $VA - \{x_2\}$. Since y_1 fixes every edge in $E_X(x_1)$, it follows that y_1 acts faithfully on M_{v_i} and holding a discussion similar to the above one with v_i in place of x_2, we may assume that Γ_{v_i} acts on M_{v_i} as a subgroup of a normaliser of a split torus of M_{v_i}. It follows that $\langle y_1 \rangle$ is normal in each Γ_{v_i}. Now let $z_1, \ldots, z_m$ be the neighbours of x_2 in $VA - \{x_1\}$. Let us consider $\Gamma_{z_i} = \Gamma \cap P_{z_i}$. If $|C_{\Gamma_{z_i}}(M_{z_i})| \leq \delta$, then there is at most one such vertex, otherwise we would contradict the minimality of covolume of Γ. Hence, we may assume that if it happens, $i = 1$, i.e., $|C_{\Gamma_{z_1}}(M_{z_1})| \leq \delta$. Then we may further assume that $T \leq P_{z_1}$. Thus $y_1, y_2 \in \Gamma_{z_1}$. If Γ_{z_1} acts on M_{z_1} as a subgroup of $K_{z_1} \in \{S_4, A_5\}$, then $|\Gamma_{z_1}| \leq 60\delta$, which is a contradiction, as always ($\frac{1}{60\delta} \geq \frac{2}{(q+1)\delta}$ for $q > 120$). Hence, Γ_{z_1} acts on M_{z_1} as a subgroup of a normaliser of a split torus of M_{z_1}. It follows that $\langle y_1 \rangle$ is a normal subgroup of Γ_{z_1}. Now for $i > 1$, there exists $y_{z_i} \in C_G(M_{z_i})$ whose order $o(y_{z_i}) > \delta$ (and thus is at least 3) and does divide $\frac{q-1}{2}$. But this element sits in the kernel of action of L_{z_i} on $E_X(z_i)$ and therefore, $y_{z_i} \in \Gamma_{x_2}$. On the other hand by the usual argument, y_2 acts faithfully on M_{z_i} and so $[y_2, y_{z_i}] = 1$. It follows that $\langle y_{z_i} \rangle$ is normal in Γ_{x_2}. Finally, as $C_{\Gamma_{x_2}}(M_2)$ stabilises (x_2, z_i), it follows that $C_{\Gamma_{x_2}}(y_{z_i}) \leq \Gamma_{z_i}$. It follows that $y_1 \in \Gamma_{z_i}$ and so $\langle y_1, y_2 \rangle \leq \Gamma_{z_i}$. Assume that y_1 acts on M_{z_i} as a subgroup of $K_{z_i} \in \{S_4, A_5\}$. Using the same argument as before we obtain that there exists $y_1' \in \langle y_1 \rangle$ with $[y_1', M_{z_i}] = 1$ for all $i > 1$ and with $o(y_1') \geq 3$. In this case we will replace y_1 by y_1' if necessary in all the previous subgroups to obtain the following conclusion: $\langle y_1 \rangle$ is normal in Γ_v for all the vertices mentioned so far, i.e., $x_1, x_2, z_1, \ldots, z_m, v_1, \ldots, v_k$. By iterating this argument we may show that $\langle y_1 \rangle \triangleleft \Gamma$ which is a contradiction.

We are now reduced to the last possible situation: Γ_{x_1} acts on M_1 as a subgroup of $N_{M_1}(M_1 \cap T)$. Notice that because of the symmetry between x_1 and x_2 to finish the analysis it remains to consider the case when Γ_{x_2} acts on M_2 as a subgroup of $N_{M_2}(M_2 \cap T)$. But in this case $\langle \Gamma_{x_1}, \Gamma_{x_2} \rangle \leq N$. Hence we may move to the next vertex y on our graph. Using the previous argument we obtain that again that the only possible case will be $\Gamma_y \leq N$, and so on and so forth. Therefore, in the end of this case, the only possible conclusion will be $\Gamma \leq N$, which is a contradiction as N is not a cocompact lattice of G, not does it contain any cocompact lattice.

6.6. **Non-edge-transitive case, Subcase 2.** We are now in the situation when T induces some non-trivial outer-diagonal automorphisms on M_i, that is, $L_i/Z(L_i) \cong PGL_2(q)$. Consider $L_i = M_iT$. As before $M_i \triangleleft L_i$ and $T_i = C_T(M_i)$. Then there exists an element $t_i \in T - T_iM_i$ such that $t_i^2 \in T_iM_i$ and t_i induces an outer diagonal automorphism on M_i. Since $q \equiv 1 \pmod 4$, if x is an involution in $L_i \cap T$, then $x \in M_iT_i$.

Recall that G does not admit any edge-transitive lattice. Therefore if $Q_i \in \mathcal{S}yl_2(L_i)$ and Q_i^2 is its unique subgroup of index 2, then $Q_i^2 \not\leq Z(G)$. It follows that $|Q_i/Q_i \cap Z(G)| \geq 4$ and so $\delta = 2$.

The minimality of covolume of Γ' can be now shown by repeating exactly the same sequence of arguments as in Subcase 1 applied to subgroups of L_i, $i = 1, 2$. It turns out that the difference in the structure of L_i (which is now a quotient of $GL_2(q)$) does not significantly affect the argument, and so we omit it here in order to avoid a fairly routine repetition.

This completes the proof of Theorem 1.4.

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