LATTICES ACTING ON RIGHT-ANGLED HYPERBOLIC BUILDINGS

ANNE THOMAS

Abstract. Let X be a right-angled hyperbolic building. We show that the lattices in $\mathrm{Aut}(X)$ share many properties with tree lattices. For example, we characterise the set of covolumes of uniform and of nonuniform lattices in $\mathrm{Aut}(X)$, and show that the group $\mathrm{Aut}(X)$ admits an infinite ascending tower of uniform and of nonuniform lattices. These results are proved by constructing a functor from graphs of groups to complexes of groups.

Introduction

Let G be a locally compact topological group, with suitably normalised Haar measure μ. A discrete subgroup $\Gamma \leq G$ is a *lattice* if the covolume $\mu(\Gamma\backslash G)$ is finite, and a *uniform lattice* if $\Gamma\backslash G$ is compact. A *tower* of lattices is a strictly increasing infinite chain of subgroups

$$\Gamma_1 < \Gamma_2 < \cdots < \Gamma_i < \cdots$$

such that each Γ_i is a lattice in G. Two basic questions are:

(1) What are the possible covolumes of lattices in G?
(2) Does G admit a tower of lattices?

These questions have been studied for many G. For example, if G is a noncompact simple real Lie group, then the covolumes of lattices in G are bounded away from 0 (Kazhdan–Margulis [24], [27]), and in most cases the set of lattice covolumes is discrete (Borel [5], Wang [35] and Thurston [32]). A strong finiteness result for lattices in semisimple groups is that of Borel–Prasad [10], which implies that for G an algebraic k-group of rank ≥ 2 over a number field k, and S a finite set of places in k, there are only finitely many arithmetic subgroups $\Gamma \leq G_s$, $s \in S$, such that $\mu_s(\Gamma\backslash G_s) \leq c$.

Nonclassical cases arise from the fact that if G is the automorphism group of a locally finite polyhedral complex, then G is naturally a locally compact group (see Section 1.1). Lattices in the automorphism groups of trees are treated in Bass–Lubotzky [4]. Lattices acting on a product of trees have been studied by, for example, Burger–Mozes [13]. Apart from these cases, essentially nothing is known.

We consider lattices in the automorphism groups of certain hyperbolic buildings. Let P be a compact, convex polyhedron in $\mathbb{H}^n$ with all dihedral angles $\frac{\pi}{2}$, and let (W, I) be the right-angled Coxeter system generated by reflections in the $(n-1)$-dimensional faces of P. A *right-angled hyperbolic building* of type (W, I) is a polyhedral complex X with a maximal family of subcomplexes called *apartments*, each isometric to the tesselation of $\mathbb{H}^n$ by copies of P, which satisfy the usual

Date: 3 March 2006.
2000 *Mathematics Subject Classification.* 22D05, 20E42.

axioms for a building. A right-angled building has nonpositive curvature. It is *locally* the product of trees, but is *not* globally a product. In dimension 2, where right-angled buildings are sometimes known as “Bourdon buildings”, P is a regular right-angled hyperbolic polygon, and the link of each vertex of X is a complete bipartite graph. The dimension of a right-angled building is at most 4, and this bound is sharp [33], [23]. In Section 1.2 we give precise definitions and recall the classification of right-angled buildings. Hyperbolic buildings have been studied by, for example, Bourdon [6], [7], Bourdon–Pajot [8], [9], Haglund [20], [21], Haglund–Paulin [22] and Gaboriau–Paulin [16].

Our results on lattices acting on right-angled buildings are as follows.

Main Theorem. Let (W, I) be a right-angled Coxeter system acting cocompactly on $\mathbb{H}^n$ with fundamental domain P, and let $\mathcal{S}^f$ be the set of $J \subseteq I$ such that the subgroup of W generated by J is finite. Let $\{q_i\}_{i \in I}$ be a family of positive integers, $q_i \geq 2$. Suppose X is the (unique) building of type (W, I), such that each $\{i\}$-residue of X has cardinality q_i, and let $G = \mathrm{Aut}(X)$.

(1) ***Uniform covolumes***
   (a) The set of covolumes of uniform lattices in G is
   $$\mathcal{V}_u(G) = \left\{ c(X)\frac{a}{b} \,\middle|\, \gcd(a,b) = 1, \text{ prime divisors of } b \text{ are} < \max_i\{q_i\} \right\}$$
   where
   $$c(X) = \sum_{\substack{J \in \mathcal{S}^f \\ |J| = \dim(X)}} \left(\prod_{j \in J} \frac{1}{q_j} \right)$$
   (b) If $q_i > 2$ for at least one i, then for each $v \in \mathcal{V}_u(G)$, there is a countably infinite number of nonconjugate uniform lattices of covolume v.

(2) ***Nonuniform covolumes***
   (a) If $q_i > 2$ for at least one i then the set of covolumes of nonuniform lattices is $(0, \infty)$.
   (b) If there exist $i, j \in I$ such that $m_{i,j} = \infty$ and $q_i, q_j > 2$, then for every $v > 0$ there exist uncountably many commensurability classes of nonuniform lattices of covolume v.

(3) ***Towers***
   (a) If $q_i > 2$ for at least one i then there exists a tower of uniform lattices in G.
   (b) If $q_i > 2$ for at least one i then there exists a tower of nonuniform lattices in G.
   (c) If q_i is composite for at least one i then there exists a tower of uniform lattices in G such that the quotient by each lattice is isometric to P.

Although the right-angled building X is a higher-dimensional object, these properties of lattices in $\mathrm{Aut}(X)$ are quite different to those of lattices in higher-rank algebraic groups. For example, the results of Kazhdan–Margulis and Borel–Prasad cited above imply that semisimple groups do not admit lattices of arbitrarily small covolume, thus do not admit towers of (any) lattices, and the theorem of Borel–Prasad implies that any $v > 0$ is the covolume of at most finitely many lattices. We note also that there exist nonpositively curved polyhedral complexes whose automorphism groups do *not* admit towers, for example the Bruhat–Tits building for $SL_3(\mathbb{Q}_p)$, whose automorphism group contains $SL_3(\mathbb{Q}_p)$ as a finite index subgroup.

In contrast, by comparing the Main Theorem with results for trees, we see that the lattices in $G = \mathrm{Aut}(X)$ share many properties with tree lattices.

Uniform covolumes. Rosenberg (Proposition 9.1.2, [29]) showed that the set of covolumes of uniform lattices acting on the m-regular tree is

$$\left\{ \frac{1}{m}\left(\frac{a}{b}\right) \middle| \gcd(a,b) = 1,\ \text{prime divisors of } b \text{ are} < m \right\}$$

and found a similar result for biregular trees (Theorem 9.2.1, [29]). One new feature for right-angled buildings is that the uniform covolumes depend on the structure of the associated Coxeter group.

Note that if all $q_i = 2$ then the set of uniform covolumes is discrete. However, if some $q_i > 2$ then the set of (uniform) covolumes is dense in $(0, \infty)$, and, in particular, G admits (uniform) lattices of arbitrarily small covolume, as is the case for trees.

Our proof of the counting result (1b) will show that this property holds for regular and biregular trees as well (see Proposition 3.3 below). This generalises Theorem 7.1(b) of [3], and answers a question put to us by Lubotzky.

Nonuniform covolumes. Bass–Lubotzky showed that for the m-regular tree, $m \geq 3$, every $v > 0$ is the covolume of some nonuniform lattice (Theorem 4.3, [4]). Rosenberg extended this result to biregular trees (Theorem 8.2.2, [29]).

Farb–Hruska [15] constructed commensurability invariants for lattices acting on the (m,n)-biregular tree, for $m, n \geq 3$, and used these to show that for every $v > 0$ there are uncountably many commensurability classes of nonuniform tree lattices (Corollary 1.2, [15]). Farb–Hruska have shown (2a) and (2b) for some 2-dimensional right-angled hyperbolic buildings.

Towers. Rosenberg showed that if T is a tree such that $\mathrm{Aut}(T)$ is nondiscrete and admits a uniform lattice, then $\mathrm{Aut}(T)$ admits a tower of uniform lattices (Theorem 3.3.1, [29]), and Carbone–Rosenberg showed that, with one exception, if $\mathrm{Aut}(T)$ admits a nonuniform lattice then it admits a tower of nonuniform lattices (Theorem 5.4, [14]).

Part (3c) of the Main Theorem addresses the following finer question about towers. Let X be a polyhedral complex and $G = \mathrm{Aut}(X)$. A subgroup of G is *homogeneous* if it acts transitively on the cells of maximum dimension in X. Does G admit a tower of homogeneous lattices?

When X is the 3-regular tree, a deep theorem of Goldschmidt [18] implies that G does not admit such a tower, since G has only finitely many conjugacy classes of homogeneous lattices. The Goldschmidt–Sims conjecture (see [17]), which remains open, is that if X is the (p,q)-biregular tree, where p and q are prime, then there are only finitely many conjugacy classes of homogeneous lattices in G. If X is the product of two trees of prime valence, Glasner [17] has shown that there are only finitely many conjugacy classes of (irreducible) homogeneous lattices. We do not know if G of the Main Theorem admits a homogeneous tower when all q_i are prime. We note that, by the Functor Theorem below, if G has only finitely many conjugacy classes of homogeneous lattices, then the Goldschmidt–Sims conjecture holds.

The key fact used to prove the Main Theorem is that if a tree is "nicely embedded" in a right-angled hyperbolic building, then any automorphism of the tree may be extended to the building. This may be expressed algebraically by means of a functor from graphs of groups to complexes of groups:

Functor Theorem. Let (W, I), P and X be as in the Main Theorem. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$, let T be the (q_{i_1}, q_{i_2})-biregular tree. Then there is a functor from the category of graphs of groups with universal covering tree T to the category of complexes of groups with universal cover X, provided either that X is "sufficiently symmetric", or that we restrict to 2-colourable graphs. Moreover, this functor takes coverings to coverings, and faithful graphs of groups to faithful complexes of groups.

We give a precise statement and proof of the Functor Theorem in Section 2.2, and then apply it to prove the Main Theorem in Section 3. An example of a sufficiently symmetric building X is one where P is regular, for instance if $\dim(X) = 2$, and all q_i are equal. For the theory of graphs of groups and their morphisms, see Section 1.3 and [30], [2] and [4]. The theory of complexes of groups (see [19], [11]) is outlined in Section 1.4.

I thank Benson Farb for introducing me to this area, for his enthusiasm and guidance, and his many comments on earlier versions of this paper. I would also like to thank G. Christopher Hruska and Frédéric Haglund for helpful discussions.

1. Background

Section 1.1 gives the basic definitions for lattices and describes a suitable normalisation of Haar measure for automorphism groups of polyhedral complexes. In Section 1.2 we define right-angled hyperbolic buildings, discuss their combinatorial structure and recall their classification. The key definitions for graphs of groups are given in Section 1.3, and the theory of complexes of groups is dealt with in more detail in Section 1.4. In Section 1.5 we apply the results of Sections 1.1, 1.2 and 1.4 to right-angled hyperbolic buildings. We refer the reader to [11] for generalities on polyhedral complexes.

1.1. Lattices and covolumes.

Let G be a locally compact topological group with left-invariant Haar measure μ. A discrete subgroup $\Gamma \leq G$ is a *lattice* if the covolume $\mu(\Gamma\backslash G)$ is finite. A lattice Γ is *uniform* if $\Gamma\backslash G$ is compact. Let S be a left G-set such that for every $s \in S$, the stabiliser G_s is compact and open. Then if $\Gamma \leq G$ is discrete, the stabilisers Γ_s are finite. We define the S*-covolume* of Γ by

$$\mathrm{Vol}(\Gamma\backslash\backslash S) := \sum_{s \in \Gamma\backslash S} \frac{1}{|\Gamma_s|} \leq \infty$$

The following theorem shows that Haar measure may be normalised so that $\mu(\Gamma\backslash G)$ equals the S-covolume.

Theorem 1.1 ([4], Chapter 1)**.** Let G be a locally compact topological group acting on a set S with compact open stabilisers and a finite quotient $G\backslash S$. Suppose further that G admits at least one lattice. Then there is a normalisation of the Haar measure μ, depending only on the choice of G-set S, such that for each discrete subgroup Γ of G we have $\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash S)$.

Let X be a connected, locally finite polyhedral complex with vertex set $V(X)$, and let $\mathrm{Aut}(X)$ be the group of cellular isometries of X. A subgroup of $\mathrm{Aut}(X)$ is said to act *without inversions* on X if its elements fix pointwise each cell that they preserve. The group $G = \mathrm{Aut}(X)$ is a locally compact topological group, with a neighbourhood basis of the identity consisting of automorphisms fixing larger and larger combinatorial balls. By the same arguments as for tree lattices ([4], Chapter 1), it can be shown that if $G\backslash X$ is finite, then a discrete subgroup $\Gamma \leq G$ is a uniform lattice if and only if its $V(X)$-covolume is a sum with finitely many terms. Using Theorem 1.1, we now normalise the Haar measure μ on $G = \mathrm{Aut}(X)$ so that for all uniform lattices $\Gamma \leq G$, the covolume of Γ is

$$\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash V(X))$$

1.2. **Right-angled hyperbolic buildings.** References for the theory of Coxeter groups and buildings are Brown [12] and Ronan [28]. On hyperbolic buildings, we follow [22].

Let P be a compact convex polyhedron in $\mathbb{H}^n$ with all dihedral angles $\frac{\pi}{2}$, and let (W, I) be the right-angled Coxeter group generated by reflections in the $(n-1)$-dimensional faces of P. We write $\mathcal{S}^f$ for the set of $J \subset I$ such that the subgroup W_J of W generated by J is finite. By convention, $W_\phi = 1$ and so the empty set ϕ is in $\mathcal{S}^f$. Let N be the finite nerve of (W, I). Each vertex of the barycentric subdivision N' of N has a type $J \in \mathcal{S}^f$. The barycentric subdivision P' of P may be identified with the simplicial cone on N'. We thus obtain a one-to-one correspondence between the vertices of P' and the types $J \in \mathcal{S}^f$. In particular, each vertex of P has the type of a $J \in \mathcal{S}^f$ with $|J| = n = \dim(X)$, and the barycentre of each $(n-1)$-dimensional face of P has the type of a unique $i \in I$. We will refer to the corresponding $(n-1)$-dimensional face of P as an i-face. A polyhedral isometry $g : P \to P$ may, by abuse of notation, be regarded as a bijection $g : I \to I$ such that $m_{i,j} = m_{g(i),g(j)}$ for all $i, j \in I$. Such an isometry takes the i-face of P to the $g(i)$-face of P.

A *hyperbolic building* of type (W, I) is a polyhedral complex X equipped with a maximal family of subcomplexes, called *apartments*. Each apartment is polyhedrally isometric to the tesselation of $\mathbb{H}^n$ by the images of P under W, and these images are called *chambers*. The apartments and chambers of X satisfy the usual axioms for Bruhat–Tits buildings. Each cell of X has a type $J \in \mathcal{S}^f$, induced by the types of P. For $i \in I$, an $\{i\}$*-residue* of X is the connected subcomplex consisting of all chambers which meet in a given i-face of X. Let q_i be the cardinality of each $\{i\}$-residue of X, that is, the number of copies of P in each $\{i\}$-residue, and let x be a vertex of X. Then x has the type of some $J \in \mathcal{S}^f$ with $|J| = n$, and the link of x in X is isometric to the join of n sets of points of cardinalities respectively q_j, for $j \in J$.

A key example of a right-angled hyperbolic building is Bourdon's 2-dimensional building $I_{p,q}$ (see [6]). Here, P is a regular right-angled hyperbolic p-gon. The link of each vertex of $I_{p,q}$ is the complete bipartite graph $K_{q,q}$, which may be thought of as the join of 2 sets of q points. Each $\{i\}$-residue of $I_{p,q}$ consists of q copies of P, glued together along a common edge.

The following result classifies right-angled buildings.

Theorem 1.2 (Proposition 1.2, [22])**.** Let (W, I) be a right-angled Coxeter system and $\{q_i\}_{i \in I}$ a family of positive integers ($q_i \geq 2$). Then there exists a unique (up to isometry) building X of type (W, I), such that for each $i \in I$, the $\{i\}$-residue of X has cardinality q_i.

In the 2-dimensional case, this result is due to Bourdon [6]. According to [22], Theorem 1.2 was proved by M. Globus, and was known also to M. Davis, T. Januszkiewicz and J. Świątkowski.

1.3. **Graphs of groups.** We give only the definitions most relevant to the proof of the Functor Theorem below. See [30], [4] and [2] for more complete treatments.

A *graph of groups* $\mathbb{A} = (A, \mathcal{A})$ consists of a connected graph A, with vertices $V(A)$ and edges $E(A)$, together with groups $\mathcal{A}_v$ for each $v \in V(A)$, and $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ for each $e \in E(A)$, and monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{i(e)}$. See [2] for the definitions of the *path group* $\pi(\mathbb{A})$, the *fundamental group of the graph of groups* $\pi_1(\mathbb{A}, v_0)$ and the *universal covering tree*. A graph of groups is *faithful* if its fundamental group acts faithfully on its universal covering tree.

Let $\mathbb{A} = (A, \mathcal{A})$ and $\mathbb{B} = (B, \mathcal{B})$ be graphs of groups. A *morphism* $\phi : \mathbb{A} \to \mathbb{B}$ consists of

1. a morphism of graphs $f : A \to B$,
2. homomorphisms of local groups $\phi_v : \mathcal{A}_v \to \mathcal{B}_{f(v)}$ and $\phi_e = \phi_{\overline{e}} : \mathcal{A}_e \to \mathcal{B}_{f(e)}$,
3. elements $\gamma_v \in \pi_1(\mathbb{B}, f(v))$ for each $v \in V(A)$, and $\gamma_e \in \pi(\mathbb{B})$ for each $e \in E(A)$, such that if $v = i(e)$ then
   (a) $\delta_e := \gamma_v^{-1}\gamma_e \in \mathcal{B}_{f(v)}$, and
   (b) $\phi_a \alpha_e = \mathrm{Ad}(\delta_e)\alpha_{f(e)}\phi_e$, where $\mathrm{Ad}(\delta_e)$ is conjugation by δ_e in $\mathcal{B}_{f(v)}$.

The morphism ϕ is a *covering* if in addition

1. each ϕ_v and ϕ_e is injective, and
2. for each $v \in V(A)$ and $e' \in E(B)$ with $i(e') = f(v)$, the map
$$\coprod_{\substack{e \in f^{-1}(e') \\ i(e) = v}} \mathcal{A}_v / \alpha_e(\mathcal{A}_e) \to \mathcal{B}_{f(v)} / \alpha_{e'}(\mathcal{B}_{e'})$$
induced by $g \mapsto \phi_v(g)\delta_e$ is a bijection.

1.4. **Complexes of groups.** We first recall some of the standard theory of complexes of groups, due to Haefliger in [19] and [11]; all references to [11] are to Chapter III.$\mathcal{C}$. Then in Section 1.4.1 we record some new definitions and results used for constructing towers, which are mostly generalisations of the corresponding notions for tree lattices. In [11], complexes of groups are over *scwols*, small categories without loops. To avoid unnecessary background, we work with complexes of groups over polyhedral complexes.

Throughout this section, if X is a polyhedral complex, then X' is the first barycentric subdivision of X. This is a simplicial complex with vertex set $V(X')$ and edge set $E(X')$. The edges of X' may be oriented naturally as follows. If $a \in E(X')$, then the vertices of a are the barycentres of (say) an n-cell σ and a k-cell τ of X, with $n > k$. We then orient a from σ to τ, and write $i(a) = \sigma$ and $t(a) = \tau$. Two edges a and b of X' are *composable* if $i(a) = t(b)$. If a and b are composable, there exists an edge $c = ab$ of X' such that $i(c) = i(b)$, $t(c) = t(a)$ and a, b and c form the boundary of a 2-simplex in X'.

A *complex of groups* $G(X) = (G_\sigma, \psi_a, g_{a,b})$ over a polyhedral complex X is given by:

1. a group G_σ for each $\sigma \in V(X')$, called the *local group* at σ;
2. a monomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$ for each $a \in E(X')$; and

(3) for each pair of composable edges a, b in X', an element $g_{a,b} \in G_{t(a)}$, such that

$$\mathrm{Ad}(g_{a,b})\psi_{ab} = \psi_a \psi_b$$

where $\mathrm{Ad}(g_{a,b})$ is conjugation by $g_{a,b}$ in $G_{t(a)}$, and for each triple of composable edges a, b, c the following cocycle condition holds

$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$$

All complexes of groups in this paper will be *simple*, meaning that each $g_{a,b}$ is trivial. Equivalently, all diagrams of monomorphisms commute.

Next we define morphisms of complexes of groups. Let $G(X) = (G_\sigma, \psi_a)$ and $H(Y) = (H_\tau, \psi_b)$ be simple complexes of groups over polyhedral complexes X and Y. Let $f : X' \to Y'$ be a simplicial map sending vertices to vertices and edges to edges (such an f is *nondegenerate*). A *morphism* $\phi : G(X) \to H(Y)$ over f consists of:

(1) a homomorphism $\phi_\sigma : G_\sigma \to H_{f(\sigma)}$ for each $\sigma \in V(X')$, and
(2) an element $\phi(a) \in H_{t(f(a))}$ for each $a \in E(X')$, such that

$$\mathrm{Ad}(\phi(a))\psi_{f(a)}\phi_{i(a)} = \phi_{t(a)}\psi_a$$

and for all pairs of composable edges (a, b) in $E(X')$,

$$\phi(ab) = \phi(a)\psi(\phi(b))$$

We note that morphisms may also be defined over degenerate maps $f : X' \to Y'$. If f is an isometry and each ϕ_σ an isomorphism then ϕ is an *isomorphism*. A morphism ϕ is a *covering* if

(1) each ϕ_σ is injective, and
(2) for each $\sigma \in V(X')$ and $b \in E(Y')$ such that $t(b) = f(\sigma)$, the map

$$\coprod_{\substack{a \in f^{-1}(b) \\ t(a) = \sigma}} G_\sigma/\psi_a(G_{i(a)}) \to H_{f(\sigma)}/\psi_b(H_{i(b)})$$

induced by $g \mapsto \phi_\sigma(g)\phi(a)$ is a bijection.

Let G be a group acting without inversions on a polyhedral complex Y. This action induces a complex of groups, as follows. Let $X = G\backslash Y$ with $p : Y \to X$ the natural projection. For each $\sigma \in V(X')$, choose $\tilde{\sigma} \in V(Y')$ such that $p(\tilde{\sigma}) = \sigma$. The local group G_σ is the stabiliser of $\tilde{\sigma}$ in G, and the ψ_a and $g_{a,b}$ are defined using further choices. The resulting complex of groups $G(X)$ is unique up to isomorphism. A complex of groups is *developable* if it is isomorphic to a complex of groups associated, as just described, to an action.

We now define the *fundamental group of a complex of groups* in two ways. Let $G(X) = (G_\sigma, \psi_a)$ be a simple complex of groups over a connected polyhedral complex X. Let $E^\pm(X')$ be the set of symbols a^+ and a^-, where $a \in E(X')$. The group $FG(X)$ then has generators

- the elements of G_σ, for $\sigma \in V(X')$, and
- the elements of $E^\pm(X)$,

and relations

- the relations in the groups G_σ,
- $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$ for all $a \in E(X')$,
- $a^+b^+ = (ab)^+$, for all composable edges a and b, and

- $\psi_a(g) = a^+ g a^-$ for all $g \in G_{i(a)}$ and all $a \in E(X')$.

For $e \in E^{\pm}(X')$, if $e = a^+$ we define $i(e) = t(a)$ and $t(e) = i(a)$. If $e = a^-$ we define $i(e) = i(a)$ and $t(e) = t(a)$. Let σ and τ be vertices of X'. An *edge path* in X' joining σ to τ is then a sequence

$$c = (e_1, e_2, \ldots, e_n)$$

of elements of $E^{\pm}(X')$ such that $i(e_1) = \sigma$, $t(e_j) = i(e_{j+1})$ for $1 \le j \le n-1$ and $t(e_n) = \tau$. A $G(X)$*-path* joining σ to τ is a sequence

$$\gamma = (g_0, e_1, g_1, \ldots, e_n, g_n)$$

where $(e_1, \ldots, e_n)$ is an edge path joining σ to τ, and $g_0 \in G_\sigma$ and $g_j \in G_{t(e_j)}$ for $1 \le j \le n$. We associate to γ as above the element $\pi(\gamma) = g_0 e_1 g_1 \cdots e_n g_n$ of $FG(X)$.

Fix a vertex $\sigma_0 \in V(X')$. A $G(X)$*-loop* at σ_0 is a $G(X)$-path joining σ_0 to itself. Two $G(X)$-loops γ and γ' at σ_0 are *homotopic* if $\pi(\gamma) = \pi(\gamma')$. Concatenation defines a group structure on the set of homotopy classes of $G(X)$-loops at σ_0. The resulting group, denoted $\pi_1(G(X), \sigma_0)$, is the *fundamental group* of $G(X)$ at σ_0. By definition, the map π identifies $\pi_1(G(X), \sigma_0)$ to a subgroup of $FG(X)$.

We now give the second definition of the fundamental group. Let T be a maximal tree in the 1-skeleton of X'. The *fundamental group* of $G(X)$ at T, written $\pi_1(G(X), T)$, is the quotient of the group $FG(X)$ by the normal subgroup generated by the elements a^+ such that a is an edge of T. If X is simply connected, the fundamental group $\pi_1(G(X), T)$ is the direct limit of the family of groups G_σ and monomorphisms ψ_a.

The fundamental groups $\pi_1(G(X), \sigma_0)$ and $\pi_1(G(X), T)$ are isomorphic, via the following maps (Theorem 3.7, [11]). Identify $\pi_1(G(X), \sigma_0)$ to a subgroup of $FG(X)$. Then the isomorphism

$$\Psi : \pi_1(G(X), \sigma_0) \to \pi_1(G(X), T)$$

is given by restricting the natural projection $FG(X) \to \pi_1(G(X), T)$. To state the inverse of Ψ, for each $\sigma \in V(X')$, let $c_\sigma = (e_1, \ldots, e_n)$ be the unique edge path in X' joining σ_0 to σ such that no two consecutive edges are inverse to each other and each e_i is contained in T. Let $\pi_\sigma = e_1 \cdots e_n \in FG(X)$. Then

$$\Theta : \pi_1(G(X), T) \to \pi_1(G(X), \sigma_0)$$

mapping the generator $g \in G_\sigma$ to $\pi_\sigma g \pi_\sigma^{-1}$ and a^+ to $\pi_{t(a)} a^+ \pi_{i(a)}$ is the inverse of Ψ.

We now discuss conditions for a complex of groups to be developable.

Theorem 1.3 (Haefliger, Theorem 4.1, [19])**.** Let $G(X)$ be a (simple) complex of groups over a connected polyhedral complex X, such that for each $\sigma \in V(X')$, the natural homomorphism from G_σ to $FG(X)$ is injective. Let T be a maximal tree in the 1-skeleton of X'. Then there is canonically a simply connected polyhedral complex $D(G(X), T)$ and an action of $\pi_1(G(X), T)$ without inversions on $D(G(X), T)$, such that $G(X)$ is naturally isomorphic to the complex of groups associated to this action.

The polyhedral complex $D(G(X), T)$ in the statement of Theorem 1.3 is called the *universal cover* of the complex of groups $G(X)$. The natural projection from $FG(X)$ to $\pi_1(G(X), T)$ is injective on the image of each G_σ (see the proof of

Theorem 3.7, [11]), so we may identify each G_σ to its image in $\pi_1(G(X),T)$. The vertices of $D(G(X),T)'$ are then the pairs

$$(gG_\sigma,\sigma) \quad \text{where } \sigma \in V(X') \text{ and } g \in \pi_1(G(X),T)$$

and the edges of $D(G(X),T)'$ are the pairs

$$(gG_{i(a)},a) \quad \text{where } a \in E(X') \text{ and } g \in \pi_1(G(X),T)$$

The initial and terminal vertices of each edge are given by

$$i(gG_{i(a)},a) = (gG_{i(a)},i(a)) \quad \text{and} \quad t(gG_{i(a)},a) = (ga^-G_{t(a)},t(a))$$

The action of $h \in \pi_1(G(X),T)$ on the vertices is

$$h \cdot (gG_\sigma,\sigma) = (hgG_\sigma,\sigma)$$

We now describe a geometric condition for developability. Let X be a connected polyhedral complex and $\sigma \in V(X')$. The *star* of σ, written $\mathrm{St}(\sigma)$, is the union of the interiors of the simplices in X' which meet σ. If $G(X)$ is a complex of groups over X then, even if $G(X)$ is not developable, each $\sigma \in V(X')$ has a *local development*. That is, we may associate to σ an action of G_σ on the star $\mathrm{St}(\tilde\sigma)$ of a vertex $\tilde\sigma$ in some simplicial complex, such that $\mathrm{St}(\sigma)$ is the quotient of $\mathrm{St}(\tilde\sigma)$ by the action of G_σ. If $G(X)$ is developable, then for each $\sigma \in V(X')$, the local development of σ is isomorphic to the star of each lift $\tilde\sigma$ of σ in the universal cover $D(G(X),T)$.

The local development $\mathrm{St}(\tilde\sigma)$ has a metric structure induced by that of the polyhedral complex X. We say that a complex of groups $G(X)$ is *nonpositively curved* if for all $\sigma \in V(X')$, $\mathrm{St}(\tilde\sigma)$ has nonpositive curvature (that is, $\mathrm{St}(\tilde\sigma)$ is locally $\mathrm{CAT}(\kappa)$ for some $\kappa \le 0$) in this induced metric. The importance of this condition is given by:

Theorem 1.4 (Haefliger, [19])**.** A nonpositively curved complex of groups is developable.

Let $G(X)$ be a developable complex of groups over a polyhedral complex X, with universal cover Y and fundamental group Γ. We say that $G(X)$ is *faithful* if the action of Γ on Y is faithful. If X is finite, all local groups are finite, and $G(X)$ is faithful, then Γ may be regarded as a uniform lattice in $\mathrm{Aut}(Y)$.

1.4.1. *Towers.* Let $G(X) = (G_\sigma,\psi_a)$ and $H(X) = (H_\tau,\psi_b)$ be simple complexes of groups over a polyhedral complex X. We say that $G(X)$ is a *full complex of subgroups* of $H(X)$ if there is a covering $\phi : G(X) \to H(X)$ over the identity map $X' \to X'$ such that each $\phi(a) = 1$. There is then, fixing $\sigma_0 \in V(X')$, a natural homomorphism

$$\pi_1(\phi,\sigma_0) : \pi_1(G(X),\sigma_0) \to \pi_1(H(X),\sigma_0)$$

which sends the homotopy class of the $G(X)$-loop $(g_0,e_1,g_1,\ldots,e_n,g_n)$, with $g_i \in G_{\sigma_i}$, to the homotopy class of the $H(X)$-loop

$$(\phi_{\sigma_0}(g_0),e_1,\phi_{\sigma_1}(g_1),\ldots,e_n,\phi_{\sigma_n}(g_n))$$

(see Proposition 3.6, [11]). Proposition 1.5 below shows that, in the presence of nonpositive curvature, the map $\pi_1(\phi,\sigma_0)$ is an injection. The proof of this proposition is similar to that for trees in Proposition 2.7(ii), [2]. The result may be generalised to coverings of developable complexes of groups [25].

Proposition 1.5. Let $G(X) = (G_\sigma, \psi_a)$ and $H(X) = (H_\tau, \psi_b)$ be simple complexes of groups over a complete connected polyhedral complex X, and fix $\sigma_0 \in V(X')$. Suppose $G(X)$ is a full complex of subgroups of $H(X)$, with respect to a covering ϕ. Assume in addition that $G(X)$ and $H(X)$ have nonpositive curvature. Then $\pi_1(\phi, \sigma_0)$ is injective.

Proof. Choose a maximal tree T in the 1-skeleton of X'. Let $\Psi' : \pi_1(H(X), \sigma_0) \to \pi_1(H(X), T)$ and $\Theta : \pi_1(G(X), T) \to \pi_1(G(X), \sigma_0)$ be the isomorphisms defined in Section 1.4, and consider

$$\Lambda = \Psi' \circ \pi_1(\phi, \sigma_0) \circ \Theta : \pi_1(G(X), T) \to \pi_1(H(X), T)$$

Since Ψ' and Θ are isomorphisms, it suffices to show that Λ is injective.

By Theorem 1.3 and the discussion following it, we may identify the groups G_σ and H_σ to their images in respectively $\pi_1(G(X), T)$ and $\pi_1(H(X), T)$. By the definitions of Ψ' and Θ, for each $g \in G_\sigma$, we then have $\Lambda(g) = \phi_\sigma(g)$. Thus, by Section 2.18 [11], there is a Λ-equivariant map of the universal covers $\widetilde{\phi} : D(G(X), T) \to D(H(X), T)$, defined by

$$(gG_\sigma, \sigma) \mapsto (\phi_\sigma(g)H_\sigma, \sigma) \quad \text{and} \quad (gG_{i(a)}, a) \mapsto (\phi_\sigma(g)H_{i(a)}, a)$$

Since $\widetilde{\phi}$ preserves adjacency of vertices and edges, condition (2) of the definition of a covering implies that $\widetilde{\phi}$ is locally an isometric embedding. Thus, as $D(G(X), T)$ and $D(H(X), T)$ are complete, simply connected and of nonpositive curvature, $\widetilde{\phi}$ is an isometric embedding.

Suppose $\Lambda(h) = 1$. Since $\widetilde{\phi}$ is injective and Λ-equivariant, h must act trivially on $D(G(X), T)$. In particular, h fixes (G_{σ_0}, σ_0), so $h \in G_{\sigma_0}$. Thus $\phi_{\sigma_0}(h) = 1$, which implies $h = 1$ since ϕ_{σ_0} is injective. $\square$

Let X be a complete, connected polyhedral complex. A *tower* on X with universal cover Y is a sequence $(G_i(X))$ of complexes of groups over X, such that

(1) for each i, $G_i(X)$ is a full complex of subgroups of $G_{i+1}(X)$ with respect to a covering ϕ_i;
(2) the image of each $(\phi_i)_\sigma$ is a proper subgroup; and
(3) Y is the universal cover of each $G_i(X)$.

By Proposition 1.5, such a sequence induces an infinite ascending chain of fundamental groups

$$\pi_1(G_1(X), \sigma_0) < \pi_1(G_2(X), \sigma_0) < \cdots < \pi_1(G_i(X), \sigma_0) < \cdots$$

If each $G_i(X)$ is a faithful complex of groups, with finite covolume, then each fundamental group in this chain is a lattice of $\mathrm{Aut}(Y)$. We thus obtain a tower of lattices in $\mathrm{Aut}(Y)$.

1.5. **Application to right-angled hyperbolic buildings.** Let X be a right-angled building of type (W, I), with chambers isometric to P and parameters $\{q_i\}$. This building may be realised as the universal cover of a simple complex of groups $G(P)$ over P [22], as we now explain.

For $i \in I$, let G_i be a group of order q_i. If $\sigma \in V(P')$ has type $J \in \mathcal{S}^f$, then the local group G_σ is $\prod_{j \in J} G_j$, with $G_\sigma = 1$ if σ is the barycentre of P. The monomorphisms along edges of P' are natural inclusions.

Let σ be a vertex of P of type J, with $|J| = n = \dim(X)$. Then the local development at σ with respect to the complex of groups $G(P)$ has link isomorphic to the join of n sets of points of cardinalities respectively q_j, for $j \in J$. Since P is right-angled, by Gromov's Link Condition (Theorem 5.2, Chapter II, [11]) each local development has nonpositive curvature, so by Theorem 1.4 the complex of groups $G(P)$ is developable. The universal cover is a building (see [16], section 3.3), and by Theorem 1.2, the universal cover is the unique right-angled building X having chambers P and parameters $\{q_i\}$. More generally, if $G(Y)$ is a developable complex of groups such that the universal cover of $G(Y)$ is a right-angled building with chambers P and parameters $\{q_i\}$, then by Theorem 1.2 the universal cover of $G(Y)$ is X.

Since the local group at the barycentre of P is trivial, $G(P)$ is a faithful complex of groups. Hence the fundamental group Γ of $G(P)$ is a uniform lattice in $\mathrm{Aut}(X)$. The covolume of Γ is

$$\sum_{\sigma \in V(P)} \frac{1}{|\Gamma_\sigma|} = \sum_{\substack{J \in \mathcal{S}^f \\ |J| = \dim(X)}} \left(\prod_{j \in J} \frac{1}{q_j} \right)$$

2. The Functor Theorem

This section contains our key technical result, the Functor Theorem below, which constructs a functor from graphs of groups to complexes of groups. Let $\mathcal{G}$ be the category of graphs of groups and morphisms, as defined by Bass [2]. Let $\mathcal{C}$ be the category of complexes of groups and morphisms, as defined by Haefliger [11]. In Section 2.1 we make precise the relationship between $\mathcal{G}$ and the subcategory of $\mathcal{C}$ consisting of complexes of groups over 1-dimensional polyhedral complexes. This is used in Section 2.2 where we state and prove the Functor Theorem.

2.1. **Relationship between graphs of groups and 1-dimensional complexes of groups.** Complexes of groups are generalisations of graphs of groups. However, Haefliger remarks that it is unclear whether the notion of morphism is the same ([11], p. 566). Let $\mathcal{C}_1$ be the subcategory of $\mathcal{C}$ consisting of complexes of groups over 1-dimensional polyhedral complexes (that is, simplicial graphs), and morphisms over *nondegenerate* polyhedral maps. The following result specifies the relationship between the category $\mathcal{G}$ of graphs of groups and the category $\mathcal{C}_1$.

Proposition 2.1. There is a functor $F : \mathcal{G} \to \mathcal{C}_1$ which is a bijection on objects, takes faithful graphs of groups to faithful complexes of groups, surjects onto the set of morphisms of $\mathcal{C}_1$ and takes coverings to coverings. However, there is no functor from $\mathcal{G}$ to $\mathcal{C}_1$ which is injective on the set of morphisms of $\mathcal{G}$. Hence these categories are not isomorphic.

Proof. We first define the functor F on objects. Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups and let $|A|$ be the geometric realisation of the graph A. We construct a complex of groups $F(\mathbb{A})$ over $|A|$. The local groups at the vertices of $|A|$ are the vertex groups of $\mathbb{A}$. For each $e \in E(A)$, let $\sigma_e = \sigma_{\overline{e}}$ be the vertex of the barycentric subdivision $|A|'$ at the midpoint of e. Then the local group at σ_e in the complex of groups is $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$. Each monomorphism $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{i(e)}$ of the graph of groups induces the same monomorphism of corresponding local groups in the complex of groups $F(\mathbb{A})$.

It is clear that F is a bijection from the set of objects of $\mathcal{G}$ to the set of objects of $\mathcal{C}_1$. From the definitions of the actions on the respective universal covers, one sees also that faithful graphs of groups are mapped to faithful complexes of groups.

To define the functor F on morphisms, let $\phi : \mathbb{A} \to \mathbb{B}$ be a morphism of graphs of groups over a graph morphism $f : A \to B$. By abuse of notation f induces a nondegenerate polyhedral map $f : |A|' \to |B|'$. We define a morphism of complexes of groups $F(\phi) : F(\mathbb{A}) \to F(\mathbb{B})$ over f. The homomorphisms on local groups are the same as for the morphism ϕ. Let a be an edge of $|A|'$. Then the monomorphism ψ_a in the complex of groups $F(\mathbb{A})$ is the same as some $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{i(e)}$ in the graph of groups $\mathbb{A}$. Put $\phi(a) = \delta_e$, and we obtain the required

$$\phi_{t(a)}\psi_a = \mathrm{Ad}(\phi(a))\psi_{f(a)}\phi_{i(a)}$$

Hence $F(\phi)$ is a morphism of complexes of groups. The map F respects composition of morphisms, and is thus a functor.

Given a morphism of $\mathcal{C}_1$, one can construct a morphism of the corresponding objects in $\mathcal{G}$ by setting all $\gamma_v = 1$, and all $\gamma_e = \phi(a)$, where the monomorphism α_e is the same as the monomorphism ψ_a. Hence F surjects onto the set of morphisms of $\mathcal{C}_1$. By checking the definition of covering in both categories, we find that F sends coverings to coverings, and surjects onto the set of coverings of $\mathcal{C}_1$.

However, since many choices of γ_v and γ_e could lead to the same collection of δ_e, there is no functor from $\mathcal{G}$ to $\mathcal{C}_1$ which is injective on the set of morphisms of $\mathcal{G}$. $\square$

2.2. **The Functor Theorem.** Let T be a tree and X a right-angled hyperbolic building. We write $\mathcal{G}(T)$ for the category of graphs of groups with universal covering tree T and $\mathcal{C}(X)$ for the category of developable complexes of groups with universal cover X. We now construct a functor from $\mathcal{G}(T)$ to $\mathcal{C}(X)$, for certain T and X.

This functor will exist only when the building X is "sufficiently symmetric", as defined below. However, the proof of the Functor Theorem will show that even if X is not sufficiently symmetric, we can construct a functor taking faithful graphs of groups to faithful complexes of groups, and coverings to coverings, if we restrict to graphs of groups whose vertices are 2-colourable and to coverings which preserve this colouring. As shown in Section 3 below, the proof of the Main Theorem requires only this restricted functor.

To explain when a right-angled building X is sufficiently symmetric, let (W, I) be the right-angled Coxeter system, P the fundamental domain and $\{q_i\}$ the parameters associated to X. Since the apartments of X are infinite, there exist $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$, and since the chambers of X are right-angled hyperbolic polyhedra, for each $i_1 \in I$, there is an $i_2 \in I$ such that $m_{i_1,i_2} = \infty$.

Consider the following symmetry conditions on X:

(1) there exists a bijection g on the set I such that $m_{i,j} = m_{g(i),g(j)}$ for all $i, j \in I$, and $g(i_1) = i_2$. This may be thought of as an isometry of P which takes the i_1-face to the i_2-face.

(2) there exists a bijection

$$h : \{i \in I \,|\, m_{i_1,i} < \infty\} \to \{i \in I \,|\, m_{i_2,i} < \infty\}$$

such that $m_{i,j} = m_{h(i),h(j)}$ for all i, j in the domain, $h(i_1) = i_2$, and $q_i = q_{h(i)}$ for all i in the domain. This may be thought of an isometry from the simplicial neighbourhood in P' of the i_1-face to the simplicial neighbourhood in P' of the i_2-face, which preserves the cardinalities q_i.

Conditions (1) and (2) are satisfied if, for example, P is regular and all q_i are equal. In dimension 2, all fundamental domains P are regular polygons. In each of dimensions 3 and 4, there is only one regular P, the dodecahedron and the 120-cell respectively (see, for example, [34]). The convex polyhedra in $\mathbb{H}^3$ are classified in [1]. Those of $\mathbb{H}^4$ have not been classified.

Conditions (1) and (2) may be satisfied for P which is not regular. For example, suppose P is 3-dimensional. We claim there exist $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$ and the i_1- and i_2-faces of P are isometric. Since the 2-dimensional faces of P are regular, it suffices to exhibit a pair of nonadjacent 2-dimensional faces with, say, 5 sides. Let a_2 be the number of 2-dimensional faces of P. For each 2-dimensional face F, let $a_1(F)$ be the number of sides of F and let $\mathrm{ex}(F) = a_1(F) - 5 \geq 0$. Elementary calculations yield

$$a_2 = 12 + \sum_{\dim(F)=2} \mathrm{ex}(F)$$

([26], p. 70). This implies that there are at least 12 faces F with 5 sides. Hence we can find two nonadjacent 2-dimensional faces of P with 5 sides. This kind of argument cannot be extended to dimension 4, since the 3-dimensional faces of P will not in general be regular.

We now state and prove the Functor Theorem.

Functor Theorem. *Let X be a right-angled hyperbolic building of type (W, I), with chambers P and parameters $\{q_i\}$. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$, let T be the (q_{i_1}, q_{i_2})-biregular tree. Suppose condition (1) above holds, and that if $q_{i_1} = q_{i_2}$ then condition (2) above holds, with g an extension of h. Then there is a functor F from $\mathcal{G}(T)$ to $\mathcal{C}(X)$, which takes faithful graphs of groups to faithful complexes of groups, and coverings to coverings.*

Proof. By Proposition 2.1, it suffices to construct a functor F from the image of $\mathcal{G}(T)$ in $\mathcal{C}_1$ to $\mathcal{C}(X)$. We first define F on objects. Let $G(Y)$ be an object of $\mathcal{C}_1$ which is in the image of $\mathcal{G}(T)$.

For each edge e of the simplicial graph Y, let P_e be a copy of P. Identify the midpoint of e with the barycentre of P_e. Suppose first that the vertices of Y may be 2-coloured with the types i_1 and i_2. If $q_{i_1} \neq q_{i_2}$, then we use the colouring induced by the valences of the vertices in the universal covering tree T. For $j = 1, 2$ identify the vertex of e of type i_j with the i_j-face of P_e. If the vertices of Y are not 2-colourable and the edge e does not form a loop in Y, then identify one vertex of e with the i_1-face of P_e, and the other vertex with the i_2-face. If the edge e does form a loop in Y then use the isometry h of condition (2) above to form P_e/h. That is, glue together the i_1- and i_2-faces of P_e using h. Then identify the vertex of Y to which both ends of e are joined with the barycentre of this glued face.

Now, glue together, either by preserving type on i_1- and i_2-faces, or if the vertices of Y are not 2-colourable by using the isometry h, the faces of the various P_e which correspond to the same vertex of Y. Let $F(Y)$ be the resulting polyhedral complex. By construction, Y' embeds in $F(Y)'$. Note also that each cell of $F(Y)$ has either the type of a unique subset $J \in S^f$, or the two types J and $h(J)$, where $i_1 \in J$ and $J \in S^f$.

We now construct a complex of groups $F(G(Y))$ over $F(Y)$. First, fix the local groups and monomorphisms induced by the embedding of Y' in $F(Y)'$. For each $i \in I$, let $G_i = \mathbb{Z}/q_i\mathbb{Z}$. For each edge e of Y, let G_e be the local group at the midpoint of e in the complex of groups $G(Y)$.

Consider $J \in S^f$ which does not contain i_1 or i_2. The local group at the face with type J of P_e or P_e/h is then

$$G_e \times \prod_{j \in J} G_j$$

The monomorphisms between such local groups are natural inclusions.

Now consider $J \in S^f$ which contains one of i_1 or i_2 (since $m_{i_1,i_2} = \infty$, J cannot contain both i_1 and i_2). Without loss of generality suppose $i_1 \in J$, and let F_e be either the i_1-face of P_e, or the glued face of P_e/h. The face of type J in P_e or P_e/h is then contained in F_e. Let v be the vertex of Y which is identified with the barycentre of F_e, and let G_v be the local group at v in the complex of groups $G(Y)$. The local group at the face of type J of P_e or P_e/h is then

$$G_v \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j$$

This is well-defined, since if the face of type J also has type $h(J)$, by condition (2) we have $q_j = q_{h(j)}$ for each $j \in J$, hence $G_j = G_{h(j)} = \mathbb{Z}/q_j\mathbb{Z}$. The monomorphism from G_v to this local group is the natural inclusion onto the first factor. For each $J' \subset J$ with $i_1 \in J'$, the monomorphism

$$G_v \times \prod_{\substack{j \in J' \\ j \neq i_1}} G_j \to G_v \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j$$

is the natural inclusion. For each $J' \subset J$ with $i_1 \notin J'$, the monomorphism

$$G_e \times \prod_{j \in J'} G_j \to G_v \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j$$

is a monomorphism $G_e \to G_v$ from the complex of groups $G(Y)$ on the first factor, and natural inclusions on the other factors.

One verifies that the link in the local development at each vertex of $F(Y)$ is the join of $n = \dim(X)$ sets of points of the appropriate cardinalities. Thus, by Section 1.5, the universal cover is X.

The tree T naturally embeds in X, and by construction, if the fundamental group of $G(Y)$ acts faithfully on T, then the fundamental group of $F(G(Y))$ acts faithfully on T, and hence on X. Thus F maps faithful graphs of groups to faithful complexes of groups.

We now define the functor F on morphisms. Let $G(Y)$ and $H(Z)$ be objects of $\mathcal{C}_1$ in the image of $\mathcal{G}(T)$, and let $\phi : G(Y) \to H(Z)$ be a morphism over a nondegenerate map $f : Y' \to Z'$. Let $F(Y)$ and $F(Z)$ be the polyhedral complexes constructed from the simplicial graphs Y and Z as above.

If the vertices of Y and Z are 2-colourable by the types i_1 and i_2, and f preserves these types, then f may be extended to a polyhedral map $F(f) : F(Y) \to F(Z)$ by preserving type on each copy of P. Otherwise, we use condition (1) to construct $F(f) : F(Y) \to F(Z)$. On each copy of P or P/h, $F(f)$ maps the face of type J to the face of type $g(J)$. This is well-defined since h is the restriction of g.

We now construct a morphism $F(\phi)$ of complexes of groups over $F(f)$. If τ is a vertex of $F(Y)'$ then the local group at τ is a direct product

$$G_\sigma \times \prod G_j$$

where σ is a vertex of Y'. The homomorphism of local groups

$$G_\sigma \times \prod G_j \to H_{f(\sigma)} \times \prod G_j$$

is ϕ_σ on the first factor, and the identity on other factors. Let b be an edge of $F(Y)'$. If ψ_b, the monomorphism along the edge b in the complex of groups $F(G(Y))$, has as its first factor a monomorphism ψ_a from the complex of groups $G(Y)$, put $F(\phi)(b) = \phi(a)$. Otherwise, put $F(\phi)(b) = 1$.

The following claims all follow from definitions: $F(\phi)$ is a morphism of complexes of groups, $F(\phi)$ respects composition, and $F(\phi)$ takes coverings to coverings. This completes the proof of the Functor Theorem. □

3. Proof of the Main Theorem

We conclude by proving the Main Theorem, stated in the Introduction. Most parts of the theorem are proved using the Functor Theorem. Let X be the building in the statement of the Main Theorem.

Our results on both uniform and nonuniform covolumes use Corollaries 3.1 and 3.2 below, concerning the covolumes of lattices in $\mathrm{Aut}(X)$ which act without inversions. We note that any lattice constructed as the fundamental group of a complex of groups acts without inversions.

Corollary 3.1 is a consequence of Theorem 1.1. If Y is a polyhedral complex, then Y_k is the set of k-dimensional cells of Y.

Corollary 3.1. Let Y be a locally finite polyhedral complex and $n = \dim(Y)$. Then there is a constant $c(Y)$, depending only on Y, such that for all lattices Γ in $G = \mathrm{Aut}(Y)$ which act without inversions,

$$\mu(\Gamma\backslash G) = c(Y)\,\mathrm{Vol}(\Gamma\backslash\backslash Y_n)$$

By Corollary 3.1, if both $\mu(\Gamma\backslash G)$ and $\mathrm{Vol}(\Gamma\backslash\backslash X_n)$ are known for just one lattice Γ acting without inversions on the building X, the constant $c(X)$ may be computed. Using the example of a uniform lattice in Section 1.5, we obtain

$$c(X) = \sum_{\substack{J\in\mathcal{S}^f \\ |J|=\dim(X)}} \left(\prod_{j\in J} \frac{1}{q_j} \right) \tag{3.1}$$

Corollary 3.2 below, which follows from Corollary 3.1 and the proof of the Functor Theorem, gives the relationship between covolumes of lattices in $\mathrm{Aut}(X)$, and covolumes of tree lattices.

Corollary 3.2. Let X be as in the Main Theorem and $G = \mathrm{Aut}(X)$. Suppose $G(Y)$ is a faithful complex of finite groups over a polyhedral complex Y, with universal cover X. Let Γ be the fundamental group of the complex of groups $G(Y)$ (with respect to some choices). For each cell $\sigma \in Y_n$, let Γ_σ be the local group at the barycentre of that cell. Then if $G(Y) = F(\mathbb{A})$, where F is the functor defined in the Functor Theorem, and $\mathbb{A} = (A, \mathcal{A})$ is a graph of groups,

$$\mu(\Gamma\backslash G) = c(X)\left(\sum_{\sigma\in Y_n}\frac{1}{|\Gamma_\sigma|}\right) = c(X)\left(\sum_{e\in E(A)}\frac{1}{|\mathcal{A}_e|}\right)$$

where $c(X)$ is as given in (3.1) above.

The sum

$$\sum_{e\in E(A)}\frac{1}{|\mathcal{A}_e|}$$

is called the *edge covolume* of the graph of groups $\mathbb{A}$.

3.1. **Uniform covolumes.**

3.1.1. *Proof of* (1a). We first show that the covolume of every uniform lattice belongs to the set $\mathcal{V}_u(G)$ of rationals given in (1a). Then we show that any $v \in \mathcal{V}_u(G)$ is the covolume of some uniform lattice.

Since X is a building, the group $G = \mathrm{Aut}(X)$ has a finite index normal subgroup $\mathrm{Aut}_0(X)$, the group of type-preserving automorphisms, which acts without inversions. Thus, any uniform lattice $\Gamma < G$ has a finite index subgroup $\Gamma \cap \mathrm{Aut}_0(X)$ which acts without inversions. Now, the set $\mathcal{V}_u(G)$ is closed under multiplication by positive integers. Hence, to show that the covolume of every uniform lattice belongs to $\mathcal{V}_u(G)$, it suffices to consider only uniform lattices which act without inversions.

Let Γ be a uniform lattice in G which acts without inversions and let

$$\mathrm{Vol}(\Gamma\backslash\backslash X_n) = \frac{a}{b}$$

where $\gcd(a, b) = 1$. By Corollary 3.1 and the value of $c(X)$ given in (3.1) above, it suffices to show that the prime divisors of b are strictly less than $\max_i\{q_i\}$. Let x be a vertex of X. Then x has the type of some $J \in \mathcal{S}^f$ with $|J| = n = \dim(X)$. Let S_k be the symmetric group on k letters. The group of type-preserving automorphisms of the link of x in X is then

$$\prod_{j\in J} S_{q_j}$$

and its subgroup which fixes an $(n-1)$-dimensional cell of the link pointwise is

$$\prod_{j\in J} S_{q_j-1}$$

By Theorem 3 of [31], this implies the restriction on the prime divisors of b. Thus the covolume of any uniform lattice in G belongs to the set $\mathcal{V}_u(G)$.

We now show that every $v \in \mathcal{V}_u(G)$ is the covolume of some uniform lattice. By Corollary 3.2 above, and the proof of the Functor Theorem, it suffices to show that, for any rational $\frac{a}{b}$ with $\gcd(a, b) = 1$ and the prime divisors of b strictly less than $\max_i\{q_i\}$, there is a faithful graph of finite groups $\mathbb{A} = (A, \mathcal{A})$ over a finite graph A such that

1. the universal covering tree of $\mathbb{A}$ is (q_{i_1}, q_{i_2})-biregular, where $m_{i_1,i_2} = \infty$,
2. the vertices of the graph A may be 2-coloured, and
3. the edge covolume of $\mathbb{A}$ equals $\frac{a}{b}$.

FIGURE 1. Rosenberg's graph for the case $q = 3$

Suppose first that all q_i are equal, and let T_q be the q-regular tree.

If all $q_i = 2$, we construct a graph of groups with universal covering tree T_2 and edge covolume $\frac{a}{1} = a$, for any integer $a \geq 1$. Let A be the graph which is a ray of a edges. The vertices of A are then 2-colourable. Let $\mathbb{A} = (A, \mathcal{A})$ be the graph of groups over A with each edge group trivial, the two vertex groups at either end of the ray both $\mathbb{Z}/2\mathbb{Z}$, and all other vertex groups trivial. Since the edge groups are trivial this graph of groups is faithful. Its universal covering tree is T_2 and it has edge covolume a.

If all $q_i = q$ are equal, and $q \geq 3$, we require a graph of groups with universal covering tree T_q and edge covolume $\frac{a}{b}$, where $\gcd(a, b) = 1$ and the prime divisors of b are strictly less than q. This is exactly the set of edge covolumes of uniform lattices in $\mathrm{Aut}(T_q)$ (Proposition 9.1.2, [29]). For $q \geq 4$, the vertices of each of the graphs in Rosenberg's proof are 2-colourable. In the case $q = 3$, though, some of the graphs of groups are over the graph shown in Figure 1. The vertices of this graph cannot be 2-coloured.

A key idea of Rosenberg's proof, which is also used below to prove (1b), is that of finite-sheeted covers. In each case, the graph A, considered as a topological space, is not simply connected. The fundamental group $\pi_1(A)$ is then a free group on finitely many generators so has subgroups of arbitrary finite index. Hence A has a finite-sheeted topological cover for any finite number of sheets. Thus, if a uniform lattice constructed over A has edge covolume $\frac{a}{b}$, we may construct a uniform lattice with edge covolume a positive integer multiple of $\frac{a}{b}$, over a finite-sheeted cover of A.

In particular, the 2-sheeted cover of the graph in Figure 1 has 4 vertices, which may be 2-coloured. As explained in Example 4, Chapter 4, [29], the edge groups of this graph of groups have order a power of 2, and this power may be chosen arbitrarily. Thus, by taking a 2-sheeted cover and then doubling the order of the edge groups, we obtain a graph of groups over a 2-colourable graph with unchanged edge covolume. This completes the proof of (1a) in the case where all q_i are equal.

Now suppose the q_i are not all equal, and let $q_{i_1} = \max_i\{q_i\}$. If there is a q_{i_2} with $m_{i_1,i_2} = \infty$ and $q_{i_1} = q_{i_2}$, then we use the same graphs of groups as in the case where all q_i are equal. Otherwise, to simplify notation, if $m_{i_1,i_2} = \infty$ put $p = q_{i_1}$ and $q = q_{i_2}$, so that $p > q \geq 2$, and let $T_{p,q}$ be the (p, q)-biregular tree.

Rosenberg, in Theorem 9.2.1 [29], shows that the set of edge covolumes of uniform lattices in $\mathrm{Aut}(T_{p,q})$ is the set of rationals $\frac{a}{b}$ such that $\gcd(a, b) = 1$ and the prime divisors of b are strictly less than p. As with regular trees, for each such $\frac{a}{b}$, there is a faithful graph of finite groups, over a finite graph, with edge covolume $\frac{a}{b}$. Since each graph is covered by $T_{p,q}$, and $p \neq q$, the vertices of each graph are 2-colourable. This completes the proof of (1a).

3.1.2. *Proof of* (1b). By the proof of the Functor Theorem and Corollary 3.2, it suffices to prove Proposition 3.3 below.

Proposition 3.3. Let T be a regular or biregular tree, other than the 2-regular tree. Let $v > 0$ be the (edge) covolume of a uniform lattice in $\mathrm{Aut}(T)$. Then there is a countably infinite number of nonconjugate uniform lattices in $\mathrm{Aut}(T)$ which have (edge) covolume v.

Proof. By Corollary 3.1, it suffices to prove this proposition for edge covolumes. Since there are only countably many finite graphs of finite groups, there are at most countably many uniform lattices in $\mathrm{Aut}(T)$.

Assume that T is the m-regular tree, for $m \geq 3$ (the proof for biregular trees is similar). Rosenberg, in the proof of Proposition 9.1.2 [29], constructs a faithful graph of finite groups $\mathbb{A} = (A, \mathcal{A})$ over a finite graph A such that

(1) the universal covering tree of $\mathbb{A}$ is T,
(2) the fundamental group Γ of $\mathbb{A}$ has edge covolume v,
(3) the vertices of the graph A may be 2-coloured (in the case where T is 3-regular, use a 2-sheeted cover),
(4) the graph A is not simply connected, and
(5) the orders of the edge and vertex groups $\mathcal{A}_e$ and $\mathcal{A}_v$ may be multiplied by powers of primes strictly less than m, and the new graph of groups is faithful.

Let n be a positive integer whose prime divisors are strictly less than m. By (4), we may obtain a uniform lattice in $\mathrm{Aut}(T)$ of edge covolume nv. Applying (5), we may obtain a uniform lattice in $\mathrm{Aut}(T)$ of edge covolume v/n. By carrying out these two processes at the same time, we obtain a new lattice Γ' of edge covolume v, which is nonconjugate to the original lattice Γ, since the graph $\Gamma' \backslash T$ is not the same as A. This may be done for countably many values of n. Hence we obtain countably many nonconjugate uniform lattices of edge covolume v. □

3.2. **Nonuniform covolumes.**

3.2.1. *Proof of* (2a). Let T be a regular or biregular tree, other than the 2-regular tree, and let $v > 0$. By Corollary 3.1, Corollary 3.2 and the proof of the Functor Theorem, it suffices to construct a nonuniform lattice Γ in $\mathrm{Aut}(T)$ of covolume v, such that the vertices of the graph $\Gamma \backslash T$ may be 2-coloured.

Bass–Lubotzky showed that for $m \geq 3$ and every $v > 0$ there is a nonuniform lattice Γ acting on the m-regular tree T_m, such that the covolume of Γ is v (Theorem 4.3, [4]). Moreover, the graph $\Gamma \backslash T_m$ is a ray, hence its vertices are 2-colourable. Rosenberg showed the analogous result for biregular trees, without the conclusion that the quotient graph is a ray (Theorem 8.2.2, [29]). Since the vertices of any graph covered by a biregular tree $T_{m,n}$, with $m \neq n$, are 2-colourable, this completes the proof of (2a) of the Main Theorem.

3.2.2. *Proof of* (2b). Let $T_{m,n}$ be the (m,n)-biregular tree, with $m, n \geq 3$. In [15], Farb–Hruska construct commensurability invariants for nonuniform lattices acting on $T_{m,n}$, using the growth types of the quotient graph and the stabilisers. These commensurability invariants also apply to nonuniform lattices constructed using the Functor Theorem. Thus, by Corollary 1.2 of [15], for each $v > 0$ there are uncountably many commensurability classes of nonuniform lattices in $\mathrm{Aut}(X)$ of covolume v.

3.3. **Towers.** We construct towers in $\mathrm{Aut}(X)$ using full subcomplexes of subgroups, as explained in Section 1.4.1. Let $T_{p,q}$ be the (p,q)-biregular tree, with at least one of p and q greater than 2. By the proof of the Functor Theorem, it suffices to find a tower of tree lattices in $\mathrm{Aut}(T_{p,q})$, constructed as a sequence of full subgraphs of subgroups, over a graph A whose vertices may be 2-coloured.

3.3.1. *Proof of* (3a). Let m and q be integers ≥ 2. Bass–Kulkarni, in the proof of Proposition 7.15 of [3], give an example of a tower of uniform tree lattices acting on the $(m+1,q)$-biregular tree. The vertices of the graph A in this example may be 2-coloured.

3.3.2. *Proof of* (3b). An *indexing* on a graph A is a map I from the edges of A to the positive integers. A graph of groups $\mathbb{A} = (A, \mathcal{A})$ over A induces the indexing $I(e) = [\mathcal{A}_{i(e)} : \alpha_e(\mathcal{A}_e)]$. We say that an indexed graph (A, I) admits a tower of lattices if there exists a sequence of full subgraphs of subgroups over A which induces a tower of tree lattices, such that each graph of groups in the sequence induces the indexing I.

Theorem 5.3 of Carbone–Rosenberg [14] gives a sufficient condition for (A, I) to admit a tower of nonuniform tree lattices. It is easy to construct examples of (A, I) which satisfy this condition, so that the tower of tree lattices acts on the regular or biregular tree, and the vertices of A are 2-colourable.

3.3.3. *Proof of* (3c). We apply Example 7.13, [3]. The graph A in this construction is a single edge, so its vertices are 2-colourable. The universal covering tree is the (mp, q)-biregular tree, where m, p and q are positive integers ≥ 2. Thus, by the proof of the Functor Theorem, if some q_i is composite, there exists a tower of uniform lattices in $\mathrm{Aut}(X)$ with quotient a single chamber.

References

[1] E. M. Andreev, *On convex polyhedra in Lobachevskii spaces*, Mat. Sb. **81** (1970), 445–478; English transl.: Math USSR-Sb. **10** (1970), 413–440.

[2] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra **89** (1993), 3–47.

[3] H. Bass and R. Kulkarni, *Uniform tree lattices*, J. Amer. Math. Soc **3** (1990), 843–902.

[4] H. Bass and A. Lubotzky, *Tree Lattices*, Birkhäuser, Boston, 2001.

[5] A. Borel, *Commensurability classes and volumes of hyperbolic 3-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **8** (1981), 1–23.

[6] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geom. funct. anal. **7** (1997), 245–268.

[7] ———, *Sur les immeubles fuchsiens et leur type de quasi-isométrie*, Ergod. Th. & Dynam. Sys. **20** (2000), 343–364.

[8] M. Bourdon and H. Pajot, *Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings*, Proc. Amer. Math. Soc. **127** (1999), 2315–2324.

[9] ———, *Rigidity of quasi-isometries for some hyperbolic buildings*, Comment. Math. Helv. **75** (2000), 701–736.

[10] A. Borel and G. Prasad, *Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 119–171.

[11] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, 1999.

[12] K. S. Brown, *Buildings*, Springer-Verlag, New York, 1989.

[13] M. Burger and S. Mozes, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. **92** (2000), 151–194.

[14] L. Carbone and G. Rosenberg, *Infinite towers of tree lattices*, Math. Res. Lett. **8** (2001), 469–477.

[15] B. Farb and G. Christopher Hruska, *Commensurability invariants for nonuniform tree lattices*, to appear in Israel J. Math.

[16] D. Gaboriau and F. Paulin, *Sur les immeubles hyperboliques*, Geom. Dedicata **88** (2001), 153–197.

[17] Y. Glasner, *A two-dimensional version of the Goldschmidt–Sims conjecture*, J. Algebra **269** (2003), 381–401.

[18] D. M. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. of Math. (2) **111** (1980), 377–406.

[19] A. Haefliger, *Complexes of groups and orbihedra*, Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky, ed), Proc. ICTP Trieste 1990, World Scientific, Singapore, 1991, 504–540.

[20] F. Haglund, *Réseaux de Coxeter-Davis et commensurateurs*, Ann. Inst. Fourier, Grenoble **48** (1998), 649–666.

[21] ———, *Existence, unicité et homogénéité de certains immeubles hyperboliques*, Math Z. **242** (2002), 97–148.

[22] F. Haglund and F. Paulin, *Constructions arborescentes d'immeubles*, Math. Ann. **325** (2003), 137–164.

[23] T. Januszkiewicz and J. Świątkowski, *Hyperbolic Coxeter groups of large dimension*, Comment. Math. Helv. **78** (2003) 555–583.

[24] D. Kazhdan and G. Margulis, *A proof of Selberg's hypothesis*, Mat. Sbornik **75** (1968), 162–168.

[25] S. Lim and A. Thomas, *Counting overlattices in automorphism groups of buildings*, in preparation.

[26] L. Potyagailo and E. Vinberg, *On right-angled reflection groups in hyperbolic spaces*, Comment. Math. Helv. **80** (2005), 63–73.

[27] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer-Verlag, New York, 1972.

[28] M. Ronan, *Lectures on Buildings*, Academic Press, Boston, 1989.

[29] G. E. Rosenberg, *Towers and Covolumes of Tree Lattices*, Ph.D. Thesis, Columbia University, 2001.

[30] J. P. Serre, *Trees*, Springer-Verlag, Berlin, 2000.

[31] A. Thomas, *Covolumes of uniform lattices acting on polyhedral complexes*, to appear in Bull. London Math. Soc.

[32] W. Thurston, *Three-Dimensional Geometry and Topology*, Princeton University Press, Princeton, 1997.

[33] E. B. Vinberg, *Absence of crystallographic reflection groups in Lobachevsky space of large dimension*, Trudy Moskov. Math. Obshch. **47** (1984), 68–102; English transl.: Trans. Mosc. Math. Soc. **1985**, 75–112.

[34] E. B. Vinberg and O. V. Shvartsman, *Discrete groups of motions of spaces of constant curvature*, Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya **29** (1988), 147–259; English transl.: in *Geometry II: Spaces of Constant Curvature*, Encyclopaedia of Math. Sci. 29, Springer-Verlag, Berlin 1993, 139–248.

[35] H. C. Wang, *Topics on totally discontinuous groups in symmetric spaces*, (W. Boothey ed) M. Dekker (1972), 460–487.

Department of Mathematics, University of Chicago, Chicago, Illinois 60637
E-mail address: `athomas@math.uchicago.edu`