Global aspects of Lorentzian manifolds with special holonomy

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Outline

1. Special Lorentzian holonomy
2. Geodesic completeness of compact pp-waves
3. Lorentzian manifolds with disconnected holonomy groups

[Joint work with D. Schliebner, arXiv:1306.0120, and with H. Baum & K. Lärz, arXiv:1204.5657, all Humboldt University Berlin]

Holonomy groups in a nutshell

- Let $(\mathcal{M}, \mathbf{g})$ be a semi-Riemannian manifold $\rightsquigarrow$ Parallel transport

$$\mathcal{P}_\gamma \quad : \quad T_{\gamma(0)}\mathcal{M} \ni X_0 \quad \stackrel{\sim}{\longmapsto} \quad X(1) \in T_{\gamma(1)}M$$

where $X(t)$ is the solution to the ODE $\nabla_{\dot\gamma(t)}X(t) \equiv 0$ with $X(0) = X_0$.

For $p \in M^n$ we define the (connected) holonomy group

$$\mathrm{Hol}_p^0(\mathcal{M}, \mathbf{g}) := \left\{\mathcal{P}_\gamma \,\middle|\, \gamma(0) = \gamma(1) = p, \gamma \sim \{p\}\right\} \subset \mathrm{O}(T_p\mathcal{M}, \mathbf{g}_p) \simeq \mathrm{O}(r, s)$$

- For $p, q \in M$: $\mathrm{Hol}_p(\mathcal{M}, \mathbf{g}) \sim \mathrm{Hol}_q(\mathcal{M}, \mathbf{g})$ conjugated in $\mathrm{O}(r, s)$.
- $\mathrm{Hol}^0_p(\mathcal{M}, \mathbf{g}) \subset \mathrm{Hol}_p(\mathcal{M}, \mathbf{g})$ normal and

$$\Pi_1(\mathcal{M}, p) \ni [\gamma] \quad \stackrel{\text{surjects}}{\twoheadrightarrow} \quad \left[\mathcal{P}_\gamma\right] \in \mathrm{Hol}_p(\mathcal{M}, \mathbf{g})\big/\mathrm{Hol}^0_p(\mathcal{M}, \mathbf{g})$$

- Ambrose-Singer: $\mathfrak{hol}_p(\mathcal{M}, \mathbf{g})$ is spanned by

$$\mathcal{P}_\gamma^{-1} \circ R_{\gamma(1)}(X, Y) \circ \mathcal{P}_\gamma \in \mathfrak{so}(T_p\mathcal{M}, \mathbf{g}_p),$$

where $\gamma(0) = p$, $R_{\gamma(1)}$ the curvature at $\gamma(1)$, $X, Y \in T_{\gamma(1)}M$.

Special Lorentzian holonomy

A Lorentzian manifold $(\mathcal{M}^{n+2}, \mathbf{g})$ has special holonomy if

1. $\mathrm{Hol}^0 \neq \mathrm{SO}^0(1, n+1)$ and
2. Hol^0 acts indecomposably on $T_p\mathcal{M}$, i.e., without non-degenerate Hol^0-invariant subspaces.

(Riemannian: indecomposable = irreducible, Berger's list)

- De Rham/ Wu decomposition
  $\Longrightarrow (\mathcal{M}, \mathbf{g})$ is not a product, not even locally.

Fundamental difference to Riemannian:

- $\mathrm{Hol}^0 \subset \mathrm{SO}^0(1, n+1)$ irreducible $\overset{\left[\substack{\text{Berger} \\ \text{DiScala,Olmos}}\right]}{\Longrightarrow}$ $\mathrm{Hol}^0 = \mathrm{SO}^0(1, n+1)$.
  Special holonomy $\Longrightarrow$
  Hol^0-invariant null line $\mathbb{L} \subset T_p\mathcal{M}$, i.e., $\mathrm{Hol}^0 \subset \mathrm{Stab}_{\mathrm{O}(1,n+1)}(\mathbb{L})$.
- $\mathrm{Nor}_{\mathrm{O}(1,n+1)}(\mathrm{Hol}^0) \subset \mathrm{Stab}_{\mathrm{O}(1,n+1)}(\mathbb{L}) \simeq (\mathbb{R}^* \times \mathrm{O}(n)) \ltimes \mathbb{R}^n \Longrightarrow$
  $\mathbb{L}$ is Hol-invariant.

Geometrically: $\mathcal{M}$ admits a parallel null line bundle, i.e., fibres are invariant under parallel transport.

Classification [Bérard-Bergery & Ikemakhen '93, TL '03]

If $(\mathcal{M}^{n+2}, \mathbf{g})$ is Lorentzian with special holonomy, then $\mathrm{Hol}^0(\mathcal{M}, \mathbf{g}) \simeq$

1. $G \ltimes \mathbb{R}^n$ or $(\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n$, where G is a Riemannian holonomy group,
2. $(A \times G^s) \ltimes \mathbb{R}^{n-k}$, where $G := \mathrm{pr}_{SO(n)}(\mathrm{Hol}^0(\mathcal{M}, \mathbf{g}))$ is a Riemannian holonomy group G and G^s its semisimple part,
   $A \subset \mathbb{R}^+ \times Z(G)$ if $k = 0$, or $A \subset Z(G) \times \mathbb{R}^k$
   (in fact, $A = \mathrm{graph}(\Psi)$ for $\Psi \in \mathrm{Hom}(Z(G), \mathbb{R}^+$ or $\mathbb{R}^k)$.

- For all possible groups there exist metrics [... Galaev '06].
- E.g.: $(\mathcal{N}^n, \mathbf{h})$ Riemannian, $H \in C^\infty(\mathbb{R}^2 \times \mathcal{N})$, $\exists p$: $\det(\nabla^{\mathbf{h}} dH)_p \neq 0 \Rightarrow$

$$\left(\mathcal{M} = \mathbb{R}^2 \times \mathcal{N}\ ,\ \mathbf{g} = \mathbf{g}^{\mathbf{h},H} := 2du(dv + Hdu) + \mathbf{h}\right)$$

  has holonomy $(\mathbb{R}^+ \times \mathrm{Hol}(\mathcal{N}, \mathbf{h})) \ltimes \mathbb{R}^{n-2}$ or $\mathrm{Hol}(\mathcal{N}, \mathbf{h}) \ltimes \mathbb{R}^{n-2}$, if $\frac{\partial H}{\partial v} = 0$.

- Are there compact or geodesically complete examples for all groups?
- What are possible full holonomy groups, i.e., $\mathrm{Hol}/\mathrm{Hol}^0 =$?

pp-waves ("plane fronted with parallel rays" [Ehlers-Kundt])

Definition: A Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ is a

- *pp-wave* if it admits a parallel null vf V and $\mathrm{R}(U, W) = 0\ \forall U, W \in V^{\perp}$.
- *standard pp-wave* if $\mathcal{M} = \mathbb{R}^{n+2} \ni (u, v, x^1, \dots, x^n)$ and

$$\mathbf{g} = \mathbf{g}^H := 2du(dv + Hdu) + \delta_{ij}dx^i dx^j \tag{1}$$

for a smooth function H with $\partial_v H = 0$.

Equivalences: $(\mathcal{M}, \mathbf{g})$ is a pp-wave

$\Leftrightarrow$ it is locally of the form (1),

$\Leftrightarrow$ $\nabla V = 0$ & $\mathrm{R}(X, Y) : V^{\perp} \to \mathbb{R}V,\ \forall\ X, Y \in T\mathcal{M}$,

$\Leftrightarrow$ $\nabla V = 0$ & the *screen bundle* $\Sigma := V^{\perp}/_{\mathbb{R}\cdot V}$ is flat w.r.t. $\nabla^{\Sigma}[\sigma] := [\nabla\sigma]$,

$\Leftrightarrow$ $\mathrm{Hol}^0(\mathcal{M}, \mathbf{g}) \subset \mathbb{R}^n$,

$\Leftrightarrow$ $\mathrm{Hol}(\mathcal{M}, \mathbf{g}) \subset \Gamma \ltimes \mathbb{R}^n$ for $\Gamma \subset \mathrm{O}(n)$ discrete,

$\Leftrightarrow$ $\nabla V = 0$ & locally, $\exists\ S_1, \dots S_n \in \Gamma(V^{\perp})$ with $g(S_i, S_j) = \delta_{ij}$ and $\nabla S_i = \alpha^i \otimes V$, where α^i local one-forms with $d\alpha^i|_{V^{\perp}\wedge V^{\perp}} = 0$.

Geodesic completeness for compact pp-waves

- Compact Lorentzian manifolds are not always geodesically complete.
- They are if: homogeneous (Marsden '72), of constant curvature (Carriére '89, Klingler '96), or have a time-like conformal vf (Romero/Sánchez '95) Are compact pp-waves complete?
- Ehlers-Kundt '62: *"Prove that complete, Ricci-flat 4-dim pp-waves are plane waves, no matter which topology one chooses!"* *(EK)*
- *plane wave* = *pp-wave with* $\nabla_X \mathrm{R} = 0 \ \forall\ X \in V^{\perp}$.

Theorem (Schliebner/TL '13)

Let $(\mathcal{M}, \mathbf{g})$ *be a compact pp-wave. Then:*

1. *Its universal cover is globally isometric to a standard pp-wave.*
2. $(\mathcal{M}, \mathbf{g})$ *is geodescically complete.*

Corollary

Every compact Ricci-flat pp-wave is a plane wave.

Thm and Corollary give a proof of (EK) in the compact case (and any dim).

Steps in the proof

1. $\nabla V = 0$ and $\mathcal{M}$ compact $\Longrightarrow \widetilde{\mathcal{M}} \simeq \mathbb{R} \times \mathcal{N}$, $\mathcal{N}$ leaf of $\widetilde{V}^\perp$.
2. Curvature condition $\Longrightarrow \nabla$ induces flat connection on $\mathcal{N}$.
3. $\exists$ complete, ∇-parallel frame fields on $\mathcal{N} \overset{[Palais]}{\Longrightarrow} \mathcal{N} \simeq \mathbb{R}^{n+1}$
4. Z a *screen vf*, i.e., Z null and $\mathbf{g}(V, Z) = 1$, γ integral curve of $\widetilde{Z} \Longrightarrow$

$$\Phi : \mathbb{R}^{n+2} \ni (u, v, x^1, \ldots, x^n) \mapsto \exp^{\widetilde{\mathbf{g}}}_{\gamma(u)} \big(v\, \widetilde{V}(\gamma(u)) + x^k S_k(\gamma(u)) \big) \in \widetilde{\mathcal{M}}$$

   diffeomorphism, $\Phi^* \widetilde{\mathbf{g}}$ is a standard pp-wave with $2H := (\Phi^* \widetilde{g})(\partial_u, \partial_u)$.
5. $\mathcal{M}$ compact $\Longrightarrow \frac{\partial^2 H}{\partial x^i \partial x^j}$ bounded $\Longrightarrow (\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ and thus $(\mathcal{M}, \mathbf{g})$ complete

Proof of corollary: $\mathrm{Ric} = 0 \Rightarrow H$ and thus $\partial_i \partial_j H$ harmonic for $\Delta^0 = \sum_{i=1}^n \partial_i^2$. $\partial_i \partial_j H$ bounded $\Rightarrow$ independent of x^i.

Holonomy groups and covering maps

$(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ semi-Riemannian, $\Gamma \subset \mathrm{Iso}(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ properly discontinuous
$\Longrightarrow$ covering map $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}}) \stackrel{\pi}{\longrightarrow} (\mathcal{M} := \widetilde{\mathcal{M}}/\Gamma, \mathbf{g})$.
For $p \in M$ and $\widetilde{p} \in \pi^{-1}(p)$:

1. injective group homomorphism
$$\iota : \mathrm{Hol}_{\widetilde{p}}(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}}) \hookrightarrow \mathrm{Hol}_p(\mathcal{M}, \mathbf{g}), \quad \widetilde{P}_{\widetilde{\gamma}} \longmapsto P_{\pi\circ\widetilde{\gamma}},$$
for $\widetilde{\gamma}$ a loop at $\widetilde{p}$, and the image is a normal subgroup.
2. surjective group homomorphism
$$\Phi \ : \ \Gamma \twoheadrightarrow \mathrm{Hol}_p(\mathcal{M}) \big/ \mathrm{Hol}_{\widetilde{p}}(\widetilde{\mathcal{M}}), \quad \sigma \longmapsto \left[P_{\pi\circ\widetilde{\gamma}}\right],$$
$\widetilde{\gamma}$ is a curve in $\widetilde{\mathcal{M}}$ from $\widetilde{p}$ to $\sigma^{-1}(\widetilde{p})$.

For a loop γ at $p \in M$, we have:
$$P_\gamma = d\sigma|_{\sigma^{-1}(\widetilde{p})} \circ \tilde{P}_{\tilde{\gamma}} \quad (\text{using } T_{\widetilde{p}}\widetilde{\mathcal{M}} \stackrel{d\pi_{\widetilde{p}}}{\simeq} T_pM),$$
$\widetilde{\gamma}$ is the lift of γ starting at $\widetilde{p}$ and ending at $\sigma^{-1}(\widetilde{p})$ with $\sigma \in \Gamma$. I.e.,
$$\left[d\sigma|_{\sigma^{-1}(\widetilde{p})} \circ \tilde{P}_{\tilde{\gamma}} = (d\sigma^{-1}|_{\widetilde{p}})^{-1} \circ \tilde{P}_{\tilde{\gamma}}\right] \in \Phi(\sigma) \in \mathrm{Hol}_p(M) \big/ \mathrm{Hol}_{\widetilde{p}}(\widetilde{\mathcal{M}}).$$

Let $(\mathcal{N}^n, \mathbf{h})$ be Riemannian, $H \in C^\infty(\mathbb{R}^2 \times \mathcal{N})$, $\exists p$: $\det(\nabla^{\mathbf{h}} dH)_p \neq 0$.

$$\left(\widetilde{\mathcal{M}} = \Omega \times \mathcal{N}\ ,\ \widetilde{\mathbf{g}} = \mathbf{g}^{\mathbf{h},H} := 2du(dv + H du) + \mathbf{h}\right)$$

$\Omega \subset \mathbb{R}^2$ open domain. Isometries of $(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ are of the form:

$$\sigma \begin{pmatrix} v \\ \mathbf{x} \\ u \end{pmatrix} = \begin{pmatrix} av + \tau(u, \mathbf{x}) \\ \rho(u, \mathbf{x}) \\ a^{-1} u + b \end{pmatrix}, \text{ with } \rho(u, .) \in \mathrm{Iso}(\mathcal{N}, \mathbf{h}),\ a, b \text{ constants, ...}$$

Theorem (Baum, Lärz,TL '12)

Let $\pi : (\widetilde{\mathcal{M}}, \mathbf{g}^{H,\mathbf{h}}) \to (\mathcal{M}, \mathbf{g}) := \widetilde{M}/\Gamma$ *be a covering map. Then, for* $\sigma \in \Gamma$ *a representative of* $\Phi(\sigma) \in \mathrm{Hol}_p(\mathcal{M}) \big/ \mathrm{Hol}_{\widetilde{p}}(\widetilde{\mathcal{M}})$ *is given by*

$$\hat{\phi}(\sigma) = \begin{pmatrix} a & 0 & 0 \\ 0 & (\mathrm{d}\rho^{-1}(u, v, .)|_{\mathbf{x}})^{-1} \circ \mathrm{P}^{\mathbf{h}}_{\sigma} & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \in \Phi(\sigma),$$

In particular, $\mathrm{Hol}_{\pi(\widetilde{q})}(M) = \left\{ \hat{\phi}(\sigma) \middle| \sigma \in \Gamma \right\} \cdot \mathrm{Hol}_p(N, h) \ltimes \mathbb{R}^n.$

Examples with disconnected holonomy groups [BLL '12]

Using certain $\Gamma \subset \mathrm{Iso}(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$ we obtain examples with disconnected $\mathrm{Hol} =$

$\mathbb{Z}^p \ltimes \mathbb{R}^n$, $(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \ltimes \mathrm{R}^n$, $(\mathbb{Z} \oplus \mathbb{Z}) \ltimes \mathrm{R}^n$, $(\mathbb{Z} \ltimes \mathrm{SU}(n)) \ltimes \mathbb{R}^{2n}$, $(\mathbb{Z}_2 \ltimes \mathrm{SU}(n)) \ltimes \mathbb{R}^{2n}$

Example with infinitely generated holonomy group

- $\mathcal{N} := \mathbb{R}^2 \setminus \mathbb{Z}^2$, flat metric $\mathbf{h} = dx^2 + dy^2$, $\Gamma := \Pi_1(\mathcal{N}) = \mathbb{Z} * \mathbb{Z} * \dots$ infinitely generated free group, $\mathrm{Hol}(\mathcal{N}, \mathbf{h})$ trivial. Fix $H \in C^\infty(\mathcal{N})$.
- $\pi : \mathbb{R}^2 \to \mathcal{N} = \mathbb{R}^2/\Gamma$ univ. cover, $\widetilde{\mathbf{h}} = \pi^* \mathbf{h}$, $\widetilde{H} := H \circ \pi$ are Γ-invariant.
- $\Omega := \left\{(v, u) \in \mathbb{R}^2 \mid u > 0\right\}$, $\widetilde{\mathcal{M}} := \Omega \times \mathbb{R}^2$, $\widetilde{\mathbf{g}} = 2du(dv + \frac{\widetilde{H}}{u^2} du) + \mathbf{h}$.
- Fix generators $(\gamma_1, \gamma_2, \dots)$ of Γ, $\underline{\lambda} := (\lambda_1, \lambda_2, \dots)$ lin. indep. over $\mathbb{Q}$, $\sigma_i(v, u, x) := (\mathrm{e}^{\lambda_i} v, \mathrm{e}^{-\lambda_i} u, \gamma_i(x))$, $\Gamma_{\underline{\lambda}} := \left\langle \sigma_i \mid i = 1, 2, \dots \right\rangle \subset \mathrm{Iso}(\widetilde{\mathcal{M}}, \widetilde{\mathbf{g}})$.
- $\Gamma_{\underline{\lambda}}$ acts properly discontinuous on $\widetilde{\mathcal{M}}$ and $\mathcal{M} = \widetilde{\mathcal{M}}/\Gamma_{\underline{\lambda}}$ is LMf with metric $\mathbf{g}$, $\mathrm{Hol}(\mathcal{M}, \mathbf{g})$ is infinitely generated by

$$\begin{pmatrix} \mathrm{e}^{\lambda_i} & w & * \\ 0 & 1_2 & * \\ 0 & 0 & \mathrm{e}^{-\lambda_i} \end{pmatrix} \in \mathrm{O}(1,3), \quad w \in \mathbb{R}^2$$