Chevalley groups and finite geometry

~~Expecte~~ 453 ① 30/9/13

[PhD thesis supervised by A. Ram (Unimelb) and J. Bamberg (UWA)]
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Thesis problem: Is there an $\boxed{\text{ovoid}}$ in the $\boxed{\text{finite Hermitian variety } H(5,q^2)}$?

~~space~~

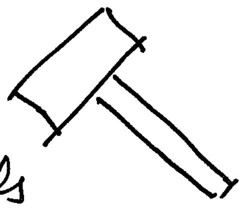
'finite ~~oval~~ 3-dimensional oval' [J. Tits Ovoïdes et translations]

No for $q=2$ [de Beule - Metsch 2006]

Unknown for $q>2$.

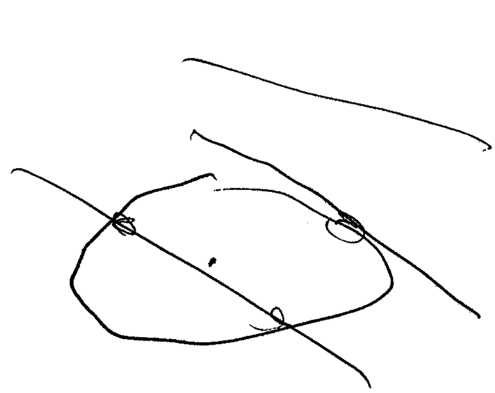
∄ an ovoid - Jeff Thas.

Idea: ~~Cref~~ Craft a hammer



@big words

Using Schubert calculus, Chevalley groups, Steinberg presentations, sheaf cohomology, intersection cohomology

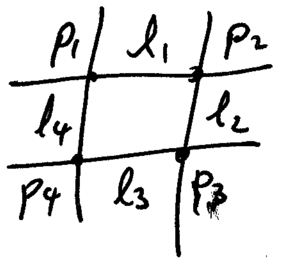


Incidence geometries

An *incidence geometry* $\mathcal{G}$ is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$.

Example

Quadrangle:



$$\mathcal{P} = \{p_1, p_2, p_3, p_4\}$$
$$\mathcal{L} = \{l_1, l_2, l_3, l_4\}$$
$$\mathcal{I} = \{(p_1, l_1), (p_1, l_4), (p_2, l_1) \\ \ldots\}$$

An *ovoid* in $\mathcal{G}$ is a set of points $\mathcal{O} \subseteq \mathcal{P}$ such that every line $l \in \mathcal{L}$ is incident with $\mathcal{O}$ exactly once.

Example For $\mathcal{G} =$ quadrangle, $\mathcal{O} = \{p_1, p_3\}$ and $\mathcal{O} = \{p_2, p_4\}$ are ovoids.

The finite Hermitian variety $H(3, q^2)$

$\mathbb{F}_{q^2} =$ a finite field with q^2 elements, q a prime power.

$V = \mathbb{F}_{q^2}^4$, a vector space over $\mathbb{F}_{q^2}$.

The <u>Frobenius automorphism</u> is

$$- : \mathbb{F}_{q^2} \longrightarrow \mathbb{F}_{q^2}$$
$$c \longmapsto \bar{c} := c^q$$

The <u>Hermitian form</u> is

$$\beta : V \times V \longrightarrow \mathbb{F}_{q^2}$$
$$\beta(v,w) = v_1\bar{w}_1 + v_2\bar{w}_2 + v_3\bar{w}_3 + v_4\bar{w}_4 .$$

A subspace $W \subseteq V$ is <u>totally isotropic</u> if $\beta(w,w') = 0$ for all $w, w' \in W$.

The <u>finite Hermitian variety</u> $H(3,q^2)$ is the incidence geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ with

$$\mathcal{P} = \left\{ \begin{matrix} \text{1-dimensional totally isotropic} \\ \text{subspaces of } V \end{matrix} \right\}$$

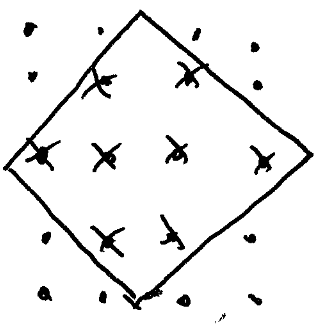
$$\mathcal{L} = \left\{ \begin{matrix} \text{2-dimensional totally isotropic} \\ \text{subspaces of } V \end{matrix} \right\}$$

$$\mathcal{I} = \{ (p,l) \in \mathcal{P} \times \mathcal{L} \mid p \subseteq l \}.$$

<u>Example of an ovoid</u>

$$\mathcal{O} = \{ \langle (v_1, v_2, v_3, 0) \rangle \mid v_1, v_2, v_3 \in \mathbb{F}_4 \} \cap \mathcal{P}$$

is an ovoid in $H(3,4)$.



Fact: Ovoids exist in $H(3,q^2)$ for $q \geq 2$.

Recall: (Thas's problem) Is there an ovoid in $H(5,q^2)$ for $q \geq 2$?

Conversion to Chevalley group world

Studying $H(3,q^2)$ = Studying the Chevalley group $U_4(\mathbb{F}_{q^2})$.

Ovoids in $U_4(\mathbb{F}_{q^2})$

~~Ovoids~~ Let $G = U_4(\mathbb{F}_{q^2})$ and let P_1 and P_2 be certain parabolic subgroups of G. A set of P_1 cosets $\mathcal{O} = \{g_1P_1, g_2P_1, \dots, g_kP_1\}$ is an ovoid if

$$\bigcup_{i \in \{1,2,\dots,k\}} g_i P_1 P_2 = G$$

and the union is disjoint.

Analogously ~~Stu~~:
Studying ovoids in $H(5,q^2)$ = Studying ovoids in $U_{\cancel{4}6}(\mathbb{F}_{q^2})$.

Let $G = U_6(\mathbb{F}_{q^2})$ and let P_1 and P_3 be certain parabolic subgroups of G. A set of P_1 cosets $\mathcal{O} = \{g_1P_1, g_2P_1, \dots, g_kP_1\}$ is an ovoid if

$$\bigcup_{i \in \{1,2,\dots,k\}} g_i P_1 P_3 = G \quad \cancel{\text{and}}$$

and the union is disjoint.

Current task · Find 'good' generators and relations for $U_4(\mathbb{F}_{q^2})$ ⑤

[Steinberg 'Notes on Chevalley groups' 1967 ... ~~off~~ by a minus sign!]

Future tasks: Find ~~g~~good generators and relations for $U_6(\mathbb{F}_{q^2})$, $Sp_4(\mathbb{F}_{q^2})$.

related ~~to other~~
~~to questions about~~