A First Course in Linear Algebra

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Hints and Short Solutions to Selected Exercises

Chapter 0 Introduction

0.1. Observe that $\frac{1}{2}(2x+4) - x \;=\; x + 2 - x \;=\; 2\,.$

0.2. The y-intercept is 2 and the x-intercept is 3. The slope of the line regarding the x-axis as horizontal is $-2/3$. Regarding the y-axis as horizontal the slope is $-3/2$.

0.3. The equation of the reflected line is $3x+2y = 6$, with slope $-3/2$, which is the reciprocal of the slope of the original line.

0.4. The equation of the rotated line is $-3x+2y = 6$, with slope $3/2$, which is the negative reciprocal of the slope of the original line.

0.5. The point of intersection is $(6/5, 6/5)$.

0.6.* The equation of the reflected line is $bx + ay = c + k(a-b)\,.$

0.7.* The equation of the rotated line is $-bx + ay = c + a(x_0 - y_0) + b(x_0 + y_0)\,.$

0.10.** Verify first that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$, and then play with inequalities.

Chapter 1 Geometric Vectors

1.1. (i) $\mathbf{v}+\mathbf{u}$ (ii) $\mathbf{u}-\mathbf{v}$ (iii) $\mathbf{u}-\mathbf{v}$ (iv) $\mathbf{u}+\mathbf{v}-\mathbf{w}$ (v) $\mathbf{u}-\mathbf{v}-\mathbf{w}$

1.2. (i) $\mathbf{x} = \mathbf{a} - \mathbf{b}$ (ii) $\mathbf{x} = \mathbf{c} - \dfrac{12}{7}\mathbf{b}$

1.4. $\left(\overrightarrow{PQ} + \overrightarrow{QR}\right) + \overrightarrow{RS} \;=\; \overrightarrow{PR} + \overrightarrow{RS} \;=\; \overrightarrow{PS} \;=\; \overrightarrow{PQ} + \overrightarrow{QS} \;=\; \overrightarrow{PQ} + \left(\overrightarrow{QR} + \overrightarrow{RS}\right)$

1.5. The rule works because ratios of corresponding sides of similar triangles are equal, and triangles which have been rotated 180° are congruent.

1.6. $\overrightarrow{OR} \;=\; \overrightarrow{OP} + \overrightarrow{PR} \;=\; \overrightarrow{OP} + \frac{1}{2}\left(\overrightarrow{PO} + \overrightarrow{OQ}\right) \;=\; \overrightarrow{OP} - \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OQ} \;=\; \frac{1}{2}\left(\overrightarrow{OP} + \overrightarrow{OQ}\right)$

1.7.* Use the previous exercise. Check that the sum of vectors representing the medians, emanating from the vertices and pointing in the direction of midpoints of opposite sides, is the zero vector.

1.9.* Let $PQRS$ be a parallelogram and suppose T is the midpoint of the diagonal PR. Verify that $\overrightarrow{QT} = \frac{1}{2}\overrightarrow{QS}$. First steps: expand $\overrightarrow{QT} = \overrightarrow{QP} + \overrightarrow{PT} = \overrightarrow{QP} + \frac{1}{2}\overrightarrow{PR}$.

1.10.** Suppose that $\overrightarrow{PU} = \alpha\overrightarrow{PR}$ and $\overrightarrow{QU} = \beta\overrightarrow{QT}$. Use the fact that $\overrightarrow{PS}$ and $\overrightarrow{PQ}$ are not parallel to deduce that $\alpha = \beta = \dfrac{r+s}{r+2s}\,.$

Chapter 2 Position Vectors and Components

2.1. (i) $5\mathbf{i}+9\mathbf{j}+4\mathbf{k}$ (ii) $3\mathbf{j}-4\mathbf{k}$ (iii) $5\mathbf{i}+15\mathbf{j}-4\mathbf{k}$ (iv) $3\mathbf{i}+6\mathbf{j}-6\mathbf{k}$ (v) $\dfrac{\mathbf{i}}{2}-\dfrac{3\mathbf{j}}{2}-4\mathbf{k}$ (vi) 13 (vii) 5 (viii) 3 (ix) 9 (x) 21 (xi) $\dfrac{5\mathbf{i}}{13}-\dfrac{12\mathbf{j}}{13}$ (xii) $-\dfrac{3\mathbf{j}}{5}+\dfrac{4\mathbf{k}}{5}$ (xiii) $\dfrac{\mathbf{i}}{3}+\dfrac{2\mathbf{j}}{3}-\dfrac{2\mathbf{k}}{3}$ (xiv) $\sqrt{6}$ (xv) $\sqrt{62}$

2.2. (i) $\overrightarrow{OP}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\overrightarrow{OQ}=-\mathbf{i}-\mathbf{j}$, $\overrightarrow{OR}=\mathbf{j}+2\mathbf{k}$, $\overrightarrow{OS}=2\mathbf{i}+3\mathbf{j}+3\mathbf{k}$, $\overrightarrow{PQ}=-2\mathbf{i}-2\mathbf{j}-\mathbf{k}$, $\overrightarrow{QP}=2\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\overrightarrow{QR}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $\overrightarrow{RS}=2\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\overrightarrow{SP}=-\mathbf{i}-2\mathbf{j}-2\mathbf{k}$

(ii) Observe that $\overrightarrow{PQ}=\overrightarrow{SR}$ so $PQRS$ is a parallelogram. Further $|\overrightarrow{PQ}|=|\overrightarrow{QR}|=3$, so $PQRS$ is a rhombus. But the diagonals may be represented by the vectors $\overrightarrow{PR}=-\mathbf{i}+\mathbf{k}$ of length $\sqrt{2}$ and $\overrightarrow{QS}=3\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ of length $\sqrt{34}\neq\sqrt{2}$, so the rhombus cannot be a square.

2.4. (i) -2, 6 (ii) $4/3$

2.6.* $\overrightarrow{OR} = \overrightarrow{OP}+\overrightarrow{PR} = \overrightarrow{OP}+\dfrac{\lambda}{\lambda+\mu}\left(\overrightarrow{PO}+\overrightarrow{OQ}\right) = \dfrac{\mu\,\overrightarrow{OP}+\lambda\,\overrightarrow{OQ}}{\lambda+\mu}$

2.7.* $\overrightarrow{QT}=\dfrac{1}{3}(\mathbf{v}-2\mathbf{u})$, $\overrightarrow{QB}=\dfrac{1}{2}(\mathbf{v}-2\mathbf{u})$,

2.8.* If $\lambda_1\mathbf{v}_1+\cdots+\lambda_n\mathbf{v}_n=\mathbf{0}$ where some $\lambda_i\neq 0$ then

$$\mathbf{v}_i = \left(\frac{-\lambda_1}{\lambda_i}\right)\mathbf{v}_1+\cdots+\left(\frac{-\lambda_{i-1}}{\lambda_i}\right)\mathbf{v}_{i-1}+\left(\frac{-\lambda_{i+1}}{\lambda_i}\right)\mathbf{v}_{i+1}+\cdots+\left(\frac{-\lambda_n}{\lambda_i}\right)\mathbf{v}_n\,.$$

Conversely, if some $\mathbf{v}_i$ is a linear combination of the other vectors, say

$$\mathbf{v}_i = a_1\mathbf{v}_1+\cdots+a_{i-1}\mathbf{v}_{i-1}+a_{i+1}\mathbf{v}_{i+1}+\cdots+a_n\mathbf{v}_n\,,$$

then

$$a_1\mathbf{v}_1+\cdots+a_{i-1}\mathbf{v}_{i-1}+(-1)\mathbf{v}_i+a_{i+1}\mathbf{v}_{i+1}+\cdots+a_n\mathbf{v}_n = \mathbf{0}\,,$$

so that the implication in the definition of linear independence fails.

2.10.** Suppose that

$$\lambda_0 f_0+\cdots+\lambda_n f_n = \mathbf{0}\,,$$

where $\lambda_0,\ldots,\lambda_n$ are real numbers and $\mathbf{0}$ denotes the zero function. Interpret this as a polynomial equation. Prove that a nonzero polynomial of degree n has at most n roots. Use the fact that there are infinitely many real numbers to deduce that $\lambda_0=\ldots=\lambda_n=0$.

Chapter 3 Dot Products and Projections

3.1. (i) $-36\,,\ 29\,,\ -14\,,\ 15\,,\ 100$ (ii) obtuse, obtuse, acute

(iii) $-\dfrac{36}{5}\,,\quad \dfrac{36}{25}\big(3\mathbf{j}-4\mathbf{k}\big)\,,\quad 5\mathbf{i}+\dfrac{192}{25}\mathbf{j}+\dfrac{144}{25}\mathbf{k}$

(iv) $\dfrac{5}{3}\big(\mathbf{i}+2\mathbf{j}-2\mathbf{k}\big)\,,\quad \dfrac{1}{3}\big(10\mathbf{i}+17\mathbf{j}+22\mathbf{k}\big)\,,\quad \dfrac{15}{122}\big(5\mathbf{i}+9\mathbf{j}+4\mathbf{k}\big)\,,\quad \dfrac{1}{122}\big(47\mathbf{i}+109\mathbf{j}-304\mathbf{k}\big)$

3.2. (i) acute, obtuse (ii) both equal to $\left(\frac{1}{2}, 1, \frac{3}{2}\right)$ (iii) $\overrightarrow{PR}\cdot\overrightarrow{QS}=0$

3.3. (i) $\mathbf{v}\cdot(a\mathbf{x}+b\mathbf{y}) \;=\; a(\mathbf{v}\cdot\mathbf{x})+b(\mathbf{v}\cdot\mathbf{y}) \;=\; a(0)+b(0) \;=\; 0$

(ii) $\mathbf{v}\cdot\mathbf{y} \;=\; \mathbf{v}\cdot\left(\dfrac{1}{b}(a\mathbf{x}+b\mathbf{y}-a\mathbf{x})\right) \;=\; \dfrac{1}{b}\Big(\mathbf{v}\cdot(a\mathbf{x}+b\mathbf{y})-a(\mathbf{v}\cdot\mathbf{x})\Big) \;=\; \dfrac{1}{b}(0-a(0)) \;=\; 0$

3.4. Expand brackets, bring scalars to the front, and evaluate.

3.5. Drop a perpendicular to create a right angled triangle.

3.6.* Given $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ and $\mathbf{w}=d\mathbf{i}+e\mathbf{j}+f\mathbf{k}$ and the geometric definition, we have, applying the Cosine Rule at the second step, and the length formula at the third step,

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{w} \;&=\; |\mathbf{v}||\mathbf{w}|\cos\theta \;=\; \frac{1}{2}\big(|\mathbf{v}|^2+|\mathbf{w}|^2-|\mathbf{v}-\mathbf{w}|^2\big)\\
&=\; \frac{1}{2}\Big(a^2+b^2+c^2+d^2+e^2+f^2-\big((a-d)^2+(b-d)^2+(c-f)^2\big)\Big)\\
&=\; \frac{1}{2}\big(2ad+2bd+2ef\big) \;=\; ad+bd+ef\ .
\end{aligned}$$

3.7.* Let $PQRS$ be a rhombus, so $\big|\overrightarrow{PQ}\big| = \big|\overrightarrow{QR}\big|$. Observe that

$$\overrightarrow{PR}\cdot\overrightarrow{QS} \;=\; \big(\overrightarrow{PQ}+\overrightarrow{QR}\big)\cdot\big(\overrightarrow{QR}+\overrightarrow{RS}\big)\ ,$$

expand brackets, and evaluate to 0.

3.8.* Let PQR be a triangle, C the midpoint of PQ and D the intersection of the perpendicular bisectors of PR and QR. Observe that

$$\overrightarrow{DC}\cdot\overrightarrow{PQ} \;=\; \big(\overrightarrow{DB}+\overrightarrow{BC}\big)\cdot\big(\overrightarrow{PR}+\overrightarrow{RQ}\big)\ ,$$

expand brackets, and evaluate to 0.

3.9.* With the notation of the previous exercise, use Pythagoras to show that $\big|PD\big| = \big|RD\big|$.

3.10.** Proving that the geometric dot product distributes over vector addition is the difficult part. Draw a good diagram and carefully label angles. Apply the Sine Rule for triangles and manipulate expressions using trigonometric formulae.

Chapter 4 Cross Products

4.1. (i) $48\mathbf{i} - 20\mathbf{j} - 15\mathbf{k}$ (ii) $-24\mathbf{i} + 10\mathbf{j} - 2\mathbf{k}$ (iii) $-2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$
(iv) $-26\mathbf{i} + 14\mathbf{j} + \mathbf{k}$ (v) $70\mathbf{i} + 81\mathbf{j} + 116\mathbf{k}$ (vi) 38 (vii) 38 (viii) $38\mathbf{v}$

4.2. (i) $\dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$ (ii) $-\dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$

4.3. (i) Both areas are $\dfrac{\sqrt{17}}{2}$, and there is no surprise since $PQRS$ is a rhombus.

(ii) $d_1 = \sqrt{2}$, $d_2 = \sqrt{34}$, $\dfrac{d_1 d_2}{4} = \dfrac{\sqrt{17}}{2}$, which coincides with the areas of the triangles in the first part, which is no surprise since the diagonals of a rhombus are mutually perpendicular.

4.5. (i) (c) (ii) (d) (iii) (e) (iv) (b) (v) (a)

4.9.* Having verified the first two equations, then $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ if and only if $(\mathbf{v} \cdot \mathbf{w})\,\mathbf{u} = (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}$ which occurs if and only if $\mathbf{u}$ and $\mathbf{w}$ are parallel, or $\mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{w}$.

4.10.** In proving distributivity, project the vector $\mathbf{u} + \mathbf{v}$ into the plane perpendicular to $\mathbf{w}$, and then rotate the projected image in this plane until it is perpendicular to $\mathbf{u} + \mathbf{v}$. Compare with rotated projections of $\mathbf{u}$ and $\mathbf{v}$.

Chapter 5 Lines in Space

5.1. (i) $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, $\left.\begin{aligned} x &= 3 + t \\ y &= 2 + 2t \\ z &= 1 + 3t \end{aligned}\right\}$ $t \in \mathbb{R}$,

$$x - 3 = \frac{y - 2}{2} = \frac{z - 1}{3}$$

(ii) $\mathbf{r} = \mathbf{k} + t\mathbf{k} = (1 + t)\mathbf{k} = s\mathbf{k}$, $\left.\begin{aligned} x &= 0 \\ y &= 0 \\ z &= s \end{aligned}\right\}$ $s \in \mathbb{R}$, $x = y = 0$

(iii) $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + t(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$, $\left.\begin{aligned} x &= 2 - t \\ y &= 2t \\ z &= -1 - 3t \end{aligned}\right\}$ $t \in \mathbb{R}$,

$$\frac{x - 2}{-1} = \frac{y}{2} = \frac{z + 1}{-3}$$

(iv) $\mathbf{r} = t(\mathbf{j} + \mathbf{k})$, $\left.\begin{aligned} x &= 0 \\ y &= t \\ z &= t \end{aligned}\right\}$ $t \in \mathbb{R}$, $x = 0$, $y = z$

(v) $\mathbf{r} = -4\mathbf{i} - 3\mathbf{j} + t(\mathbf{j} - \mathbf{k})$, $\left.\begin{aligned} x &= -4 \\ y &= -3 + t \\ z &= -t \end{aligned}\right\}$ $t \in \mathbb{R}$, $x = -4$, $y + 3 = -z$

(vi) $\mathbf{r} = \mathbf{i} + \mathbf{k} + t(\mathbf{i} - \mathbf{j})$, $\left.\begin{aligned} x &= 1 + t \\ y &= -t \\ z &= 1 \end{aligned}\right\}$ $t \in \mathbb{R}$, $x - 1 = -y$, $z = 1$

5.2. (i) $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - \mathbf{k})$, $\left.\begin{aligned} x &= 3+t \\ y &= 2 \\ z &= 1-t \end{aligned}\right\}$ $t \in \mathbb{R}$,

$x - 3 = 1 - z$, $y = 2$

(ii) $\mathbf{r} = \mathbf{k} + t(\mathbf{i} - \mathbf{k})$, $\left.\begin{aligned} x &= t \\ y &= 0 \\ z &= 1-t \end{aligned}\right\}$ $t \in \mathbb{R}$, $x = 1 - z$, $y = 0$

(iii) $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + t(2\mathbf{i} + 7\mathbf{j} - 4\mathbf{k})$, $\left.\begin{aligned} x &= 2+2t \\ y &= 7t \\ z &= 1-4t \end{aligned}\right\}$ $t \in \mathbb{R}$,

$$\frac{x-2}{2} = \frac{y}{7} = \frac{z-1}{-4}$$

(iv) $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(\mathbf{i} + \mathbf{j})$, $\left.\begin{aligned} x &= 1+t \\ y &= 1+t \\ z &= 1 \end{aligned}\right\}$ $t \in \mathbb{R}$, $x = y$, $z = 1$

5.3. The lines are identical since they are both parallel to $4\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ and contain $(1, 2, 10)$.

5.4. Vectors in the directions of the lines are $2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}-\mathbf{k}$ respectively, which are not parallel. If the lines intersect then

$$\frac{1+3s+3}{2} = \frac{6-2s}{3} = \frac{15-s-4}{5}$$

for some s. But the first equation implies $s = 0$ and the second implies $s = -3/7$, which is a contradiction. This proves that the lines are skew.

5.5. (i) $\mathcal{L}_1 :$ $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(5\mathbf{i} - 4\mathbf{j} - 2\mathbf{k})$, $\left.\begin{aligned} x &= 1+5t \\ y &= 1-4t \\ z &= 1-2t \end{aligned}\right\}$ $t \in \mathbb{R}$,

$$\frac{x-1}{5} = \frac{y-1}{-4} = \frac{z-1}{-2}$$

$\mathcal{L}_2 :$ $\mathbf{r} = 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k} + s(-3\mathbf{i} + 8\mathbf{j} + 6\mathbf{k})$, $\left.\begin{aligned} x &= 5-3s \\ y &= -5+8s \\ z &= -3+6s \end{aligned}\right\}$ $s \in \mathbb{R}$,

$$\frac{x-5}{-3} = \frac{y+5}{8} = \frac{z+3}{6}$$

(ii) $T = (7/2, -1, 0)$

(iii) The coordinates of T are the averages of those of P and R and also of Q and S, so T is the midpoint of PR and QS. This is not surprising since $PQRS$ is a parallelogram.

5.6.* (i) same line, zero rotation (ii) $90°$ rotation (iii) $180°$ rotation

(iv) $90° - \theta$ rotation where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

5.7.* Observe that $\overrightarrow{OR} = \lambda\overrightarrow{OP} + (1-\lambda)\overrightarrow{OQ} = \overrightarrow{OQ} + \lambda\overrightarrow{QP}$.

(i) $0 \le \lambda \le 1$ (ii) $\lambda > 1$ (iii) $\lambda < 0$ (iv) $\lambda = 1/3$ or $\lambda = -1$

5.8.* $\sqrt{2275}/13$ **5.9.**** $4/\sqrt{3}$, $(2, 25/3, -22/3)$, $(2/3, 29/3, -6)$

5.10.* Interpret $\mathbf{r}'$ as the velocity of the particle on $\mathcal{C}$ at time t. Split the limit into components.

Chapter 6 Planes in Space

6.1. $-x+y+3z=-25$ **6.2.** (i) $x+y+z=0$ (ii) $101x+22y-8z=256$

6.3. $\left.\begin{array}{rcl} x &=& 3t \\ y &=& -1+t \\ z &=& -7+7t \end{array}\right\} \; t\in\mathbb{R}\,,$ $\quad \dfrac{x}{3} \;=\; y+1 \;=\; \dfrac{z+7}{7}$ **6.4.** $-\dfrac{\sqrt{2}}{3}$

6.5. Take $Q=(d/a,0,0)$ if $a\neq 0$, $Q=(0,d/b,0)$ if $b\neq 0$ and $Q=(0,0,d/c)$ if $c\neq 0$. Take $\mathbf{n}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ in all cases. If $a=b=c=d=0$ then the equation is satisfied by all points in space. If $a=b=c=0\neq d$ then no points satisfy the equation.

6.6. If $Q=R$ then $\overrightarrow{PQ}\cdot\overrightarrow{PR}=|\overrightarrow{PQ}|^2>0$. If $Q\neq R$ then as one travels from Q to R in a straight line, the distance to P is decreasing, so the angle QPR must be acute, so $\overrightarrow{PQ}\cdot\overrightarrow{PR}>0$.

6.7.* $32\sqrt{3}/9\,,\ \left(-\dfrac{113}{27},\dfrac{86}{27},\dfrac{29}{27}\right)$ **6.8.*** $\left(\mathbf{i}+\dfrac{\partial f}{\partial x}\mathbf{k}\right)\times\left(\mathbf{j}+\dfrac{\partial f}{\partial y}\mathbf{k}\right) \;=\; \mathbf{k}-\dfrac{\partial f}{\partial y}\mathbf{j}-\dfrac{\partial f}{\partial x}\mathbf{i}$

6.9.* The equation describes a sphere of radius r centred at the origin. The tangent plane has equation $x_0x\pm\sqrt{r^2-x_0^2}\,y \;=\; r^2$, when $x_0^2+y_0^2=r^2$, or otherwise

$$z-z_0 \;=\; \frac{\pm x_0}{\sqrt{r^2-x_0^2-y_0^2}}(x-x_0)+\frac{\pm y_0}{\sqrt{r^2-x_0^2-y_0^2}}(y-y_0)\,.$$

6.10.* The intersection points are $\left(0,\dfrac{\pm100}{t-10},\dfrac{t\sqrt{101}}{t-10}\right)$. As $t\to-\infty$, $z\to\sqrt{101}$.

Chapter 7 Systems of Linear Equations

7.1. $(1,2,-3)$ **7.2.** $x_1=-\dfrac{4t}{3}\,,\ x_2=1+\dfrac{t}{3}\,,\ x_3=t$

7.3. $x_1=1-s-\dfrac{t}{2}\,,\ x_2=\dfrac{1}{2}-\dfrac{t}{2}\,,\ x_3=s\,,\ x_4=t\,,\ x_5=\dfrac{1}{2}$

7.4. $x_1=-2s\,,\ x_2=\dfrac{1}{2}(3+3s-2t)\,,\ x_3=\dfrac{1}{2}(1+9s-4t)\,,\ x_4=s\,,\ x_5=t$

7.5. $x=y=z=0$ **7.6.** $x_1=x_2=0\,,\ x_3=x_4=t$

7.8* (i) Different systems produce different augmented matrices, and every augmented matrix arises from some system. (ii) To operate on the system and then form the augmented matrix has the same effect as forming the augmented matrix and then performing the corresponding elementary row operation.

7.9.* (i) $f(n)=n^2$ (ii) $g(n)=n^2(n+1)$, including operations with zeros

7.10.** Each reduced row echelon matrix corresponds to a homogeneous system. As the matrix varies, so does the solution set of the corresponding homogeneous system. Since the solution sets are invariant under elementary row operations, the reduced echelon forms cannot be row equivalent.

Chapter 8 Matrix Operations

8.1. (i) $\begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -3 \\ -4 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} -5 & 5 \\ 4 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 7 & -1 \\ -4 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$

(vi) $\begin{bmatrix} 2 & 5 \\ 12 & 3 \end{bmatrix}$ (vii) $\begin{bmatrix} -6 & -3 \\ 4 & 11 \end{bmatrix}$ (viii) -3 (ix) $\begin{bmatrix} -13 \\ -6 \\ -1 \end{bmatrix}$ (x) -64

8.2. $\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$

8.3. $(X+Y)^2 = X^2+2XY+Y^2 \iff X^2+XY+YX+Y^2 = X^2+XY+XY+Y^2$
$\iff YX = XY \iff -XY+YX = 0$
$\iff X^2-XY+YX-Y^2 = X^2-Y^2 \iff (X+Y)(X-Y) = X^2-Y^2$

8.4. $A0_{n\times n} = 0_{m\times m}A = 0_{m\times n}$ and $0_{n\times n} \neq 0_{m\times m} \neq 0_{m\times n}$

8.5. (i) $c = 3(2x+3y) - 5(x-4y) = x + 29y$, $d = 2(2x+3y) + 3(x-4y) = 7x - 6y$

(ii) $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 29 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+29y \\ 7x-6y \end{bmatrix}$

8.6* Suppose $\mathbf{x}_1 + \lambda(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{x}_1 + \mu(\mathbf{x}_1 - \mathbf{x}_2)$. Then $(\lambda-\mu)(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$ so that $(\lambda-\mu)z = 0$ for all entries z in $\mathbf{x}_1 - \mathbf{x}_2$. But at least one value of z is nonzero, since $\mathbf{x}_1 \neq \mathbf{x}_2$. It follows that $\lambda - \mu = 0$, that is, $\lambda = \mu$.

8.7.* (i) $\begin{bmatrix} 1 & nk \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} k^n & nk^{n-1} \\ 0 & k^n \end{bmatrix}$ (iii) $\begin{bmatrix} k^n & nk^{n-1} & \frac{n(n-1)}{2}k^{n-2} \\ 0 & k^n & nk^{n-1} \\ 0 & 0 & k^n \end{bmatrix}$

8.8.* $(YZ)^T = \left[\sum_j y_{ij}z_{jk}\right]^T = \left[\sum_j y_{kj}z_{ji}\right] = \left[\sum_j z_{ji}y_{kj}\right] = [z_{ji}][y_{kj}] = Z^TY^T$

8.9.* $A(B+C) = \left[\sum_j a_{ij}(b_{jk}+c_{jk})\right] = \left[\sum_j a_{ij}b_{jk}\right] + \left[\sum_j a_{ij}c_{jk})\right] = AB + AC$,

$(A+B)C = (C^T(A^T+B^T))^T = (C^TA^T + C^TB^T)^T = AC + BC$

8.10.* $(AB)C = \left[\sum_j a_{ij}b_{jk}\right][c_{kl}] = \left[\sum_{jk} a_{ij}b_{jk}c_{kl}\right] = [a_{ij}]\left[\sum_k b_{jk}c_{kl}\right] = A(BC)$

Chapter 9 Matrix Inverses

9.1. Let A be $p \times q$ and B be $r \times s$. Since AB and BA are defined, $q = r$ and $s = p$. But $p = n$, since $AB = I_n$, and $r = n$ since $BA = I_n$, whence $p = q = r = s = n$.

9.2. (i) $\frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$ (ii) $\frac{1}{18}\begin{bmatrix} -1 & 3 \\ 4 & 6 \end{bmatrix}$ (iv) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 2 & -1 \\ -1 & -2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 33 & -17 & 9 \\ -11 & 6 & -3 \\ -4 & 2 & -1 \end{bmatrix}$ (viii) $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (ix) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

9.3. Observe that $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$ and similarly $(B^{-1}A^{-1})(AB) = I$.

9.4. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We consider the case $a \neq 0$. (The case $c \neq 0$ is similar.) Then

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & (ad-bc)/a & -c/a & 1 \end{array}\right].$$

If $ad - bc = 0$ then the left hand side has a row of zeros so M is not invertible. If $ad - bc \neq 0$ then we proceed further:

$$\begin{aligned} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] &\sim \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{array}\right] \\ &\sim \left[\begin{array}{cc|cc} 1 & 0 & d/(ad-bc) & -b/(ad-bc) \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{array}\right], \end{aligned}$$

so that M^{-1} exists and equals $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

9.5. Matching dimensions as in Exercise 9.1, we see that B is $n \times n$ and C is $m \times m$. Let i be an integer from 1 to n. Let A be any $m \times n$ matrix with first row consisting of 0 everywhere except for 1 in the ith place. Then the first row of $A = AB$ is also $\begin{bmatrix} b_{i1} & b_{i2} & \cdots & b_{in} \end{bmatrix}$, so that $b_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$, which proves $B = I_n$. A similar argument using the first column of A shows $C = I_m$.

9.6* Draw diagrams of matrices for each type of elementary row operation, and carefully keep track of labels of rows and columns.

9.7.* The first part is immediate by inspection and Exercise 9.6. Suppose A is invertible so by the recipe for finding A^{-1} we have $I = E_m \dots E_1 A$ for some elementary matrices $E_1, \dots, E_m$. Then, by Exercise 9.3,

$$E_1^{-1} \dots E_m^{-1} \;=\; (E_m \dots E_1)^{-1} I \;=\; (E_m \dots E_1)^{-1} E_m \dots E_1 A \;=\; IA \;=\; A$$

which proves, by the first part, that A is a product of elementary matrices.

9.8.* Suppose $E_1, \dots, E_m$ are elementary such that $E_m \dots E_1 A$ has a row of zeros. If A is invertible then $I = (E_m \dots E_1 A) A^{-1} E_1^{-1} \dots E_m^{-1}$ has a row of zeros, which is nonsense.

9.9.* Suppose $AB = I$. By the recipe for finding inverses, if a row of zeros appears then a contradiction is reached as in the previous exercise. Hence there must be elementary matrices $E_1, \dots, E_m$ such that $E_m \dots E_1 A = I$. Hence $B = IB = E_m \dots E_1 AB = E_m \dots E_1 I = E_m \dots E_1$, yielding $AB = BA = I$.

9.10.* Suppose $m < n$. (The case $n < m$ is similar.) Inspecting dimensions shows that A is $n \times m$ and B is $m \times n$. Since $n > m$, row reducing A must produce a row of zeros. Hence there are elementary $n \times n$ matrices $E_1, \dots, E_k$ such that $E_k \dots E_1 A$ has a row of zeros. But then the invertible matrix $E_k \dots E_1 = E_k \dots E_1 I_n = E_k \dots E_1 AB$ must have a row of zeros, and a contradiction is reached in the same way as in Exercise 9.8.

Chapter 10 Determinants

10.1. (i) -1 (ii) 1 (iii) 14 (iv) 126 (v) 82 (vi) $bd(c-d)$ (vii) $(t^2-4)(t+4)$ (viii) -4

10.2. (i) anticlockwise (ii) clockwise **10.3.** (i) inside (ii) outside (iii) on the boundary

10.4. The given equations have a unique solution if and only if the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ \ell \end{bmatrix}$$

has a unique solution, which occurs if and only if the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, which occurs if and only if its determinant $ad-bc$ is nonzero.

10.5* (i) Suppose that A has a row of zeros. The induction starts trivially. Suppose that the result holds for square matrices of size less than n where $n \geq 2$. If the first row of A is zero then $\det A = \sum(-1)^{1+j}\, 0 \det A_{1j} = 0$. If the ith row of A is zero where $i > 1$ then, by the inductive hypothesis,

$$\det A = \sum(-1)^{1+j} a_{1j} \det A_{1j} = \sum(-1)^{1+j} a_{1j}\, 0 = 0\,,$$

and the result follows by induction. Suppose now that A has a column of zeros. Again the induction starts trivially and we make a corresponding inductive hypothesis. If it is the kth column of A which is zero then

$$\begin{aligned} \det A &= \sum_{j\neq k}(-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{1+k} a_{1k} \det A_{1k} \\ &= \sum_{j\neq k}(-1)^{1+j} a_{1j}\, 0 + (-1)^{1+k}\, 0 \det A_{1k} = 0\,, \end{aligned}$$

and again we are done by induction.

(ii) The lower triangular case is a simple induction. In the case that A is upper triangular, A_{11} is also upper triangular and A_{1j} has its first column all zero for $j > 1$, so that, by part (i), $\det A = a_{11} \det A_{11}$, and again the result follows by a simple induction.

(iii) The proof is by induction on the size of B.

10.6.* This is a simple induction using the fact that A_{12} is another matrix of the same shape.

10.7.* The first two cases follow immediately from **10.5**(ii), since the elementary matrices are upper or lower triangular. The last case follows by **10.5**(iii) and **10.6** and a simple induction.

10.8.** If B is the result of interchanging two rows of A then $\det B = -\det A$ by an elaborate induction that splits up into cases depending on whether the first row is involved in the interchange. It follows quickly that if two rows of A are identical then $\det A = 0$. If B is the result of multiplying a row of A by the scalar λ then $\det B = \lambda \det A$, by a simple induction. Finally if B is the result of adding a scalar multiple of one row of A to another row then $\det B = \det A$, using the previous results of this exercise and the fact that if C, D, E are identical square matrices except that for some i the ith row of E is the sum of the ith rows of C and D then $\det E = \det C + \det D$, which itself follows by a simple induction.

10.9.[**] (i) The first step is to prove that A is invertible if and only if $\det A \neq 0$, and this follows from the previous exercise and the fact that a matrix is invertible if and only if it can be row reduced to the identity matrix. If A is not invertible then it is easy to show AB also is not invertible, so $\det AB = 0 = \det A \det B$. If A is invertible then A is a product of elementary matrices and the equality $\det AB = \det A \det B$ now follows by the previous exercise and a simple induction.

(ii) If A is not invertible then A^T also is not invertible, so from the proof of part (i), $\det A = 0 = \det A^T$. If A is invertible then it is a product of elementary matrices and the result follows by a simple induction after first checking that $\det E = \det E^T$ where E is elementary by **10.7**.

(iii) The result about expanding along the ith row follows by using the matrix B obtained from A by interchanging rows to bring the ith row to the top and the fact that interchanging rows multiplies the determinant by -1, so that $\det B = (-1)^{i-1} \det A$. The result about expanding down any column follows from the result about rows and part (ii).

10.10.[*] Suppose A is a square matrix with nonzero determinant. Row reduce A, so that $E_k \dots E_1 A = J$ where $E_1, \dots, E_k$ are elementary and J is in reduced row echelon form. If $J \neq I$ then J has a row of zeros so has zero determinant by **10.5**(ii), so, by the multiplicative property and the fact that elementary matrices are invertible,

$$\det A \;=\; \det(E_k \dots E_1)^{-1} \det J \;=\; \det(E_k \dots E_1)^{-1}\, 0 \;=\; 0 \,,$$

which is a contradiction. Hence $J = I$ and $A = E_k \dots E_1$ is invertible.

Chapter 11 Eigenvalues and Eigenvectors

11.1. (i) eigenvalues 1, 2 with eigenspaces $\left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$, $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$ respectively

(ii) eigenvalues 2, 3 with eigenspaces $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$, $\left\{ \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$ respectively

(iii) eigenvalues 1, 2, 3 with eigenspaces $\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$, $\left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$, $\left\{ \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$ respectively

(iv) eigenvalue 2 with eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$

(v) eigenvalues -1, 1, 3 with eigenspaces $\left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$, $\left\{ \begin{bmatrix} t \\ -t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$, $\left\{ \begin{bmatrix} t/2 \\ t/2 \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$ respectively

(vi) eigenvalues 1, 2 with eigenspaces $\left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$, $\left\{ \begin{bmatrix} t/2 \\ t \\ s \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$ respectively

11.2. If $M\mathbf{v} = \lambda\mathbf{v}$ then $M^k\mathbf{v} = M^{k-1}M\mathbf{v} = M^{k-1}\lambda\mathbf{v} = \lambda M^{k-1}\mathbf{v} = \cdots = \lambda^k\mathbf{v}$.

11.3. Suppose $M\mathbf{v} = \lambda\mathbf{v}$ where $\mathbf{v}$ is a nonzero vector. If $\lambda = 0$ then $M\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, so $\mathbf{v} = M^{-1}M\mathbf{v} = M^{-1}\mathbf{0} = \mathbf{0}$, which is a contradiction. Hence $\lambda \neq 0$ and

$$\mathbf{v} = I\mathbf{v} = M^{-1}M\mathbf{v} = M^{-1}\lambda\mathbf{v} = \lambda M^{-1}\mathbf{v}\ ,$$

so that $M^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$. This proves λ^{-1} is an eigenvalue of M^{-1} with eigenvector $\mathbf{v}$.

11.4. The formula still holds for $n \leq 0$ since $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$ always.

11.5* Since the determinant is unchanged under transposition,

$$\det(M^T - \lambda I) \;=\; \det(M^T - \lambda I^T) \;=\; \det(M - \lambda I)^T \;=\; \det(M - \lambda I)\ ,$$

so the characteristic polynomials of M^T and M have the same solutions.

11.6.* The Conjugation Principle can be expressed by the equation $Z = XYX^{-1}$ where Z is difficult, Y is easy, and X changes the environment or conditions to facilitate the performance of Y. In the proof template, Z represents the final proof, X the expansion step, X^{-1} the contraction step, and Y the 'easy' steps in between. In the train example, Z represents the process of getting from Bondi Junction to Redfern, X hopping on the train, X^{-1} hopping off the train, and Y the process of sitting on the train until the appropriate time to alight. In the cake example, $Z = XYWX^{-1}W^{-1}$ where Z is the process of baking the cake, Y is the process of mixing the ingredients, X the process of putting the cake in the oven and W the process of putting on oven mitts so as not to get burnt in the act X^{-1} of taking the cake out of the oven. To complete the process one performs W^{-1}, which is to remove the oven mitts.

11.7.* This is a simple induction using as inductive hypothesis that the determinant of a square matrix with entries which are polynomials in λ is also a polynomial in λ. The determinant expansion along the first row then becomes a sum of polynomials, which is itself a polynomial.

11.8.* This follows by row reducing the noninvertible square matrix A, which guarantees a row of zeros. Solving the associated homogeneous system yields a parametric solution. Picking one nonzero instance gives a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$.

11.9.* By a simple induction, $M^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$. If $M^n = I$ then $n\theta = 2\pi k$ for some integer k, so that $\pi = n\theta/(2k) \in \mathbb{Q}$, which is absurd, since π is irrational.

11.10.* The matrix $M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has eigenvalues $\cos\theta + i\sin\theta$ and $\cos\theta - i\sin\theta$ with eigenspaces $\left\{ \begin{bmatrix} it \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{C} \right\}$ and $\left\{ \begin{bmatrix} -it \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{C} \right\}$ respectively.

Chapter 12 Diagonalising a Matrix

12.1. (i) $\begin{bmatrix} \frac{1+(-1)^k}{2} & \frac{-1+(-1)^k}{2} \\ \frac{-1+(-1)^k}{2} & \frac{1+(-1)^k}{2} \end{bmatrix}$ (ii) $\begin{bmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2^{k+1} - 3^k & -2^k + 3^k \\ 2^{k+1} - 2(3^k) & -2^k + 2(3^k) \end{bmatrix}$

12.2. (i) $\begin{bmatrix} 1 & 2^k-1 & 2^k-1 \\ 0 & 2^k & 2^k-3^k \\ 0 & 0 & 3^k \end{bmatrix}$ (ii) $\begin{bmatrix} 3(-1)^k-3^k-1 & (-1)^k-1 & 1-2(-1)^k+3^k \\ 1-3^k & 1 & 3^k-1 \\ 3(-1)^k-2(3^k)-1 & (-1)^k-1 & 1-2(-1)^k+2(3^k) \end{bmatrix}$

(iii) $\begin{bmatrix} 2-2^k & 2^k-1 & 0 \\ 2-2^{k+1} & 2^{k+1}-1 & 0 \\ 0 & 0 & 2^k \end{bmatrix}$

12.3. The columns add to 1 so the matrix is regular stochastic. The steady state vector is

$$\begin{bmatrix} \frac{4}{9} \\ \frac{5}{9} \end{bmatrix} \quad \text{and} \quad M^n = \begin{bmatrix} \frac{4}{9}+\frac{5}{9}\left(\frac{1}{10}\right)^n & \frac{4}{9}-\frac{4}{9}\left(\frac{1}{10}\right)^n \\ \frac{5}{9}-\frac{5}{9}\left(\frac{1}{10}\right)^n & \frac{5}{9}+\frac{4}{9}\left(\frac{1}{10}\right)^n \end{bmatrix} \to \begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix}.$$

12.4.* The entries of M^2 are all positive. The steady state vector is $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ and $M^n =$

$$\begin{bmatrix} \frac{1}{3}+\frac{2}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n \\ \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}+\left(\frac{1}{2}\right)^{n+1}+\frac{1}{6}\left(-\frac{1}{2}\right)^n & \frac{1}{3}-\left(\frac{1}{2}\right)^{n+1}+\frac{1}{6}\left(-\frac{1}{2}\right)^n \\ \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}-\left(\frac{1}{2}\right)^{n+1}+\frac{1}{6}\left(-\frac{1}{2}\right)^n & \frac{1}{3}+\left(\frac{1}{2}\right)^{n+1}+\frac{1}{6}\left(-\frac{1}{2}\right)^n \end{bmatrix} \to \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

12.5* The identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is stochastic with steady state vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
The matrix $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is stochastic with a unique steady state vector $\mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.
The vector $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ has the property that $\lim\limits_{n\to\infty} M^n\mathbf{x}$ does not exist.

12.6.* and **12.7.*** These are special cases of **12.9.****

12.8.* If P^T can be row reduced to a matrix with a row or zeros then the associated homogeneous system has a nonzero solution, which implies that $\mathbf{v}_1^T, \ldots, \mathbf{v}_n^T$ are linearly dependent, contradicting that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Hence P^T can be row reduced to the identity matrix, so that P^T and P are invertible.

12.9.** Suppose $\mu_1\mathbf{v}_1 + \ldots + \mu_n\mathbf{v}_n = \mathbf{0}$. Apply M and the definition of eigenvector to get another equation. Combine the two equations to eliminate one of the vectors, and then apply an inductive hypothesis, exploiting the assumption that the eigenvalues are distinct, to deduce that $\mu_1 = \ldots = \mu_n = 0$.

12.10.*** Let M be a complex 2×2 matrix. If the eigenvalues are distinct then M is diagonalisable by the previous exercise. Similarly if M has one eigenvalue whose eigenspace is two dimensional then again M is diagonalisable. If remains therefore to suppose M has one eigenvalue λ and its eigenspace is one dimensional. Using the fact that $(M-\lambda I)^2 = 0$, and by solving equations, one can find vectors $\mathbf{v}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ such that $M\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. In the real case, either the eigenvalues are real and the result follows quickly, or the eigenvalues form a complex conjugate pair, and then the result follows by finding the respective one-dimensional complex eigenspaces and then separating the diagonalising matrix equation into real and imaginary parts.