Abstract

In the late 1970s, Gavin observed that Riesz product measures based on powers of 3 are ergodic for the action of the triadic rationals. This caused us to examine the role of Riesz product type constructions in ergodic theory and led us to explore the theory of G-measures. I will give an overview of our work, and show how it has now led to a structure theorem for all non-singular ergodic dynamical systems up to orbit equivalence: Hamachi and I recently proved that every such system is orbit equivalent to a uniquely ergodic G-measure on a Bratteli-Vershik space, realised as an induced transformation on a closed set of an infinite product space.

G-measures

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1 Riesz Products

Recall that F.Riesz (1927) introduced certain measures on the circle $\mathbb{T}$ by choosing sequences $-1 \leq a_i \leq 1$ and $\phi_i \in \mathbb{T}$, and defining:

$$\mu = \lim_{n\to\infty} \prod_{k=0}^{n} (1 + a_k \cos 2\pi(3^k t + \phi_k))dt$$

Riesz showed that there was a unique weak*-limit measure μ, and that these measures, which became known as Riesz products, were often singular with respect to Haar measure dt.

Before coming to Australia, Gavin and Bill Moran did some beautiful work on Riesz products, showing that they were absolutely continuous with respect to Lebesgue measure if and only if $\sum a_i^2 < \infty$, and were otherwise singular; and also producing *dichotomy* theorems like those of Kakutani for infinite product measures.

Around 1978, Gavin observed that if we consider the action of the triadic rationals on the circle, ie $\gamma : t \mapsto t + \frac{p}{3^n}$, for various values of n and p (relatively prime to 3), then $\mu \circ \gamma \sim \mu$ and in fact, the measures μ are ergodic for the action of the group of triadic rationals Γ. This neatly explains the dichotomy, for ergodic measures are either equivalent or singular.

Note that

$$\frac{d\mu \circ \gamma}{d\mu}(t) = \prod_{k=0}^{n-1} \frac{(1 + a_k \cos 2\pi(3^k(t + \frac{p}{3^n}) + \phi_k))}{(1 + a_k \cos 2\pi(3^k t + \phi_k))}$$

If we replace t with its ternary expansion, then μ can be pushed forward to a measure on the infinite product space $X = \prod_{k=0}^{\infty}\{0, 1, 2\}$, and that in this presentation, Γ acts as the group of finite coordinate changes.

2 G-measures

Let $l(i) \geq 2$ be a sequence of integers and consider the infinite product space $X = \prod_{i=1}^{\infty} \mathbb{Z}_{l(i)}$, where we write $\mathbb{Z}_{l(i)} = \{0, \cdots, l(i)-1\}$. Let $X_m^n = \prod_{i=m}^{n} \mathbb{Z}_{l(i)}$ and $X^n = X_1^n$. We shall denote $|X^n| = s(n) = \prod_{i=1}^{n} l(i)$.

We have the finite coordinate change group $\Gamma = \coprod_{i=1}^{\infty} \mathbb{Z}_{l(i)}$. This has the same orbits as the odometer T, which acts on X by the standard method:

$Tx = y$ if y is the smallest element greater than x in the lexicographic order, and if $\ell = (l(1)-1, l(2)-1, \cdots, l(n)-1, \cdots)$, then $T\ell = 0 = (0, 0, 0, \cdots)$.

Gavin's and my first paper *Ergodic Measures are of Weak Product Type* showed that any measure on X which is ergodic for the action of the finite coordinate changes (or alternatively the odometer), was equivalent to a measure ν where, for all l there exists $m > l$ such that

$$\nu(A \cap B) = \nu(A)\nu(B), \quad \text{for all } A \in \mathcal{C}_l,\ B \in \mathcal{C}^m.$$

We had the feeling that the rate of growth of the "gap" $m - l$ was critical. We constructed the Radon derivative $\frac{d\nu}{d\mu}$ as an infinite product $\prod_{l=0}^{\infty} g_l(x)$ where $g_l(x)$ depends on the coordinates from l to m. This was uncannily reminiscent of Riesz products, so we decided to study this kind of measure.

Let g_i be a function on X^i such that $\frac{1}{l(i)} \sum_{y=0}^{l(i)-1} g_i(y+x) = 1$ for all x. Let

$$G_n(x) = \Pi_{i=1}^{n} g_i(\theta^{i-1} x),$$

where θ denotes the shift map.

We say that a probability measure μ on X is a G-**measure** if for μ-almost every $x \in X$

$$\frac{d\mu}{d\mu^n}(x) = G_n(x). \tag{1}$$

Here μ^n denotes the measure $\frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} \mu \circ \gamma$.

In fact, every measure is a G-measure!

In 1991 we showed that, provided the functions G_i are continuous, the existence of a unique G-measure is equivalent to the convergence (everywhere, or uniformly) of the sequence of functions

$$\frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} f(\gamma(x)) G_n(\gamma(x)) \tag{2}$$

for all $f \in C(X)$ and $x \in X$.

If (2) holds, then the limit is equal to $\int f d\mu$ for the unique G-measure. We showed that a unique G-measure is necessarily ergodic for the finite coordinate change action on X. In this case, we say that μ **is uniquely ergodic**.

In the case when $l(j)$ is constant (say $= l$ for all j), we may identify X_n with X via the shift map. If μ is shift invariant i.e. $\mu \circ \theta = \mu$ then all the functions g_i are identical under this identification, to a single function g, and we say that we have a g-measure.

3 Unique ergodicity

We gave various conditions on the functions g_i such that we have unique ergodicity; for example, uniform Lipshitz conditions. Anthony Quas gave a beautiful construction that, for g measures, circle continuity of g was not sufficient to guarantee unique ergodicity.

In a recent paper, Örjan Stenflo and I gave what I believe is the weakest condition under which there's a uniquely ergodic G-measure. We show that there is a unique G-measure provided that

$$\sum_{n=1}^{\infty}\prod_{m=1}^{n} cff_G(s(m)) = \infty, \tag{3}$$

where

$$cff_G(s(m)) = \inf_n \inf_{1\leq j_l \leq l(n+k), 1\leq k\leq m-1} \sum_{i=1}^{l(n)} \inf_y g_n(ij_1 \dots j_{m-1}y)$$

Theorem 3.1 *Let G be a family of g-functions satisfying condition (3). Then there exists unique G-measure, μ, i.e.*

$$\lim_{n\to\infty} \frac{1}{|\Gamma_n|} \sum_{\gamma\in\Gamma_n} f(\gamma(x))G_n(\gamma(x)) = \int f d\mu \tag{4}$$

for all $f \in C(X)$ and $x \in X$.

This condition is slightly stronger than the weakest known condition for uniqueness in g-measures of this type, see Stenflo (2003).

4 Ergodic theory and the structure theorem

Let $(X, \mathcal{B}, \mu, T)$ be a non-singular conservative **measurable dynamical system** with $\mu(X) = 1$. Let $\omega_i(x) = \frac{d\mu \circ T^i(x)}{d\mu}$. We shall assume also that the system is **ergodic**: every invariant set of positive measure must have complement of measure zero.

Two such systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are said to be **metrically isomorphic** if there exists a bimeasurable invertible mapping $\Phi : X \to Y$ such that $\Phi \circ T = S \circ \Phi$ and $T \circ \Phi^{-1} = \Phi^{-1} \circ S$ a.e. and $\nu \circ \Phi^{-1} \sim \mu$.

They are **orbit equivalent** if there exists Φ as above with for all n and a.e. x, $\Phi \circ T^n x = S^{m(x,n)} \circ \Phi x$. Then $m(x, n)$ is an integer-valued cocycle.

Theorem 4.1 *(D. and Hamachi) Every ergodic non-singular dynamical system* $(X, \mathcal{B}, T, \mu)$ *is orbit equivalent to a Markov odometer. Furthermore, when considered as a G-measure, the Markov odometer may be taken to be:*

- *uniquely ergodic*
- *a minimal transformation for the topology of* X
- *an induced transformation of a full odometer*

I need to explain the notion of **Markov odometer** on a **Bratteli-Vershik diagram**, and unique ergodicity in this context.

An **ordered Bratteli-Vershik diagram**consists of:

- A vertex set V which is a disjoint union of finite sets $V^{(n)}, n \geq 0$.
- An edge set E which is a disjoint union of finite sets $E^{(n)}, n \geq 1$
- Source mappings $s = s_n : E^{(n)} \to V^{(n-1)}$
- range mappings $r = r_n : E^{(n)} \to V^{(n)}$.
- a partial order so that edges e and e' are comparable if and only if $r(e) = r(e')$.

The space $X = X(V, E)$ of infinite paths is:

$$X = \{(x_n)_{n\geq 1} : x_n \in E^{(n)},\ t(x_n) = s(x_{n+1}),\ \forall n \geq 1\}$$

A partial order on X is

$$x < y, \ \text{ if } \ \exists n \geq 1, \text{ such that } x_n < y_n, \text{ and } x_i = y_i, \ \forall i > n.$$

There is a natural **topology** on X by taking as basic cylinder sets $[f_1, ..., f_k]_1^k = \{(e_1, e_2, ...) \in X : e_1 = f_1, e_2 = f_2, ..., e_k = f_k\}$. These sets also define a σ**-algebra** on X.

We define $T : X \to X$ as follows. If x is some path which is not the maximal path, then at least one of its edges is not in E_{max}, choose such an edge with the smallest k and let f_k be its successor, let $(f_1, ..., f_k)$ be the unique path in E_{min} from v_0 to $r(x_k)$, and let $Tx = y$, where $y = (f_1, ..., f_k, x_{k+1}, x_{k+2}, ...)$.

If x_{max} is the unique maximal path, then we take $Tx_{max} = x_{min}$, where x_{min} is the unique minimal path.

T is the **Vershik transformation** of X. It's an obvious generalisation of the odometer.

Given a vertex $v \in V^{(k)}$, let $\Gamma_k(v)$ be the permutations of the finite paths finishing at v. Can be extended to X and has the same orbits as T.

Markov measures are defined by sequences of Markov transition matrices in Bratteli-Vershik systems.

We say that a matrix $P^{(n)} = \{P^{(n)}_{v,e}\}_{(v,e)\in V(n-1)\times E(n)}$ is a **stochastic matrix** if it satisfies the following two conditions:
(i) $P^{(n)}_{v,e} > 0 \ \Leftrightarrow s(e) = v$
(ii) $\sum_{\{e\in E(n): s(e)=v\}} P^{(n)}_{v,e} = 1 \ \ \forall v \in V^{(n-1)}$.

Given a sequence $P^{(n)}$ of stochastic matrices and a probability measure ν_0 on $V^{(0)}$ such that

$$\nu_0(v) > 0, \ \ \forall v \in V^{(0)},$$

we define a measure μ on cylinder sets by

$$\mu([e_1, e_2 \dots e_n]_1^n) = \nu_0(s(e_1))P^{(1)}_{s(e_1),e_1}P^{(2)}_{s(e_2),e_2}\dots P^{(n)}_{s(e_n),e_n}.$$

The dynamical system $(X, \mathcal{B}, T, \mu)$ is said to be a **Markov odometer**.

Let μ be a T-quasi-invariant measure on X. For a fixed k, define the tail measure $\mu^{(k)}$ by setting, for $n \geq k$,

$$\mu^{(k)}([a_1, \ldots, a_n]_1^n) = \frac{1}{|\Gamma_k(r(a_k))|} \sum\nolimits_{\gamma \in \Gamma_k} \mu^{(k)}(\gamma [a_1, \ldots, a_n]_1^n).$$

Then $\mu^{(k)}$ is a Γ_k-invariant measure which is equivalent to μ. Let $G_k = d\mu / d\mu^{(k)}$ and $g_k = d\mu^{(k-1)} / d\mu^{(k)}$.

Then G_k satisfies

$$\frac{1}{|\Gamma_k(r(a_k))|} \sum\nolimits_{\gamma \in \Gamma_k} G_k(\gamma a) = 1$$

for all infinite paths $a \in X$.

Further, $G_k(a) = g_1(a) g_2(a) \ldots g_k(a)$, where for each i, g_i satisfies the two conditions:
(i) (*invariance*) $g_i(a)$ is independent of $(a_0, \ldots, a_{i-1})$, and
(ii) (*normalisation*) $\frac{1}{|E^{(i)}(v,w)|} \sum_{\{e \in E^{(i)}(v,w)\}} g_i(a_1, \ldots, a_{i-1}, e, a_{i+1}, \ldots) = 1$.

Here, v denotes $r(a_{i-1})$, w denotes $s(a_{i+1})$, and $E^{(i)}(v,w)$ denotes $\{e \in E^{(i)} : s(e) = v, r(e) = w\}$

A measure on X satisfying

$$G_k = d\mu/d\mu^{(k)}$$

is called a **G-measure**.

It is clear that, for a fixed normalised compatible family G, the set of G-measures is a convex set inside the set of all T-quasi-invariant probability measures. The extreme points in this convex set are T-ergodic.

If there is just one element in this convex set, the unique G-measure, μ is T-ergodic, and we say that we have a **uniquely ergodic G-measure**.

Proposition 4.2 *Let $G = \{G_n\}$ be a normalised compatible family of continuous functions. The following are equivalent.*
(i) There is a unique G-measure - which is therefore T-ergodic.
(ii) For every $f \in C(X)$ the sequence

$$A_k(f)(x) = \frac{1}{|\Gamma_k(r(x_k))|} \sum\nolimits_{\gamma \in \Gamma_k(r(x_k))} G_k(\gamma x) f(\gamma x)$$

converges uniformly to a constant.
(iii) For every $f \in C(X)$ the sequence

$$A_n(f)(x) = \frac{1}{|\Gamma_k(r(x_k))|} \sum\nolimits_{\gamma \in \Gamma_k(r(x_k))} G_k(\gamma x) f(\gamma x)$$

converges pointwise (for every $x \in X$) to a constant.

We generalised the Brown-Dooley conditions, and it's clear that the D-Stenflo conditions also generalise to this setting.

On the traditional odometer, Markov measures are clearly of weak product type.

Hamachi and I gave an explicit construction of a Markov odometer which is not orbit equivalent to a product measure, but where the number of vertices grows extremely quickly. I have a feeling that the rate of growth is somehow a crucial ingredient in measuring the complexity of the system. Perhaps bounded B-V systems are all orbit equivalent to products....

5 Critical dimension and entropy

Let $(X, \mathcal{B}, \mu, T)$ be a non-singular conservative ergodic dynamical system with $\mu(X) = 1$. Let

$$X_{\alpha'} = \{x \in X : \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i(x)}{n^{\alpha'}} > 0\},$$

and notice that $X_{\alpha'}$ is an invariant set. The supremum over the set of α' for which $\mu(X_{\alpha'}) = 1$ is called the **lower critical dimension** α of $(X, \mathcal{B}, \mu, T)$.

Let

$$X_{\beta'} = \{x \in X : \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \omega_i(x)}{n^{\beta'}} = 0\}.$$

Let β be the infimum of the set $\{\beta' : \mu(X'_\beta) = 1\}$, the **upper critical dimension**.

Theorem 5.1 *(D. and Mortiss) The upper and lower critical dimensions are invariants for metric isomorphism.*

If μ is defined on an infinite product space X, we define the **upper** and **lower average coordinate entropy** of μ:

$$\overline{h}_{AC} = \limsup_{n\to\infty} \frac{1}{\log(s(n))} \sum_{i=0}^{n-1} H(\mu_i) \tag{5}$$

and

$$\underline{h}_{AC} = \liminf_{n\to\infty} \frac{1}{\log(s(n))} \sum_{i=0}^{n-1} H(\mu_i)$$

where $H(\mu_i) = -\sum_{j=0}^{l(i)-1} \mu_i(\{j\}) \log \mu_i(\{j\})$ is the usual entropy of the nth coordinate measure.

Theorem 5.2 *(D. and Mortiss) For the odometer action T on $(\prod_{i=1}^{\infty} \mathbb{Z}_{l(i)}, \otimes_{i=1}^{\infty} \mu_i)$ and for $2 \leq l(i) \leq m < \infty$, the lower critical dimension is given by the formula*

$$\alpha = \liminf_{n \to \infty} -\frac{\sum_{i=1}^{n} \log \mu_i(x_i)}{\log(s(n))} = \underline{h}_{AC}(\mu),$$

for μ-almost every x.

The upper critical dimension is given by

$$\beta = \limsup_{n \to \infty} -\frac{\sum_{i=1}^{n} \log \mu_i(x_i)}{\log s(n)} = \overline{h}_{AC}(\mu)$$

for μ almost all $x \in X$.

The analogous theorem holds for Markov odometers.

If the AC entropy limit exists then the upper and lower critical dimensions of the associated product odometer actions are equal. In this case, we will refer to either as the **critical dimension**. In this case, we also have

$$\lim_{n \to \infty} -\frac{\sum_{i=1}^{n} \log \mu_i(x_i)}{\log(s(n))} = \alpha, \quad \text{a.e.}$$

which is an analogue of the **Shannon-MacMillan-Breiman Theorem** for Bernoulli shifts.

Theorem 5.3 *(Katok's covering theorem) Suppose that μ is an infinite product measure of critical dimension α. Then we have*

$$\alpha = \lim_{n\to\infty} \frac{1}{\log(s(n))} \log \inf_{\mu(A)\geq 1-\delta} c_n(A)$$

for all $\delta \in (0,1)$.

We say that an orbit equivalence map $\Phi : X \to Y$ is a **Hurewicz map** if for almost all x,

$$0 < \liminf \frac{\sum_{k=0}^{n} \frac{d\mu \circ T^k}{d\mu}(x)}{\sum_{k=0}^{n} \frac{d\mu \circ T^{m(x,k)}}{d\mu}(x)} \leq \limsup \frac{\sum_{k=0}^{n} \frac{d\mu \circ T^k}{d\mu}(x)}{\sum_{k=0}^{n} \frac{d\mu \circ T^{m(x,k)}}{d\mu}(x)} < \infty$$

Theorem 5.4 *(i) Hurewicz maps preserve upper and lower critical dimension.*

(ii) A map which preserves upper and lower critical dimension of all measures is Hurewicz.

We conjecture that, within each fixed orbit equivalence class, and inside Bratteli-Vershik systems of bounded width, the critical dimension is a complete invariant for Hurewicz equivalence.

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