402- HW8 Solutions

(1) Prove that the composition of two glide reflections is a translation $\Leftrightarrow$ the reflection lines for the two glide reflections are parallel.

What do you get if they are not parallel?

$$G_1 = T_{AB} \circ r_\ell, \qquad G_2 = T_{CD} \circ r_m \qquad \vec{AB} = \vec{v} \parallel \ell$$

$$= r_\ell \circ T_{AB} \qquad = r_m \circ T_{CD} \qquad \vec{CD} = \vec{w} \parallel m$$

$$\Rightarrow G_2 \circ G_1 = T_{CD} \circ r_m \circ r_\ell \circ T_{AB}$$

(a) "$\Leftarrow$" $m \parallel \ell \Rightarrow r_m \circ r_\ell = T_{2v}$, $\vec{v}$ vector from ℓ to m.

So $G_2 \circ G_1 = T_{CD} \circ T_{2v} \circ T_{AB} = T_{CD + 2v + AB}$.

(b) "$\Rightarrow$" $m \nparallel \ell \Rightarrow r_m \circ r_\ell = R_{a,\alpha}$, $\{a\} = m \cap \ell$

is a rotation

$\therefore R_{a,\alpha} \circ T_{AB} = T_{R_{a,\alpha}(AB)} \circ R_{a,\alpha}$

So $G_2 \circ G_1 = T_{CD + R_{a,\alpha}(AB)} \circ R_{a,\alpha}$

This is a rotation

(2) Let f be any isometry.

Prove that $f \circ r_m \circ f^{-1} = r_{f(m)}$

Method 1

- write f as a composition of n reflections: $r_n \circ r_{n-1} \circ \dots \circ r_2 \circ r_1$

f^{-1} is then also a composition of n reflections: $r_1 \circ r_2 \circ \dots \circ r_n$

so $f \circ r_m \circ f^{-1}$ is a composition of $2n+1$ reflections

$\Rightarrow$ it is odd

$\Rightarrow$ it is orientation reversing, so either a glide reflection or a reflection.

Note: if $P \in f(m)$, so $P = f(Q)$ for $Q \in m$,

$f \circ r_m \circ f^{-1}(P) = f \circ r_m(Q) = f(Q) = P$.

$\Rightarrow f \circ r_m \circ f^{-1}$ fixes the line $f(m)$.

So it is not a glide reflection, it is $r_{f(m)}$.

Method 2

- write f as a composition of reflections

$f = r_n \circ \dots \circ r_1$, $\quad f^{-1} = r_1 \circ \dots \circ r_n$ (can assume $n \leq 3$ but it's not necessary)

then $f \circ r_m \circ f^{-1} = (r_n \circ \dots \circ r_1) \circ r_m \circ (r_1 \circ \dots \circ r_n)$

$= (r_n \circ \dots \circ r_2) \circ (r_1 \circ r_m \circ r_1) \circ (r_2 \circ \dots \circ r_n)$

$= (r_n \circ \dots \circ r_2) \circ r_{r_1(m)} \circ (r_2 \circ \dots \circ r_n)$

$= \dots$

$= r_{r_n \circ \dots \circ r_2 \circ r_1(m)}$

$= r_{f(m)}$.

EXERCISE (5):

Exercise 5.8.1. Write $R_{A,\angle EAB}$ and $R_{B,\angle ABE}$ using ~~as~~ reflections across m, n, $\overleftrightarrow{AB}$.

(m = bisector of $\angle ABE$; n = bisector of $\angle EBA$.)

↳ $R_{A,\angle EAB} = r_{\overleftrightarrow{AB}} \circ r_n$

$R_{B,\angle ABE} = r_m \circ r_{\overleftrightarrow{AB}}$.

Exercise 5.8.2: Use 5.8.1 to prove that O is fixed under $R_{B,\angle ABE} \circ R_{A,\angle EAB}$, and that the composition is a rotation.

↳. $R_{B,\angle ABE} \circ R_{A,\angle EAB} = r_m \circ r_{\overleftrightarrow{AB}} \circ r_{\overleftrightarrow{AB}} \circ r_n$

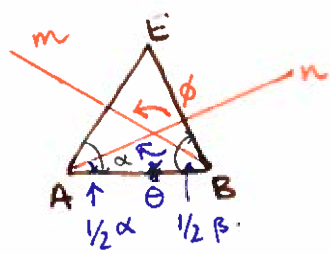
$= r_m \circ r_n$

= rotation about O.

So $R_{B,\angle ABE} \circ R_{A,\angle EAB} = R_{O,\gamma}$.

Exercise 5.8.3: Show that $\gamma = (\angle EAB + \angle ABE) \pmod{360}$.

↳ $\gamma = 2\phi$ where ϕ is the angle from n to m.



Let $\alpha = m\angle EAB$

$\beta = m\angle ABE$

$\Rightarrow \frac{1}{2}\alpha + \frac{1}{2}\beta + \theta = 180$

so $\theta = 180 \mp \frac{1}{2}\alpha - \frac{1}{2}\beta$

$\Rightarrow \phi = -\theta \equiv \frac{1}{2}\alpha + \frac{1}{2}\beta \mod 360°$

$\Rightarrow \gamma = 2\phi = \alpha + \beta = m\angle EAB + m\angle ABE$.

Exercise 5.8.4: Write the rotation at A in terms of r_{AL}, r_{AB}.
the rotation at B in terms of r_{BM}, r_{BN}.
Prove that the composition is translation by twice the vector between AL and BN.

$$R_{A,\angle CAB} = r_{\overleftrightarrow{AB}} \circ r_{\overleftrightarrow{AL}}$$

$$R_{B,\angle BAC} = R_{B,2\angle MBN} = r_{\overleftrightarrow{BN}} \circ r_{\overleftrightarrow{AB}}.$$

So $R_{B,\angle BAC} \circ R_{A,\angle CAB} = r_{\overleftrightarrow{BN}} \circ r_{\overleftrightarrow{AB}} \circ r_{\overleftrightarrow{AB}} \circ r_{\overleftrightarrow{AL}}$

$$= r_{\overleftrightarrow{BN}} \circ r_{\overleftrightarrow{AL}}.$$

Since $\overleftrightarrow{BN} \parallel \overleftrightarrow{AL}$ this is translation by 2 times the vector from $\overleftrightarrow{AL}$ to $\overleftrightarrow{BN}$.

PICTURE: