MATH 402 Worksheet 1

Wednesday 5 September, 2018

Warm-up: Definitions in Hilbert's system.

(a) List the five undefined terms in Hilbert's system of Euclidean geometry.

(b) Give careful definitions of the following, using only these five terms and terms defined from them in lectures:

- A right angle
- When two lines are *perpendicular*
- A triangle ΔABC
- An isosceles triangle; the base and the legs of an isosceles triangle

This worksheet is about an axiom called "Pasch's Axiom" that Euclid used implicitly without stating it. Moritz Pasch (1843–1930) first noticed this, and formulated the axiom to fix the holes in Euclid's proofs. Hilbert incorporated Pasch's Axiom into his axiomatic system as one of his "Between-ness Axioms".

Pasch's Axiom 1. *Let A, B, C be three non-collinear points, forming a triangle ΔABC. Let ℓ be a line which does not pass through A, B or C. Then if ℓ passes through side $\overline{AB}$ it must pass through either a point on $\overline{AC}$ or a point on $\overline{BC}$, but not both.*

Exercise 1. Draw some pictures of triangles ΔABC and lines ℓ that illustrate the axiom. Does it seem like a reasonable axiom to you?
Now drop the assumption that ℓ doesn't pass through any of the points A, B, C. Draw a picture where the conclusion of the axiom doesn't hold. (Hint: it is impossible for the line to pass through only one side of the triangle. Can you draw a picture where it passes through all three sides?)

This axiom allows us to formalize the notion that lines divide the plane into two pieces (one on each side of the line), and similarly that figures like triangles divide the plane into two pieces (inside the figure, and outside the figure).

Exercise 2. Let A and B be two points, and let ℓ be a line that does not pass through A or B. Looking in section 2.2 of the textbook, carefully make the following definitions:

(a) A and B are *on the same side* of ℓ.

(b) A and B are *on opposite sides* of ℓ.

Draw yourself a sketch to see that this matches your intuition.

Exercise 3. Let A, B, C be three non-collinear points, forming a triangle ΔABC. Let ℓ be a line which doesn't pass through any of these points. Use Pasch's axiom to prove the following statements:

(a) If A and B are on the same side of ℓ, and B and C are also on the same side of ℓ, then A and C are on the same side of ℓ too.

(b) If A and B are on opposite sides of ℓ, and B and C are also on opposite sides of ℓ, then A and C are on the same side of ℓ.

Exercise 4. Now assume that A, B, C are collinear (i.e. they all line on some line m). Let ℓ be a line which doesn't pass through any of these points. Use Hilbert's other between-ness axioms to show that the above results still hold, i.e.

(a) If A and B are on the same side of ℓ, and B and C are also on the same side of ℓ, then A and C are on the same side of ℓ too.

(b) If A and B are on opposite sides of ℓ, and B and C are also on opposite sides of ℓ, then A and C are on the same side of ℓ.

You do not need to hand your work in, but you are expected to complete it. If you get stuck or are unsure about your answers, come to office hours. This material is examinable and will not be covered in ordinary lecture format, so you must make sure that you understand it as it is presented here.