MATH 402 Worksheet 4

Friday 12 October, 2018

This worksheet is about reflections, and how they can be composed to give isometries called *translations* and *rotations*. Recall the following definitions from last class.

Definition 1. An isometry that is composed of two reflections where the lines of reflection are parallel (or coincident) is called a *translation*.
An isometry that is composed of two reflections where the lines of reflection are not parallel is called a *rotation*.

We will see (in this worksheet and in lectures) that these definitions give rise to functions that behave like our intuitive notions of "translations" and "rotations".

Remember the following facts about reflections:

(1) To prove that an isometry f is the reflection r_ℓ, it suffices to prove that f fixes the line ℓ, but doesn't fix everything.
(2) $r \circ r = \text{id}$.
(3) If $P \neq P'$, there is a unique reflection r taking P to P'.

Exercise 1. This exercise is about how reflections behave when we compose them, and will help us to understand translations and rotations.

a. Let r_ℓ and r_m be two reflections, with line of reflection ℓ and m, respectively. Prove that $r_m \circ r_\ell \circ r_m = r_{\ell'}$, where ℓ' is the image of ℓ under r_m.
   Hints:
   - *Let A, B be distinct points of ℓ. Show that $r_m(A)$ and $r_m(B)$ are fixed points of $r_m \circ r_\ell \circ r_m$.*
   - *Find a point C which is not a fixed point of $r_m \circ r_\ell \circ r_m$.*

b. Now assume ℓ_1 and ℓ_2 are two non-coincident lines which intersect at a point O. Let $P_1 \in \ell_1$, $P_2 \in \ell_2$ be two other points, and let m be the bisector of the angle $\angle P_1 O P_2$.
   i. Prove that $r_m(\ell_1) = \ell_2$.
      (Hint: choose P_1 and P_2 to have the same distance from O. What is $r_m(P_1)$?)
   ii. Use part **a.** to prove that $r_m \circ r_{\ell_1} = r_{r_m(\ell_1)} \circ r_m = r_{\ell_2} \circ r_m$.

Exercise 2. Using the definition of a translation given above, prove that a translation which is not the identity has no fixed points.

Exercise 3. Let R be a rotation given by $r_m \circ r_\ell$ where m and ℓ are two lines intersecting at the point O. Prove that O is the only fixed point of R.

You do not need to hand your work in, but you are expected to complete it. If you get stuck or are unsure about your answers, come to office hours. This material is examinable and will not be covered in ordinary lecture format, so you must make sure that you understand it as it is presented here.