MATH 595

Homework 1

Exercise 1. Let $X = \mathbb{A}^1 = \mathrm{Spec}\mathbb{C}[t]$. Let $\Delta : X \to X^2$ be the diagonal embedding, and let $j : U \to X$ be its open complement. Let $\mathcal{M}$ be a $\mathcal{D}$-module on X^2. Recall that we produced a map

$$\gamma : j_*(j^*\mathcal{M})^r \to i_*(i^*\mathcal{M})^r,$$

by setting $\gamma(mdx\wedge dy) = 0$, $\gamma((x-y)^{-1}mdx\wedge dy) = \overline{m}dt\otimes 1$, and proceding by induction, using compatibility of γ with the action of ∂_x. Carry out this induction step, and check that the resulting map is indeed a map of $\mathcal{D}$-modules (i.e. linear in $\mathbb{C}[x, y, \partial_x, \partial_y]$).

Exercise 2. Let A be a discrete set. Consider the colimit

$$\operatorname*{colim}_{I\in \mathrm{fSet}^{\mathrm{op}}} A^I.$$

(You can take the colimit in the category of sets, for this example.) Prove that it is equivalent to the set $\mathrm{fSet}(A)$ of finite non-empty subsets of A. (Write down a map in each direction—possibly on the level of S-points, and then prove that it gives a natural transformation—and check that the compositions are identity maps.)

In particular, this is why $\mathrm{Ran}(X)(S)$ is the set of all finite non-empty sets of maps from S to X.

Exercise 3. In lecture on 31 October, I sketched a proof that for $\mathcal{Y}$ a prestack and S an affine scheme, we have an equivalence of ∞-groupoids

$$\mathrm{Hom}_{\mathrm{PreStk}}(S, \mathcal{Y}) \simeq \mathcal{Y}(S).$$

In lecture, I gave a morphism in each direction. Check that the compositions are (equivalent to) the identity.

Exercise 4. In lecture on 5 November, I wrote down a pairing

$$\langle\cdot,\cdot\rangle : \mathcal{D}_X^{\leq n} \otimes \mathcal{O}_{X^{(n)}} \to \mathcal{O}_X,$$

which I claimed gives a perfect pairing (for X a smooth variety). Prove that it is indeed a perfect pairing when $X = \mathbb{A}^1 = \mathrm{Spec}\mathbb{C}[t]$.

Exercise 5. Practice with $U(\alpha)$ and $\Delta(\beta)$.

a. Suppose $\alpha : I \to J$ factors through K as $\gamma\circ\beta$. For each $j \in J$ we then have $\beta_j : I_j \to K_j$. Prove that $U(\beta) \subset U(\alpha)$ and $U(\beta) \subset \prod_{j\in J} U(\beta_j)$. (Recall that we used this in stating the compatibility condition between different $c(\alpha)$.)

b. Suppose $\gamma : I \to K$ factors through J as $\gamma = \beta\circ\gamma$. Show that $U(\beta)$ is contained in $X^J \times_{X^I} U(\gamma)$, viewed as a subset of X^J. (Recall that we used this is stating the compatibility condition between $\nu(\bullet)$ and $c(\bullet)$.)

Exercise 6. Practice with the Beilinson–Drinfeld Grassmannian. Fix a finite set I and recall the definition of $\mathcal{G}r_{G,X^I}(S)$.

a. Check carefully that this gives a functor in S.

b. Prove that there is a map π_{X^I} of prestacks from $\mathcal{G}r_{G,X^I}$ to the stack Bun_G, where Bun_G sends a scheme S to the groupoid of principal G-bundles on $S \times X$.

c. What is the fibre of the map π_{X^I} over the trivial G-bundle? (More precisely, the trivial G-bundle determines a map $\mathrm{pt} \to \mathrm{Bun}_G$. What is the fibre product $\mathrm{pt} \times_{\mathrm{Bun}_G} \mathcal{G}r_{G,X^I}$?)