MATH 595 Tuesday 6 February

First results on cohomology

(1) **Chapter III, Exercise 2.1(a).**

Let $X = \mathbb{A}^1$ be the affine line over an infinite field k; let $P, Q \in X$ be distinct closed points; and let $U = Z \setminus \{P, Q\}$. Prove that $H^1(X, \mathbb{Z}_U) \neq 0$.

(2) **Chapter III, Exercise 2.4.** *Mayer-Vietoris sequence*

Look at Exercise 2.3 for the definition and properties of *cohomology with supports* in a closed subset $Y \subset X$: $H^i_Y(X, \cdot)$ are the derived functors of $\Gamma_Y(X, \cdot)$. (You can do this exercise if you like, for practice—it is not too tricky, but it is long; for now just assume the properties hold and work on Exercise 2.4.)

In particular, for this question, you will need to understand that if $Y \subset Y' \subset X$, then there is an injective map

$$\Gamma_Y(X, \mathscr{F}) \to \Gamma_{Y'}(X, \mathscr{F}).$$

You also need to know that when $Y' = X$, and $\mathscr{F}$ is flasque, the cokernel of this map is isomorphic to $\Gamma(U, \mathscr{F})$. That is, we have a short exact sequence:

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(U, \mathscr{F}) \to 0.$$

With these facts in mind, consider the following situation: Y_1 and Y_2 are two closed subsets of X. Prove that there is a long exact sequence of cohomology with supports:

$$\ldots \to H^i_{Y_1 \cap Y_2}(X, \mathscr{F}) \to H^i_{Y_1}(X, \mathscr{F}) \oplus H^i_{Y_2}(X, \mathscr{F}) \to H^i_{Y_1 \cup Y_2}(X, \mathscr{F}) \to H^{i+1}_{Y_1 \cap Y_2}(X, \mathscr{F}) \to \ldots$$

Hint: First work with an injective sheaf $\mathscr{I}$. From the stuff on cohomology with supports above, write down a short exact sequence whose last term is $\Gamma(U_1 \cup U_2, \mathscr{I})$. On the other than, because $\mathscr{I}$ is a sheaf, you have another short exact sequence for which $\Gamma(U_1 \cup U_2, \mathscr{I})$ is the *first* term. Write these two sequences together in the shape of an L, and now try to fill in the diagram to get a 3 by 3 grid with all rows and column exact...