Fri. 15 Feb. 2019

From last time: Critical points.

Example: Let $f(x,y) = x\sin y$.

Find all (x_0, y_0) which are critical points of f.

$\boxed{?}$ How many are there with $0 \le y_0 \le 2\pi$?

— Announcements — .

Recall: Theorem: If $f:\mathbb{R}^2 \to \mathbb{R}$ has a local max/min at (a,b), then (a,b) is a critical point.

Examples: (1) $f(x,y) = x^2 + y^2$

local min. at $(0,0)$.

$\nabla f(x,y) = \langle 2x, 2y \rangle \Rightarrow \nabla f(0,0) = \langle 0, 0 \rangle$.

(2) $f(x,y) = -x^2 - y^2$

local max. at $(0,0)$

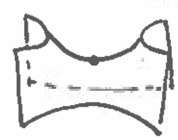
$\nabla f(x,y) = \langle -2x, -2y \rangle \longmapsto \langle 0,0 \rangle$ at $(0,0)$.

(3) $f(x,y) = -|x| - |y|$

local max at $(0,0)$

$\nabla f(0,0)$ undefined.

(4) $f(x,y) = x^2 - y^2$.



Note: $\nabla f(0,0) = \langle 0,0 \rangle$

But $(0,0)$ is neither a local max nor a local min.

Notice: The tangent plane to f at (a,b) is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

So it is horizontal $\Leftrightarrow \nabla f(a,b) = \langle 0,0 \rangle$.

For the rest of today, assume that the second order partial derivatives of f exist and are continuous.

Let (a,b) be a critical point.

Question: How can we tell if this is a local max, local min, or neither?

13.2

Definition: We write $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b) f_{yx}(a,b)$

$= f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2$.

Theorem: [Second derivative]

For f, (a,b), D as above:

(1) if $D > 0$ and $f_{xx}(a,b) > 0$, there is a local min. at (a,b)

(2) if $D > 0$ and $f_{xx}(a,b) < 0$ there is a local max at (a,b)

(3) if $D < 0$, there is neither a local max nor a local min. at (a,b).

Remarks (1) A critical point which is not a local max/min is called a saddle point

(2) When $D = 0$, we could have either local min/max or saddle point.
(or if $D > 0$, $f_{xx} = 0$).

(3) Note that $(f_{xy}(a,b))^2 \geq 0$, so if $D > 0$, we must have

$f_{xx}(a,b) f_{yy}(a,b) > 0$

$\Rightarrow f_{xx}$ and f_{yy} have the same sign.

Examples: 1) $f(x,y) = x^2 + y^2$, $(a,b) = (0,0)$.

$D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$, $f_{xx}(0) = 2 > 0$.

$\Rightarrow$ local min ✓

2) $f(x,y) = -x^2 - y^2$, $(a,b) = (0,0)$.

~~Find~~ $D. = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$, $f_{xx}(0) = -2 < 0$.

$\Rightarrow$ local max ✓

3) $f(x,y) = x^2 - y^2$, $(a,b) = (0,0)$

$D = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$. $\Rightarrow$ saddle point.

4) if $f(x,y) = x \sin y$ $(a,b) = (0, \pi)$,

find D.

Why does this work?

Recall: Taylor series.

For $f: \mathbb{R} \to \mathbb{R}$ nice enough, we have

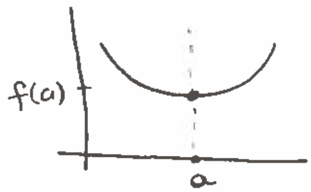
$$f(a+h) = f(a) + f'(a)h + \tfrac{1}{2}f''(a)h^2 + \underbrace{\varepsilon_2(h)}$$

very small: $\lim_{h \to 0} \frac{\varepsilon_2(h)}{h^2} = 0$

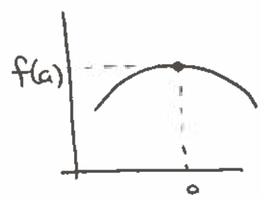
If a is a critical point,

$$f(a+h) = f(a) + \tfrac{1}{2}f''(a)h^2 + \varepsilon_2(h)$$

So near a the graph of f looks like



if $f''(a) > 0$



if $f''(a) < 0$.

Taylor series for $f(x,y)$ (nice enough):

$$f(a+h, b+k) = \underbrace{\overbrace{f(a,b) + f_x(a,b)\cancel{a}\,h + f_y(a,b)k}^{\text{linearization}} + f_{xx}(a,b)h^2 + f_{xy}(a,b)hk + f_{yy}(a,b)k^2}_{g(x,y)} + \underbrace{\varepsilon_2(h,k)}_{\text{very small!}}$$

- $g(x,y)$ is the only quadratic polynomial which ~~etc~~ satisfies
  - $g(a,b) = f(a,b)$
  - first & second order derivatives of g at (a,b) agree with those of g.

if (a,b) is a critical point,

~~graphs~~ $g(h,k) = f(a,b) + f_{xx}(a,b)h^2 + f_{xy}(a,b)hk + f_{yy}(a,b)k^2$.

So it's enough to understand the max/min/saddle points of functions like

at $(x,y) = (0,0)$

$$g(x,y) = \alpha x^2 + \beta xy + \gamma y^2$$

Here $g_x(x,y) = 2\alpha x + \beta y$, $g_y(x,y) = 2\gamma y + \beta x$.

$\Rightarrow g_{xx}(x,y) = 2\alpha$, $g_{xy}(x,y) = \beta = g_{yx}(x,y)$; $g_{yy}(x,y) = 2\gamma$.

So $D = \begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix} = 4ac - b^2$.

• $f(0,0) = 0$. So to see if $(0,0)$ is a local max/min, we need to see where g is positive / negative.

Suppose $a \neq 0$, and suppose $\exists (x,y) \neq (0,0)$ with $g(x,y) = (0,0)$

<u>Quadratic formula:</u> $\Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} y$.

· if $D > 0$, there are no solutions,

so $(0,0)$ is the only point where $g = 0$.

$\Rightarrow$ ~~$g(x,y)$ is either strictly positive on $(x,y) \neq (0,0)$ or neg~~.

$\Rightarrow$ either 1) $g(x,y) > 0$ for all $(x,y) \neq (0,0)$,

and $(0,0)$ is a local min

or 2) $g(x,y) < 0$ for all $(x,y) \neq (0,0)$

and $(0,0)$ is a local max.

<u>Note:</u> if $f_{xx}(0,0) = 2a > 0$, then $f(1,0) = a > 0$

So we must be in CASE 1

if $f_{xx} < 0$, we must be in CASE 2.

$\Big\}$ just as predicted.

· if $D < 0$, there are two lines of solutions $g(x,y) = 0$.

$\Rightarrow$ g is both positive and negative

$\Rightarrow$ must have a saddle point at $(0,0)$.

$\square$