Last time: vector fields

Let C be the line segment from $(0, 0)$ to $(1, 2)$. Consider the vector field $\mathbf{F}(x, y) = \langle 1, 2y \rangle$.

What is $\int_C \mathbf{F} \cdot d\mathbf{r}$?

(a) 9

(b) 5

(c) 0

(d) 20

(e) I don't know what to do.

(If you're done, sketch the curve and the vector field, and check whether your answer is a reasonable one.)
Correct answer: (b)

Solution:

Let C be parametrized by $\mathbf{r}(t) = \langle t, 2t \rangle,\ t \in [0, 1]$.
We have $\mathbf{F}(x, y) = \langle 1, 2y \rangle$.

- $\mathbf{F}(\mathbf{r}(t)) = \langle 1, 4t \rangle$.
- $\mathbf{r}'(t) = \langle 1, 2 \rangle$.

$$\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 1, 4t \rangle \cdot \langle 1, 2 \rangle dt$$

$$= \int_0^1 1 + 8t \ dt$$

$$= [t + 4t^2]_0^1$$

$$= 5.$$

Computing the integral of a vector field using the unit tangent vector

Consider the circle $C = \{x^2 + y^2 = 1\}$ oriented clockwise. Use the formula

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

to find $\int_C \langle y, -x \rangle \cdot d\mathbf{r}$, *without* choosing a specific parametrization of C.

(a) π

(b) $-\pi$

(c) 2π Correct answer

(d) -2π

(e) I don't know how.

If you're done, choose a parametrization and check your answer by computing the integral using the original definition.

Solution

Note that at a point $P = (x, y)$ of the circle, $\mathbf{F}(P)$ is a unit vector (check the definiton) and $\mathbf{T}(P)$ is also a unit vector (by construction).

Also, both are tangent to the circle and point clockwise.

So $\mathbf{F}(P) = \mathbf{T}(P)$ and $\mathbf{F}(P) \cdot \mathbf{T}(P) = |\mathbf{F}(P)|^2 = 1$.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot \mathbf{T} ds \\
&= \int_C ds \\
&= L = 2\pi.
\end{aligned}$$

Practice with the fundamental theorem of line integrals

Let C be a circle in $\mathbb{R}^2$ with centre P and radius r. Let $f(x, y) = 3x^2 + sin(x + y)$, and let $\mathbf{F} = \nabla f$.

What is $\int_C \mathbf{F} \cdot d\mathbf{r}$?

(a) Not enough information: I can't do it unless you tell me the starting and ending points of the path.

(b) Not enough information: I can't do it because you haven't told me the orientation of the circle.

(c) I think I can do it, but I need more time to compute this integral.

(d) It's zero.

Correct answer: (d)

Is the vector field conservative?

We're going to look at the vector field describing wind velocity.
Discuss with your neighbour: is this vector field conservative?
https://earth.nullschool.net/
(Remember the options below:)

(a) Yes, we think it is.
(b) No, we think it's not.
(c) We don't agree/we don't know.

Answer: the vector field is not conservative. You can find circles around which the integral is not zero.