<u>Last time</u> – <u>integrating vector fields over surfaces.</u>

- $\iint_S \vec{F}\cdot d\vec{S} := \iint_S \vec{F}\cdot\vec{n}\, dS$
- if $\vec{r}(u,v)$, $(u,v)\in D$ is a parametrization of S and $\vec{r}_u\times\vec{r}_v$ is positively oriented, then
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\, dA$$

[1] Let $S=\{3x+2y+z \overset{=1}{\cancel{\geq 0}},\ x\geq 0,\ y\geq 0,\ z\geq 0\}$ oriented upward.

S is parametrized by $\vec{r}(u,v) = \langle u, v, 1-3u-2v\rangle$ where $(u,v)\in D = \{u\geq 0,\ v\geq 0,\ 3u+2v\leq 1\}$.

- Is $\vec{r}_u\times\vec{r}_v$ positively oriented?
- use this to calculate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = \langle 1,0,1\rangle$.

[see slides]

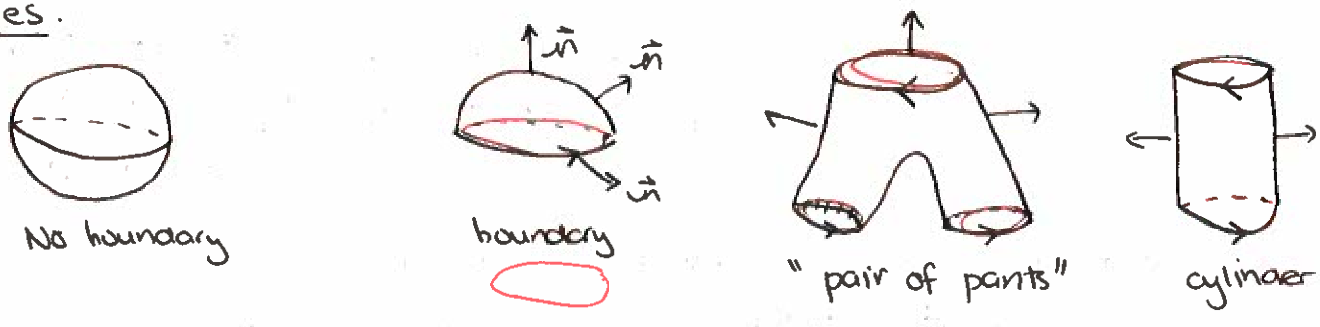
<u>Today:</u> Stoke's theorem: §16.8.

Let S be an oriented surface.

The boundary curve ∂S of S is the set of points that

- nearby, S looks like $\{y\geq 0\}\subset\mathbb{R}^2$ (instead of $\mathbb{R}^2$).

<u>Examples.</u>



The orientation on $\underset{\vec{n}}{S}$ induces an orientation on the boundary

- point your head in the direction of $\vec{n}$
- orient ∂S so that S is to your left as you walk along ∂S

Assumptions: • S is "nice"
- S is piecewise smooth
- ∂S is one or more simple closed paths.

• $\vec{F}$ has continuous first order partial derivatives on an open region $R \subset \mathbb{R}^3$ containing S.

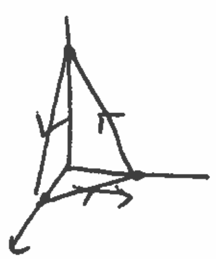
Stokes' Theorem: $$\iint_S \underbrace{\text{curl}\,\vec{F}}_{\text{derivative}} \cdot d\vec{S} = \int_{\underbrace{\partial S}_{\text{boundary}}} \vec{F} \cdot d\vec{r}$$

• use this to integrate $\text{curl}\,\vec{F}$ over S if ∂S is simpler than S or to integrate $\vec{F}$ over ∂S if $\text{curl}\,\vec{F}$ is simpler.

Why? – Fundamental theorem of calculus.

Example: Let $\vec{F} = \langle 1, x + \sin y^2, y - e^{z^3} \rangle$

$S = \{3x + 2y + z = 1, \ x \geq 0, y \geq 0, z \geq 0\}$. oriented upwards.



$C = \partial S$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Stokes' theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \cancel{\vec{F}}\ \text{curl}\,\vec{F} \cdot d\vec{S}$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 1 & x + \sin y^2 & y - e^{z^3} \end{vmatrix} = \vec{i}(1-0) - \vec{j}(0-0) + \vec{k}(1-0)$$

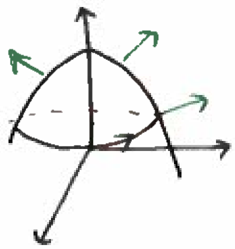
$= \langle 1, 0, 1 \rangle$. simpler than $\vec{F}$!

But we already calculated the integral of $\vec{G} = \langle 1, 0, 1 \rangle$ over S in the first example: $\frac{1}{3}$.

Example: $\vec{F} = \langle y, -x, xe^{y^2} \rangle$, $S = \{z = 10 - x^2 - y^2$ and $z \geq 1\}$ oriented upwards.

Find $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$.

35.3



$\partial S = \{1 = 10 - x^2 - y^2\}$.

$x^2 + y^2 = 9$ oriented counter clockwise.

↳ parametrization $\vec{r}(t) = \langle 3\cos t, 3\sin t, 1\rangle$, $0 \le t \le 2\pi$.
(matching orientation).

Stokes' Theorem: $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle 3\sin t, -3\cos t, 3\cos t\, e^{9\sin^2 t}\rangle \langle -3\sin t, 3\cos t, 0\rangle d$$

$$= \int_0^{2\pi} -9\, dt = -18\pi.$$

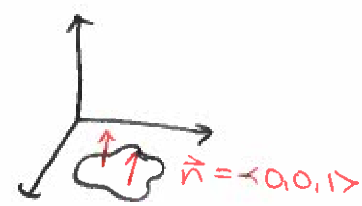
Example Assume $\vec{F} = \langle P(x,y), Q(x,y), 0\rangle$ and $S \subset \{z = 0\}$,
oriented upward.

$$\int_{\partial S} P\,dx + Q\,dy = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}) \cdot d\vec{S}$$

$$= \iint_S (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS$$

$$= \iint_S \langle 0, 0, Q_x - P_y\rangle \cdot \vec{k}\, dS$$

$$= \iint_S Q_x - P_y\, dA.$$



↳ this is Green's Theorem!

[L]. Let $\vec{F} = \langle x\sin z, y\sin z, e^{x+y}\rangle$ and $S = \{x^2 + y^2 + z^2 = 9\}$
oriented outwards.

Find $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$.

§ CURL.

Let C be an oriented curve with unit tangent vector $\vec{T}$.

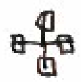
Recall: The circulation of $\vec{F}$ around C is

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r}.$$

It measures if fluid flows with C or against C. 35.4

Look so that a unit vector $\vec{v}$ points towards you.

Place a tiny paddle wheel on S with normal $\vec{v}$



By Stokes' Theorem, the circulation around ∂S is

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot d\vec{S}$$

$$= \iint_S (\operatorname{curl}\vec{F})\cdot \vec{v}\, dS$$

So: the paddle wheel

- rotates counterclockwise $\Leftrightarrow (\operatorname{curl}\vec{F})\cdot\vec{v} > 0$
- rotates clockwise $\Leftrightarrow (\operatorname{curl}\vec{F})\cdot\vec{v} < 0$.
- $|\operatorname{curl}\vec{F}\cdot\vec{v}|$ gives speed of rotation.
  - doesn't rotate $\Leftrightarrow \operatorname{curl}\vec{F}\cdot\vec{v} = 0$
  - fastest speed of rotation is when $\vec{v} = \pm \operatorname{curl}\vec{F}$.