INTRO TO QUANTUM GROUPS AND THEIR REPRESENTATION THEORY

NOTES FOR A TALK BY CRAIG SMITH

Plan of action:

- Define the quantum enveloping algebras and introduce (restricted) integral form
- Introduce induction functors to define (dual) Weyl/Verma modules
- Discuss representation theory at generic q
- Start on the representation theory at q a root of unity

Lots of Notation

First things first, we need to set some notation. So, here goes...

Fix a field $k = \bar{k}$ of characteristic zero and let $K = k(q)$ be rational functions in q.

Let $\mathfrak{g}$ be a Lie algebra over k defined by the following data:

- A weight lattice Φ, a free $\mathbb{Z}$-module;
- Simple roots $\alpha_i \in \Phi$ indexed by $i \in I$ that form a basis of the root lattice $\Psi \subset \Phi$;
- A symmetric bilinear form $(\cdot,\cdot) : \Phi \times \Phi \to \mathbb{Q}$ such that $(\alpha_i, \alpha_i) \in 2\mathbb{N}$, $(\alpha_i, \alpha_j) \leq 0$ for $i, j \in I, i \neq j$;
- Simple coroots $h_i \in \Phi^* = \mathrm{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$ such that $h_i(\alpha) = \frac{2(\alpha_i,\alpha)}{(\alpha_i,\alpha_i)}$ for $i \in I, \alpha \in \Phi$.

Remark Then $\mathfrak{g}$ can be generated by E_i, F_i, H_i for $i \in I$ with the Serre relations

$$[H_i, H_j] = 0, [E_i, F_i] = \delta_{ij} H_i, [H_i, E_j] = h_i(\alpha_j) E_j, [H_i, F_j] = -h_i(\alpha_j) F_j,$$

and for $i \neq j$,

$$(\mathrm{ad} E_i)^{1-h_i(\alpha_j)} E_j = 0, (\mathrm{ad} F_i)^{1-h_i(\alpha_j)} F_j = 0.$$

We will denote by

- $\Phi_+ = \{\alpha \in \Phi \mid h_i(\alpha) \geq 0 \text{ for all } i \in I\}$ be the dominant weights;
- $\Psi_+ = \{\sum_{i \in I} n_i \alpha_i \mid n_i \geq 0\} \subset \Psi$ the positive root lattice, $\Psi_- = -\Psi_+$ the negative roots;
- $\geq$ the partial ordering on Φ given by $\alpha \geq \beta$ if and only if $\alpha - \beta \in \Psi_+$;

- W the *Weyl group* attached to this data, generated by the *simple reflections* $s_i(\alpha) = \alpha - \frac{2(\alpha_i,\alpha)}{(\alpha_i,\alpha_i)}\alpha_i = \alpha - h_i(\alpha)\alpha_i$ for $i \in I$;
- w_0 its unique element of highest length;

- $R = W \cdot \{\alpha_i \mid i \in I\}$ the roots of $\mathfrak{g}$, $R_+ = R \cap \Phi_+$ the positive roots;
- $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$ is the half-sum of the positive roots;

- $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$;
- $q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$ for $i \in I$.
- and for integers $m, n \in \mathbb{Z}$ let

$$[n]_i = [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} = q_i^{n-1} + q_i^{n-3} + ... + q_i^{-n+3} + q_i^{-n+1}$$

$$[n]_i! = [n]_{q_i}! = [n]_i[n-1]_i[n-2]_i...[2]_i[1]_i,$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_i = \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} = \frac{[m]_i!}{[n]_i![m-n]_i!} = \prod_{t=1}^{n} \frac{q_i^{m-n+t} - q_i^{-(m-n+t)}}{q_i^t - q_i^{-t}};$$

The Quantum Enveloping Algebras and Integral Forms

Definition We may now define the *quantised enveloping algebra* $U_q = U_q(\mathfrak{g})$ to be the algebra generated over K by E_i, F_i, H_i, H_i^{-1} for $i \in I$ with the defining relations (the quantum Serre relations)

- $H_iH_i^{-1} = 1 = H_i^{-1}H_i$;
- $H_iH_j = H_jH_i$;
- $H_iE_jH_i^{-1} = q^{h_i(\alpha_j)}E_j$;
- $H_iF_jH_i^{-1} = q^{-h_i(\alpha_j)}F_j$;
- $E_iF_j - F_jE_i = \delta_{i,j}\frac{H_i - H_i^{-1}}{q_i - q_i^{-1}}$;

for all $i, j \in I$, and for $i \neq j$

- $0 = \sum_{r=0}^{1-h_i(\alpha_j)}(-1)^r E_i^{(r)}E_jE_i^{(1-h_i(\alpha_j)-r)}$;
- $0 = \sum_{r=0}^{1-h_i(\alpha_j)}(-1)^r F_i^{(r)}F_jF_i^{(1-h_i(\alpha_j)-r)}$;

where

- $E_i^{(r)} = \frac{1}{[r]_i!}E_i^r$, $F_i^{(r)} = \frac{1}{[r]_i!}F_i^r$.

This is given a Hopf algebra structure by setting

$$\begin{aligned}
\Delta :&U_q \to U_q \otimes U_q, & \varepsilon :&U_q \to K & S :&U_q \to U_q,\\
&E_i \mapsto E_i \otimes 1 + H_i \otimes E_i, & &E_i \mapsto 0 & &E_i \mapsto -H_i^{-1}E_i,\\
&F_i \mapsto F_i \otimes H_i^{-1} + 1 \otimes F_i, & &F_i \mapsto 0 & &F_i \mapsto -F_iH_i,\\
&H_i \mapsto H_i \otimes H_i, & &H_i \mapsto 1 & &H_i \mapsto H_i^{-1}.
\end{aligned}$$

Let us denote by

- $U_q^{>0}$ (respectively $U_q^{<0}$) the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i \mid i \in I\}$ (respectively $\{F_i \mid i \in I\}$);
- U_q^0 the subalgebra generated by $\{H_i^{\pm 1} \mid i \in I\}$;
- $U_q^{\geq 0}$ (respectively $U_q^{\leq 0}$) the subalgebra generated by $U_q^{>0}$ (respectively $U_q^{<0}$) and U_q^0.

Fact There is a *triangular decomposition* of U_q. That is, multiplication defines isomorphisms of K vector spaces

$$U_q^{>0} \otimes U_q^0 \otimes U_q^{<0} \to U_q$$

$$U_q^{<0} \otimes U_q^0 \otimes U_q^{>0} \to U_q$$

Remark For a fixed $i \in I$ we can see that the subalgebras of $U_q(\mathfrak{g})$ generated by $E_i, F_i, H_i^{\pm 1}$, denoted $U_q(\mathfrak{g})_i$, are isomorphic to $U_q(\mathfrak{sl}_2)$.

The problem with working over $K = k(q)$ is that we cannot specialise to any value of q that is algebraic over k, including the roots of unity we are interested in. Instead, we must work with an *integral form* of U_q.

Definition An integral form of U_q is a Hopf subalgebra H over $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ such that $U_q(\mathfrak{g}) \cong H \otimes_{\mathcal{A}} K$. We may then specialise to a general $\varepsilon \in k^{\times}$, $H_\varepsilon = H \otimes_{\mathcal{A}} k$ via the map $\mathcal{A} \to k$, $q \mapsto \varepsilon$.

There are two good choices for such integral forms, the restricted and the unrestricted integral forms. We will only be interested in the restricted integral form, which is good since I know nothing about the unrestricted integral form.

Definition The restricted integral form of U_q, denoted $U_{\mathcal{A}}^{\text{res}}$, is the $\mathcal{A}$ subalgebra of U_q generated by $E_i^{(r)}$, $F_i^{(r)}$, K_i and K_i^{-1} for $i \in I$, $r \geq 0$. Since we won't be studying the unrestricted integral form, we will drop the 'res' and simply write $U_{\mathcal{A}}$. Let us define $U_{\mathcal{A}}^{\bullet} = U_{\mathcal{A}} \cap U_q^{\bullet}$ for $\bullet \in \{>0, \geq 0, <0, \leq 0, 0\}$.

(Non-Trivial) Fact $U_{\mathcal{A}}$ is an integral form of U_q and a free $\mathcal{A}$-module, and has a similar triangular decomposition. This is proven by constructing (using an action of the braid group) a free $\mathcal{A}$-basis of $U_{\mathcal{A}}$ that is also a K-basis of U_q. Given an enumeration of the positive roots $R_+ = \{\beta_1, .., \beta_N\}$, there exist $E_{\beta_i}, F_{\beta_i} \in U_{\mathcal{A}}^{\text{res}}$ such that $K_j E_{\beta_i} K_j^{-1} = q^{(\alpha_j, \beta_i)} E_{\beta_i}$, $K_j F_{\beta_i} K_j^{-1} = q^{-(\alpha_j, \beta_i)} F_{\beta_i}$, and that

$$\{(E_{\beta_N})^{(l_N)} (E_{\beta_{N-1}})^{(l_{N-1})} ... (E_{\beta_1})^{(l_1)} \mid l_i \geq 0\},$$

$$\{(F_{\beta_N})^{(l_1)} (F_{\beta_{N-1}})^{(l_2)} ... (F_{\beta_1})^{(l_N)} \mid l_i \geq 0\}$$

are $\mathcal{A}$-bases of $U_{\mathcal{A}}^{>0}$ and $U_{\mathcal{A}}^{<0}$ respectively, and are K-bases of $U_q^{>0}$ and $U_q^{<0}$ respectively. Then, using a triangular decomposition we obtain the required bases.

Induction and (Dual) Verma and Weyl modules

Definition For a $U_{\mathcal{A}}$ module M we denote by $\mathcal{O}(M)$ and $F(M)$ the $\mathcal{A}$ submodules

$$\mathcal{O}(M) = \bigoplus_{\lambda\in\Phi} M_\lambda \text{ where } M_\lambda := \{m \in M \mid H_i m = q^{h_i(\lambda)} m \text{ for all } i \in I\},$$

the direct sum of the weight spaces of M, and

$$F(M) = \{m \in \mathcal{O}(M) \mid E_i^{(r)} m = 0 = F_i^{(r)} m \text{ for all } i \in I \text{ and for } r \gg 0\},$$

the integrable part of M. Note that these are in fact $U_{\mathcal{A}}^{\mathrm{res}}$ submodules of M. We say that M is *integrable* if $M = F(M)$.

Let $\mathcal{O}_{\mathcal{A}}$ be the category of $U_{\mathcal{A}}$ modules M such that $M = \mathcal{O}(M)$ and that are locally $\mathcal{A}$-finite (finitely generated as an $\mathcal{A}$ module) $U_{\mathcal{A}}^{\geq 0}$ modules. Let $\mathcal{O}_{\mathcal{A}}^{\mathrm{int}}$ be the full subcategory of integrable modules in $\mathcal{O}_{\mathcal{A}}$.

By the commutation relations we see that, for $M \in \mathcal{O}_{\mathcal{A}}$, $E_i^{(r)} M_\lambda \subset M_{\lambda+r\alpha_i}$ and $F_i^{(r)} M_\lambda \subset M_{\lambda-r\alpha_i}$. An element $m \in M_\lambda$ is said to be *a weight vector of weight* λ and if, in addition, $E_i^{(r)} m = 0$ for every $i \in I$ and every $r > 0$ then we say m is a *highest weight vector*. A $U_{\mathcal{A}}$ module in $\mathcal{O}_{\mathcal{A}}$ that is generated by a highest weight vector is called a *highest weight module*.

We may analogously define integrable $U_{\mathcal{A}}^{\geq 0}$ modules, where we only insist on $E_i^{(r)} m = 0$ for $r \gg 0$ in the definition of F, and analogous categories $\mathcal{O}_{\mathcal{A}}^{\geq 0}$, $\mathcal{O}_{\mathcal{A}}^{\geq 0,\mathrm{int}}$. Then we have an induction functor $\mathrm{Ind} : \mathcal{O}_{\mathcal{A}}^{\geq 0} \to \mathcal{O}_{\mathcal{A}}$, $N \mapsto U_{\mathcal{A}} \otimes_{U_{\mathcal{A}}^{\geq 0}} N$.

For $\lambda \in \Phi$ we denote by $\mathcal{A}_\lambda$ the $U_{\mathcal{A}}^{\geq 0}$ module structure on $\mathcal{A}$ where $E_i^{(r)}$ act by 0 and K_i act by $q^{h_i(\lambda)}$ on $\mathcal{A}$. Caution: As Kobi remarked in the talk, this is not enough to define $\mathcal{A}_\lambda$ since $U_{\mathcal{A}}^0$ is not only generated by $H_i^{\pm}$, but also by $\begin{bmatrix} H_i; m \\ n \end{bmatrix} := \prod_{t=1}^n \frac{H_i q_i^{m-t+1} - H_i^{-1} q_i^{-(m-t+1)}}{q_i^t - q_i^{-t}}$ that arise from the commutation of the $E_i^{(r)}$ and $F_i^{(r)}$. We define the *Verma module with highest weight* λ to be $M_{\mathcal{A}}(\lambda) := \mathrm{Ind}(\mathcal{A}_\lambda) = U_{\mathcal{A}} \otimes_{U_{\mathcal{A}}^{\geq 0}} \mathcal{A}_\lambda$. This is a highest weight module of weight λ generated by $x_\lambda = 1 \otimes 1$, and in fact $M_{\mathcal{A}}(\lambda)$ is universal in the sense that it surjects uniquely up to scalar onto any other highest weight module of weight λ.

Using this induction functor, we may also define another induction functor from $\mathcal{A}$-finite integrable $U_{\mathcal{A}}^{\geq 0}$ modules to $\mathcal{A}$-finite integrable $U_{\mathcal{A}}$ modules as follows.

Proposition/Definition Let N be an $\mathcal{A}$-finite integrable $U_{\mathcal{A}}^{\geq 0}$ module. Let $M = U_{\mathcal{A}} \otimes_{U_{\mathcal{A}}^{\geq 0}} N$. Then the set of $U_{\mathcal{A}}$ submodules L of M such that M/L is $\mathcal{A}$-finite has a unique minimal element, L_0 say. Then we define $D(N) = M/L_0$.

Proof. We will make use of the following Lemma:

Lemma. *If M is an integrable $U_{\mathcal{A}}$ module and λ is a weight of M then $w\lambda$ is a weight of M for all $w \in W$.*

This is proven by reducing to the case of w being a simple reflection and $\mathfrak{g}$ being $\mathfrak{sl}_2$, where it is straightforward to prove.

The weights of N form a finite set $X \subset \Phi$. Note that $M \cong U_{\mathcal{A}}^{\leq 0} \otimes_{U_{\mathcal{A}}^0} N$, so the weights of M are in $X' = X + \Psi_-$, which only contains finitely many dominant weights. Now each weight in Φ is in the same orbit as a dominant weight, so $X'' = X' \cap (W(X' \cap \Phi_+))$, the largest W-stable subset of X', is finite. If L is a $U_{\mathcal{A}}$ submodule of M such that M/L is a finitely generated $\mathcal{A}$ module then, as the weights of M/L are stable under W, they are contained in X''. So L contains $M' = \bigoplus_{\lambda \notin X''} M_\lambda$. Conversely, if L_0 is the $U_{\mathcal{A}}$ submodule generated by M' then the weights of M/L_0 are contained in X'' and hence are finite, and each weight space of M is a finitely generated $\mathcal{A}$ module, so M/L_0 is a finitely generated $\mathcal{A}$ module. □

We denote by $V_{\mathcal{A}}(\lambda)$ the $\mathcal{A}$-finite induced module $D(\mathcal{A}_\lambda)$, which we call the *Weyl module* of highest weight λ. Note that, if $\lambda \notin \Phi_+$ then $X'' = \emptyset$ and so $V_{\mathcal{A}}(\lambda) = \{0\}$. $V_{\mathcal{A}}(\lambda)$ is a highest weight module of weight λ, generated by the image of x_λ, v_λ say, and is universal in the sense that it surjects onto any $\mathcal{A}$-finite highest weight module of weight λ.

Fact Both $M_{\mathcal{A}}(\lambda)$ and $V_{\mathcal{A}}(\lambda)$ are free $\mathcal{A}$ modules.

More explicitly, $M_{\mathcal{A}}(\lambda)$ is isomorphic to the quotient of $U_{\mathcal{A}}$ by the left ideal generated by $E_i^{(r)}$ and $H_i - q^{h_i(\lambda)} \cdot 1$ for $i \in I$ and $r \geq 1$, where x_λ is the image of 1 in the quotient. Then $V_{\mathcal{A}}(\lambda)$ is the quotient of $M_{\mathcal{A}}(\lambda)$ by the submodule generated by $F_i^{(s_i)} x_\lambda$ for $i \in I$ and $s_i > h_i(\lambda)$. This follows from the relation

$$E_i^{(r)} F_i^{(s)} = \sum_{0 \leq t \leq r,s} F_i^{(s-t)} \left(\prod_{l=1}^{t} \frac{K_i q_i^{2t-r-s-l+1} - K_i^{-1} q_i^{-(2t-r-s-l+1)}}{q_i^l - q_i^{-l}} \right) E_i^{(r-t)}$$

For example, if we take $\mathfrak{g} = \mathfrak{sl}_2$, so $\Phi = \mathbb{Z}$ with simple root is 2, then we may give explicit descriptions of $M_{\mathcal{A}}(\lambda)$ as follows:

$$\begin{aligned}
M_{\mathcal{A}}(\lambda) &= \bigoplus_{i\geq 0} \mathcal{A}F^{(i)}x_\lambda, \\
K(F^{(i)}x_\lambda) &= q^{\lambda-2i}F^{(i)}x_\lambda, \\
F^{(j)}(F^{(i)}x_\lambda) &= \begin{bmatrix} i+j \\ i \end{bmatrix} F^{(i+j)}x_\lambda, \\
E^{(j)}(F^{(i)}x_\lambda) &= \begin{cases} \begin{bmatrix} \lambda-(i-j) \\ j \end{bmatrix} F^{(i-j)}x_\lambda & \text{if } i-j \geq 0, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

Note that, by definition, $\begin{bmatrix} \lambda-(i-j) \\ j \end{bmatrix} = \prod_{t=1}^{j} \frac{q^{\lambda-i+t}-q^{-(\lambda-i+t)}}{q^t-q^{-t}}$ is zero if and only if zero appears as one of these terms, which only happens when $\lambda+j \geq i > \lambda$. This shows that $\bigoplus_{i>\lambda} \mathcal{A}F^{(i)}x_\lambda$ is a submodule. Quotienting out by this submodule gives the Weyl module:

$$\begin{aligned}
V_{\mathcal{A}} &= \bigoplus_{i=0}^{\lambda} \mathcal{A}F^{(i)}v_\lambda, \\
K(F^{(i)}v_\lambda) &= q^{\lambda-2i}F^{(i)}v_\lambda, \\
F^{(j)}(F^{(i)}v_\lambda) &= \begin{cases} \begin{bmatrix} i+j \\ i \end{bmatrix} F^{(i+j)}v_\lambda & \text{if } i+j \leq n, \\ 0 & \text{otherwise,} \end{cases} \\
E^{(j)}(F^{(i)}v_\lambda) &= \begin{cases} \begin{bmatrix} \lambda-(i-j) \\ j \end{bmatrix} F^{(i-j)}v_\lambda & \text{if } \lambda \geq i-j \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For $M \in \mathcal{O}_{\mathcal{A}}$ we define the dual module of M to be $M^* = \bigoplus_{\lambda\in\Phi} M'_\lambda$ where $M'_\lambda = \mathrm{Hom}_{\mathcal{A}}(M_\lambda, \mathcal{A})$ is the dual to M_λ. Since $M^* \subset \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})$ we may endow it with the $U_{\mathcal{A}}$ module structure arising from the antipode S (namely $(x\cdot\phi)(m) = \phi(S(x)\cdot m)$). Unfortunately, this is no longer in $\mathcal{O}_{\mathcal{A}}$ - it's kind of upside-down - however if the roles of E_i and F_i were interchanged it would be. Therefore we must twist the action using the automorphism ω of $U_{\mathcal{A}}$ determined by $E_i^{(r)} \mapsto F_i^{(r)}$, $F_i^{(r)} \mapsto E_i^{(r)}$ and $H_i \mapsto H_i^{-1}$. So the action on M^* is given by $(x\cdot\phi)(m) = \phi(S(\omega(x))\cdot m)$ and gives a dual module in $\mathcal{O}$. Then we call $M^*_{\mathcal{A}}(\lambda)$ the *dual Verma module* and $V^*_{\mathcal{A}}(\lambda)$ the *dual Weyl module*. If $x^*_\lambda \in M'_{\mathcal{A}}(\lambda)_\lambda$ is such that $x^*_\lambda(x_\lambda) = 1$ then $(E_i^{(r)}\cdot x^*_\lambda)(m) = x^*_\lambda(S(F_i^{(r)})\cdot m) = 0$ for $r > 0$ since $S(F_i^{(r)})$ is a scalar multiple of $H_i^r F_i^{(r)}$ and so reduces the weight of m, but λ is of highest weight. So there is a morphism $M_{\mathcal{A}}(\lambda) \to M^*_{\mathcal{A}}(\lambda)$, and in fact this factors as

$$M_{\mathcal{A}}(\lambda) \twoheadrightarrow V_{\mathcal{A}}(\lambda) \to V^*_{\mathcal{A}}(\lambda) \hookrightarrow M^*_{\mathcal{A}}(\lambda).$$

We will see that, in the case where q is generic (and so we work over K instead of $\mathcal{A}$) this is an isomorphism but this is not the case at roots of unity.

Representation Theory at Generic q

Let us first deal with q transcendental over k, where things are nice.

Let $M(\lambda) = M_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} K$ be the Verma modules of U_q of weight λ and let $V(\lambda) = V_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} K$ be the Weyl modules. Let $\mathcal{O}$ and $\mathcal{O}^{\text{int}}$ be defined analogously for U_q.

Theorem 0.1 (Lusztig)**.** *Every $M \in \mathcal{O}^{int}$ can be written as a direct sum of Weyl modules $V(\lambda)$ for $\lambda \in \Phi$, which are simple and distinct for different λ.*

In order to prove the above, we will make use of the *Quantum Casimir Operator*, specifically the one found in Lusztig's 'Introduction to Quantum Groups'. We will not need an explicit description of the operator, so we shall just summarise its properties.

Given any U_q module $M \in \mathcal{O}$, we have an operator $\Omega : M \to M$, the *Quantum Casimir Operator*, that commutes with the action of U_q such that:

- The eigenvalues of Ω are of the form q^c for various integers c;
- There is a function $G : \Phi \to \mathbb{Z}$ such that Ω acts by scalar $q^{G(\lambda)}$ on the Verma module $M(\lambda)$;
- If $\lambda \geq \lambda'$ and $G(\lambda) = G(\lambda')$ then $\lambda = \lambda'$.

Proof. First we show that Weyl modules $V(\lambda)$ are simple. In fact, we show that any integrable quotient M of a Verma module $M(\lambda)$ is simple (and hence $V(\lambda)$ is the unique integrable quotient of $M(\lambda)$). Suppose we have a proper non-trivial submodule M' of M. Then $M'_\lambda = \{0\}$, and we can find a maximal $\lambda' \in \Phi$ such that $M'_{\lambda'} \neq \{0\}$. Let $m \in M'_{\lambda'}$ be nonzero, so $E_i m = 0$ for all $i \in I$. So there is a morphism $M(\lambda') \to M$. Since Ω acts by $q^{G(\lambda)}$ on M and by $q^{G(\lambda')}$ on $U_q \cdot m$, and since $\lambda \geq \lambda'$, we have $\lambda = \lambda'$ giving a contradiction.

Now we prove that every $M \in \mathcal{O}^{\text{int}}$ is a direct sum of simple $V(\lambda)$, for which it is enough just to show it is a sum of these modules. By writing M as a direct sum of generalised eigenspaces of Ω, we may assume that $(\Omega - q^c)$ is locally nilpotent on M. Let $P = \{m \in M \mid E_i m = 0 \text{ for all } i \in I\}$, which decomposes as $P = \bigoplus P_\lambda$ for $P_\lambda = P \cap M_\lambda$. The submodules of M generated by $m \in P_\lambda$ are integrable quotients of Verma modules, and so are each isomorphic to some Weyl module. So the submodule M' generated by P is a sum of Weyl modules. Let us show that $M'' = M/M'$ is trivial. If it weren't, we would be able to find λ maximal such that $M''_\lambda \neq \{0\}$, and a nonzero $m \in M''_\lambda$ and a representative $\tilde{m} \in M$ of m. Since $(\Omega - q^c)$ acts locally nilpotently on M'', and since Ω acts by $q^{G(\lambda)}$ on the submodule generated by m, we have $c = G(\lambda)$. By assumption, not all $E_i\tilde{m} = 0$ for $i \in I$, so we may find $\lambda' \geq \lambda$, $\lambda' \neq \lambda$, such that $M_{\lambda'} \neq \{0\}$ and a nonzero $m' \in M_{\lambda'}$. Then by a similar argument to before $c = G(\lambda')$, giving a contradiction. So M'' is trivial and we have our result. □

Representation Theory at $q^n = 1$

Unfortunately, we don't get such a nice picture when q takes the value of a root of unity.

Let $\varepsilon \in k$ be a primitive l^{th} root of unity ($l \neq 2$), and let $U_\varepsilon = U_{\mathcal{A}} \otimes_{\mathcal{A}} k$ given by the map $\mathcal{A} \to k$, $q \mapsto \varepsilon$, and likewise $V_\varepsilon(\lambda)$.

Let us consider, for simplicity, $\mathfrak{g} = \mathfrak{sl}_2$. If l is odd, the Weyl modules $V(\lambda)$ for $0 \leq \lambda < l - 1$ are simple (and if l is even a similar statement may be made with $\frac{l}{2}$ in place of l). A proof using the Quantum Casimir operator works, but uses the fact that $1, q^2, .., q^{2n}$ are distinct. But if we look at $V_\varepsilon(\lambda)$ for $\lambda > l$, we see that $\text{Span}_k\{F^{(i)}v_\lambda \mid i = 0, .., l-1\}$ is a submodule. Clearly it is closed under the action of H and $E^{(r)}$ for $r \geq 0$, and the fact that it is closed under the action of $F^{(r)}$ follows from

$$F^{(j)}(F^{(i)}v_\lambda) = \begin{cases} \begin{bmatrix} i+j \\ i \end{bmatrix}_\varepsilon F^{(i+j)}v_\lambda & \text{if } i+j \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

and the fact that $\begin{bmatrix} i+j \\ j \end{bmatrix}_\varepsilon = 0$ if $i + j \geq l$ since $[l]_\varepsilon = 0$. So $V_\varepsilon(\lambda)$ is no longer simple, and in fact this new submodule does not split. So the category $\mathcal{C}$ is no longer semisimple. (Note that, if l is even, $\varepsilon^{\frac{l}{2}} = \varepsilon^{-\frac{l}{2}}$ and so $[\frac{l}{2}]_\varepsilon = 0$.

For a general $\mathfrak{g}$ and l odd we have:

Theorem 0.2 (Andersen, Polo & Wen)**.** *The modules $V_\varepsilon(\lambda)$ are irreducible if $(\lambda + \rho, \alpha) < l$ for all positive roots $\alpha \in R_+$.*

The set of $\lambda \in \Phi_+$ with $(\lambda + \rho, \alpha) < l$ for all positive roots $\alpha \in R_+$ is called the *principal alcove.* A similar statement is true for l even, where we use $\frac{l}{2}$ in place of l.

Run away before people start throwing things