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THE PROBLEM OF THE BROKEN ROD AND ERNESTO CESÀRO'S EARLY WORK IN PROBABILITY

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Abstract

A rod of length 1 is broken into n pieces at $n-1$ randomly chosen points. What is the probability that k segments are of length greater than x? This problem and its solution go back to 1873 and are closely associated with the foundation of the *Société Mathématique de France*. The most complete solution was, remarkably, given a little later by Ernesto Cesàro, an Italian writing in French at the very beginning of his research career, which was not generally associated with geometric probability. Various solutions have been rediscovered in the 20th century in a number of important probabilistic-model settings. We review the pre-history of the problem in the late 19th century, where credit for its solution belongs. We also provide a modern solution to the problem of the broken rod and a verification of Cesàro's expression for the solution.

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1. Introduction

The problem of the broken rod can be formulated as follows.

Problem 1. A rod of length 1 is broken into n pieces at $n-1$ randomly chosen points. What is the probability, denoted by $\mathrm{P}_{[k]}$, that exactly k segments are of length greater than x, where $k = 0, 1, 2, \ldots, n$ and $kx \le 1$?

In modern statistical language, the $n-1$ random points are the order statistics

$$Y_1, Y_2, \ldots, Y_{n-1}, \qquad 0 < Y_1 < Y_2 < \cdots < Y_{n-1} < 1,$$

of $n-1$ independently distributed random variables, $X_1, X_2, \ldots, X_{n-1}$, each uniformly distributed on $(0, 1)$. We shall use the notation

$$W_{j+1} = Y_{j+1} - Y_j, \qquad j = 0, 1, \ldots, n-1,$$

for the successive segment (interval) lengths, and define $Y_0 = 0$ and $Y_n = 1$.

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A relatively modern solution method was given in Kendall and Moran (1963, p. 32), which results in the following expression (after misprints have been corrected):

$$\mathrm{P}_{[k]} = \binom{n}{k}\left((1-kx)^{n-1} - \binom{n-k}{1}(1-(k+1)x)^{n-1} + \cdots + (-1)^s\binom{n-k}{s}(1-(k+s)x)^{n-1}\right), \qquad (1)$$

where the series on the right-hand side stops at the last term for which $1-(k+s)x$ is positive, and $k+s \leq n$.

This result has numerous statistical applications. One of these is a statistical test of significance (see Fisher (1929), (1940)) in harmonic (time-series) analysis based on $1 - \mathrm{P}_{[0]} = \mathrm{P}(\max_i W_i > x)$. Baticle (1933), Garwood (1940), and Kendall and Moran (1963, p. 33) described other applications. The initial importance of this result is attested to by several tabulations of the significance point x for specified significance values $\mathrm{P}(\max_i W_i > x)$, and for significance points for the second-largest W_i (see Fisher (1929), (1940) and Davis (1941)).

The motivation for the present paper is the following cautious statement by Kendall and Moran (1963, p. 32).

> In a simplified form this problem was originally given by Whitworth (1901, Problem 667, p. 361) but is possibly earlier in origin.

In fact, the 5th edition of Whitworth's book (1901) has been reprinted as Whitworth (1965), where the relevant problems (numbers 666 and 667) are on p. 331. These are stated as follows.

Problem 666. A line of length c is divided into n segments by $n-1$ random points. Find the chance that no segment is less than a given length a, where $c > na$ (say $c - na = ma$).

Problem 667. In the last question find the chance that r of the segments shall be less than a and $n-r$ greater than a.

These problems have the same number in the 4th edition (Whitworth (1886)), after which Whitworth (1897) published solutions, given on pp. 195–197. These depend on first considering the rod divided into N segments of equal length, using heavily combinatorial arguments and letting $N \to \infty$. Garwood (1940) in an Appendix (pp. 73–77) (from where almost all of the citations on the broken-rod problem given by Kendall and Moran (1963, p. 33) are originally given), first followed Baticle's (1933) combinatorial method to solve (p. 74) Problem 1; and then gave his own proof. This latter proof is essentially that of Jordan (1872), following on from that of Halphen (1872) in the first volume of *Bulletin de la Société Mathématique de France.*

The problem of the broken rod (Problem 1) and a preceding simpler problem (Problem 2, below), are thus intimately connected with the founding in 1872 of the *Société Mathématique de France* and with one of its founding Council members Irenée Jules Bienaymé (1796–1878), who commented on another aspect of Jordan's (1872, p. 281) work. In the French publications on the problem, a significant role was played by Ernesto Cesàro (1859–1906), an Italian student of Eugène Catalan in Liège, Belgium. We concentrate on Cesàro's work.

Neither Lévy (1939) nor Baticle (1933) mentioned earlier writers, although both published in *Comptes Rendus de l'Académie des Sciences Paris;* nor did Whitworth. To be fair, Lévy was more concerned with asymptotic theory, and cited Gumbel (1937) in this context.

Problem 2. A rod of length 1 is broken into n pieces at $n-1$ randomly chosen points. What is the probability that the n segments can form a closed convex polygon? (Halphen did not actually use the word 'convex', which is implicit.)

To form such a polygon it is necessary and sufficient for each segment's length to be less than half the length of the rod (Halphen (1872, p. 222)). The probability sought is therefore

$$\begin{aligned}\mathrm{P}\big(W_1 < \tfrac{1}{2}, W_2 < \tfrac{1}{2}, \ldots, W_{n-1} < \tfrac{1}{2}, W_n < \tfrac{1}{2}\big) \qquad (2)\\ = 1 - \mathrm{P}\Big(\bigcup_{i=1}^{n}\{W_i \geq \tfrac{1}{2}\}\Big)\\ = 1 - \mathrm{P}\Big(\bigcup_{i=1}^{n}\{W_i > \tfrac{1}{2}\}\Big)\\ = 1 - \sum_{i=1}^{n}\mathrm{P}\{W_i > \tfrac{1}{2}\},\end{aligned}$$

since the events $\{W_i > \frac{1}{2}\}$, $i = 1, \ldots, n$, are mutually exclusive and the random variables W_i, $i = 1, \ldots, n$, are identically distributed. We now obtain the result

$$\mathrm{P}\big(W_1 < \tfrac{1}{2}, W_2 < \tfrac{1}{2}, \ldots, W_{n-1} < \tfrac{1}{2}, W_n < \tfrac{1}{2}\big) = 1 - n\,\mathrm{P}\{W_1 > \tfrac{1}{2}\}. \qquad (3)$$

Although $\mathrm{P}\{W_1 > \frac{1}{2}\}$ can be evaluated directly as $1/2^{n-1}$ (see Halphen (1872, pp. 222–223)), (2) can be calculated by evaluating $\mathrm{P}_{[k]}$ given by (1), with $k = 0$ and $x = \frac{1}{2}$.

Problem 2 is thus subsumed by Problem 1. It is clear that in dealing with the prehistory of Problems 1 and 2, we are dealing with part of the very early history of geometric probability. In this sense, this paper is a companion to Seneta *et al.* (2001) about Buffon's needle.

2. A modern solution to Problem 1

As a focus for the subsequent discussion, we outline a modern (and complete) solution to Problem 1. We begin, as in Kendall and Moran (1963, p. 32), with the now well-known identity

$$\mathrm{P}_{[k]} = S_k - \binom{k+1}{1}S_{k+1} + \binom{k+2}{2}S_{k+2} + \cdots + (-1)^{n-k}\binom{n}{n-k}S_n, \qquad 0 \leq k \leq n, \quad (4)$$

which is most easily obtained using indicator functions (Moran (1968, p. 27)) and the notion of expectation. Here $S_0 = 1$ and, for $1 \leq h \leq n-1$, we have

$$S_h = \sum_{1 \leq i_1 < i_2 < \cdots < i_h \leq n} \mathrm{P}(A_{i_1} A_{i_2} \cdots A_{i_h}),$$

where A_i, $i = 1, \ldots, n$, are arbitrary events in some probability space and $\mathrm{P}_{[k]}$ is the probability that precisely k of these events occur.

In our specific problem, $A_i = \{W_i > x\}$, $i = 1, \ldots, n$, and, clearly,

$$S_r = \binom{n}{r}\mathrm{P}(W_1 > x, W_2 > x, \ldots, W_r > x), \qquad r = 1, \ldots, n. \qquad (5)$$

Now, from the theory of order statistics, it is well known that $Y_1, Y_2, \ldots, Y_n$, $1 \le r \le n-1$, has a joint probability density function (PDF)

$$g(y_1, y_2, \ldots, y_r) = \frac{(n-1)!}{(n-1-r)!}(1-y_r)^{n-1-r}, \qquad 0 < y_1 < y_2 < \cdots < y_r < 1. \tag{6}$$

Since $Y_i = W_1 + W_2 + \cdots + W_i$, $i = 1, \ldots, r$, where $r \le n-1$, the joint PDF of $W_1, \ldots, W_r$ is

$$h(w_1, \ldots, w_r) = \frac{(n-1)!}{(n-1-r)!}(1 - w_1 - w_2 - \cdots - w_r)^{n-1-r},$$

which is positive in (the simplex) $\{w_i,\ i = 1, \ldots, r;\ w_i > 0,\ i = 1, \ldots, r;\ \sum_{i=1}^{r} w_i < 1\}$. Thus, for $1 \le r \le n-1$,

$$\begin{aligned}
&\mathrm{P}(W_1 > x, W_2 > x, \ldots, W_r > x) \\
&\quad = \int_x^{1-(r-1)x} \\
&\quad \times \cdots \times \left(\int_x^{1-w_1\cdots-w_{r-2}-x} \times \left(\int_x^{1-w_1-\cdots-w_{r-1}} h(w_1, \ldots, w_r)\, \mathrm{d}w_r \right) \mathrm{d}w_{r-1} \right) \\
&\quad \times \cdots \times \mathrm{d}w_1,
\end{aligned}$$

provided that $rx < 1$. The r successive repeated integrals can be easily evaluated to give the final answer

$$\mathrm{P}(W_1 > x, W_2 > x, \ldots, W_r > x) = (1 - rx)^{n-1}. \tag{7}$$

The result also holds in the case $r = n$ if $nx < 1$, but requires a separate multiple integration based on $(y_1, \ldots, y_{n-1})$ in the PDF (6) with $r = n-1$, using $\mathrm{P}(W_1 > x, W_2 > x, \ldots, W_n > x) = \mathrm{P}(Y_1 > x, Y_2 - Y_1 > x, \ldots, Y_{n-1} - Y_{n-2} > x, Y_{n-1} < 1 - x)$.

If $rx \ge 1$, $r = 1, 2, \ldots, n$, then

$$\mathrm{P}(W_1 > x, W_2 > x, \ldots, W_r > x) = 0,$$

since $W_r \le 1 - \sum_{i=1}^{r-1} W_i$, and

$$\{W_i > x,\ i = 1, \ldots, r-1\} \Longrightarrow W_r \le 1 - \sum_{i=1}^{r-1} W_i < 1 - (r-1)x < x.$$

Combining the result for $r = n$ with (4), (5), and (7), leads to the result (1) after noting that

$$\binom{n}{k+j}\binom{k+j}{j} = \binom{n}{k}\binom{n-k}{j}. \tag{8}$$

Fisher's (1929), (1940) characteristically geometrical proof of (7) is condensed by Kendall and Moran (1963, p. 32). It is similar in concept to the geometric treatment of Problem 2 in the case $n = 3$ by Lalanne (1878) in its latter part (pp. 357–358), as summarized by Mansion (1878).

3. The prehistory

By the prehistory of Problems 1 and 2 we shall mean the writings preceding the citations in Kendall and Moran (1963) which begin with Whitworth in 1901 (reprinted as Whitworth (1965)).

The beginning of this prehistory seems to be the paper of Lemoine (1872) who proposed Problem 2 in the case $n = 3$, when the polygon is a triangle. He considered the rod divided conceptually into $2m$ equal segments and, taking the limit as $m \to \infty$, found that the probability sought is $\frac{1}{4}$. This approach by discrete approximation was used later by Whitworth (1897) for Problem 1. At about the same time, and in virtually the same manner as Problem 2 for $n = 3$, the same problem was proposed and solved in Laurent (1873, pp. 62–63). Laurent's treatment was not noticed until a footnote by Catalan on the problem in Mansion (1878, p. 386) drew attention to it.

Following Lemoine (1872), Halphen (1872) solved Problem 2 for general n in two ways. The first way is given essentially by evaluating $\mathrm{P}\{W_1 > \frac{1}{2}\}$ in (3). In our notation,

$$\begin{aligned}\mathrm{P}\{W_1 > \tfrac{1}{2}\} &= \mathrm{P}\{Y_1 > \tfrac{1}{2}\} \\ &= \mathrm{P}\{X_1 > \tfrac{1}{2}, X_2 > \tfrac{1}{2}, \ldots, X_{n-1} > \tfrac{1}{2}\} \\ &= \left(\mathrm{P}\{X_1 > \tfrac{1}{2}\}\right)^{n-1} \\ &= \left(\tfrac{1}{2}\right)^{n-1},\end{aligned}$$

since the X_i are independently and identically distributed on $(0, 1)$. Halphen's second evaluation proceeded along the lines of the multiple integral in our Section 2. Halphen (1872) effectively found the joint PDF of $W_1, W_2, \ldots, W_{n-1}$ to be

$$h(w_1, \ldots, w_{n-1}) = (n-1)!$$

over the simplex $\{w_i,\ i = 1, \ldots, n-1;\ w_i > 0,\ i = 1, \ldots, n-1;\ \sum_{i=1}^{n-1} w_i < 1\}$, from which he evaluated $\mathrm{P}(W_1 > \frac{1}{2})$ by integrating out $w_2, \ldots, w_{n-1}$, to find the marginal PDF of W_1.

Jordan (1872), still in the first volume of *Bulletin de la Société Mathématique de France*, proposed Problem 1 for the first time. He solved it by evaluating (5) using (7) and (6). Jordan (1872) cited and generalized the first evaluation of $\mathrm{P}(W_1 > \frac{1}{2})$ given by Halphen (1872) in obtaining (7). He then derived (4) from an expression for the probability that *at least* k of the events $\{A_i\}$, $i = 1, \ldots, n$, occur, a result which he ascribed to his own paper in *Comptes Rendus de l'Académie des Sciences Paris* (9 décembre 1867), a surprisingly early reference to such combinatorial expressions. All this is stated in about one page of Jordan (1872), after which he goes on to a second similar problem on the sphere, in relation to which Bienaymé intervenes. In any case, we may consider Problem 1 to have been solved, in principle, by Jordan (1872), although the explicit expression (involving the change in number of summands, depending on k) is avoided.

The next episode in the story is an extract by Lalanne (1878) in *Comptes Rendus de l'Académie des Sciences Paris,* confined to the statutory three printed pages, from a larger and otherwise unpublished paper. In the second part of this extract (pp. 357–358), Lalanne cited Lemoine (1872) and gave an elegant geometric solution to Problem 2 for $n = 3$, in contrast to Lemoine's more combinatorial solution. This part of Lalanne's extract excited the interest of Mansion (1878), who reproduced it in the *Nouvelle Correspondance Mathématique*, edited by Catalan. A footnote by Catalan himself (see Mansion (1878, p. 386)) added that Lalanne had presented his method at a Congress in Paris, and that Laurent (1873) had found another proof. These comments by Mansion and Catalan elicited a Letter to the Editor (Catalan) of the *Nouvelle Correspondance* from Lemoine (1879), who announced that Halphen's generalization of his original paper stimulated him (Lemoine) to communicate a paper to the Havre Congress (of the *Association Française pour l'Avancement des Sciences*) in 1877, on the day on which Catalan presided over the assembly. (The general president of the mathematical section was J. J. Sylvester.) Lemoine wrote further that '...je compte rédiger, au premier moment de loisir, et publier ce petit travail'. Indeed, Lemoine published such a paper (Lemoine (1882)) a few years later. We shall return to it shortly.

Lemoine's letter to the *Nouvelle Correspondance* attracted the attention of Cesàro, then at the very beginning of his research career.

Ernesto Cesàro had been in Liège, Belgium, since 1873, where he had rejoined his older brother Giuseppe, and was studying at the École des Mines. He had been studying mathematics with Catalan, when he was forced to return to the family home at Torre Annunziata, near Naples, after the death of his father in 1879. Eventually he won a scholarship to allow him to continue his studies at Liège and in 1882 he returned to Belgium. However, by May 1883 he had had a personal dispute with a Professor Deschamps in Liège, and by June was back at Torre Annunziata, in spite of Catalan's advice. Catalan, as well as being Cesàro's mentor and guide in mathematics, advised him on his numerous personal problems, his health, and his financial difficulties. Hoping to continue his work in Italy, Cesàro obtained a scholarship at the University of Rome, which he entered in 1884. The scholarship was for the continuation of his 'études d'ingénieur', for, although he had already published a number of research papers, he had not yet obtained any diploma. We note that in Cesàro (1882) he is described as 'élève-ingénieur de l'École des Mines (Liège)', and in Cesàro (1883) as 'élève ingénieur, à Rome'. This particularly turbulent period in Cesáro's life is well reflected in Catalan's letters A12–A27 to him (Butzer *et al.* (1999)). A biography by Novỳ and Folta (1971) paid almost no attention to Cesàro's work in probability; and the biography by J. J. O'Connor and E. F. Robertson (see http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/) paid none at all. Both biographies refer to his first mathematical paper 'Sur diverses questions d'arithmétique' published (with Catalan's help) in 1882 in *Mémoires de la Société Royale des Sciences de Liège.* We note, for the record, his important but neglected later work (Cesàro (1891)) on continuous probability, now written in Italian. Moreover, while the selected works (*Opere Scelte*) of Cesàro (1964), (1965) contain an almost complete list of his mathematical works, including the probabilistic ones, none of the later work (Cesàro (1891)) is reprinted.

Cesàro's (1880) paper, presumably written in Italy, is proposed as a solution to Questions 370 and 371 in the *Nouvelle Correspondance Mathématique*. This is the original form of Question 371

Question 371. On casse, au hasard, en trois morceaux, une barre de longueur l. Quelle est la probabilité P que deux de ces trois morceaux soient plus petits qu'une longueur donnée a? (E. Lemoine)

This is, in fact, not Lemoine's version (with $n = 3$) of Problem 2, but is a case (with $n = 3$) of Problem 1. Cesàro seems to put forward a unified solution to Questions 371 and 370, possibly taking these questions as equivalent. Question 370 says that, given a rod AB of length l, choose a point C at random between A and B; and then choose, at random, a point D between C and B. What is the probability P that two of the lengths AC, CD, and DB will be smaller than a given length a? It appears that (regarding Question 371) Cesàro interpreted the question as meaning *at least* two of the three pieces being smaller than a given length a. The complementary event whose probability may be sought is the event that at most one piece is of length greater than or equal to a. Cesàro's (1880, p. 77) answer (to Question 371) for P is

$$P = \begin{cases} 6x^2, & 0 \le x \le \frac{1}{3}, \\ 2(3x^2 - (3x-1)^2), & \frac{1}{3} \le x \le \frac{1}{2}, \\ 1, & \frac{1}{2} \le x \le 1, \end{cases}$$

where $x = a/l$. This answer is correct, as is seen by calculating $P_{[0]} + P_{[1]}$ from (1) with $n = 3$. Cesàro's method consists of dividing the rod into equal segments of length ε, and letting $\varepsilon \to 0+$, while treating the three cases above separately and 'algebraically'.

The two problems given by Questions 370 and 371 are not equivalent, and this is possibly why, on p. 77 of Cesàro (1880), there is a cautionary footnote by Catalan, who anticipates objections and critical comments.

> Cette solution donnera lieu, peut-être, à diverses remarques. En les attendant, nous rappelons que *les Auteurs sont responsables de leurs articles.*

Cesàro (1882) treated precisely the same problem as Lemoine (Question 371), essentially Problem 1 with $n = 3$, by geometric methods similar to Lalanne's (1878) geometric treatment of Problem 2. He also gave Lemoine credit for posing Question 371; and for alluding to Problem 2 in Lemoine (1879).

Cesàro's (1883) next article first tackled Problem 1 in the case $n = 3$, asking for the probability that at least p pieces are of length less than or equal to a, when $p = 1$ and $p = 3$, the case $p = 2$ having been solved earlier in Cesàro (1880) by algebraic means. Geometric methods are used by Cesàro (1883) for the problem with $n = 4$ pieces, but then for arbitrary n the methodology is again algebraic. We consider his final expression for at most p pieces being of length less than x in Section 4. Here it suffices to say that Cesàro's (1883) account is very incomplete. (This account is not only incomplete, but with a serious misprint at least on p. 234 (the case $p = 3$). In a letter (B1) to Catalan of 8 October 1880 (Butzer *et al.* (1999)), Cesàro already laments 'Je m'étonne que mes notes soient toujours remplies de fautes typographiques'.) The reason that Cesàro's account is incomplete is given in a footnote by Paul Mansion, editor of *Mathesis* (see Cesàro (1883, p. 233)) as follows.

> Cet article et le précédent (*Mathesis*, t.II, p. 177–179) sont extraits d'un mémoire inédit où M. Cesàro se sert, en la simplifiant, d'une méthode employée par MM. Lemoine et Lalanne et, sans doute, par d'autres, pour étudier des questions de calcul des probabilités. Voir un nouvel article de M. Lemoine sur des questions de ce genre, traitées de cette manière, dans le *Bulletin de la Société mathématique de France*, t. XI, p. 13–25. (P.M.).

And so the reader is advised to turn for revelation to Lemoine (1882), the upstart novice Cesàro (1882), (1883) being rather shabbily dismissed. Lemoine, in a tone designed to reclaim all priority, began as follows.

> J'ai donné à la Société mathématique, l'année de sa fondation, un petit problème de probabilités qui a eu ensuite le grand honneur d'attirer l'attention de plusieurs mathématiciens. MM. Halphen et Jordan en ont ici même, successivement, donné des généralisations; divers géomètres s'en sont occupés dans la *Correspondance mathématique* de M. Catalan, dans les *Nouvelles Annales,* dans le Journal *Mathesis* et enfin M. Lalanne dans les *Comptes rendus de l'Académie des Sciences.* C'est après la lecture de ce dernier travail que j'ai eu l' idée des développements qui vont suivre. L'énoncé du problème primitif était celui-ci: *On casse au hasard une barre en trois morceaux, quelle est la probabilité pour que l'on puisse former un triangle avec ces trois morceaux?*

That is, the primary problem is Problem 2 with $n = 3$. Lemoine went on to consider variants of this triangle problem, where the triangle is subject to certain constraints, for example, with acute angles only, using Lalanne's geometric approach; but Lemoine does not go beyond the case $n = 3$. The last sentence of the paper is tentative.

> Je suppose qu'en employant les conventions de la Géometrie à n dimensions, on étendrait cette méthode à un nombre de variables autre que 2 et 3.

However, Cesàro is already proceeding boldly, though algebraically, to n dimensions.

We suppose that Cesàro (1880) in the above description is ungraciously relegated to one of the *divers géomètres*. In fact (as we shall see in Section 4), in spite of Lemoine's posturing and Mansion's deference to Lemoine, there is a strong case for crediting the young, inexperienced and enthusiastic Cesàro, then at the beginning of his career, with the complete solution to Problem 1.

What of the aftermath? The notable historian of probability of the times was Emanuel Czuber (1851–1925). In his book on geometric probability, Czuber (1884) alluded to the works of C. Jordan, É. Lemoine and L. Lalanne, but clearly had no precise knowledge of Jordan's paper. The part of Czuber's book devoted to the broken rod problem is limited to the case $n = 3$, with details on Lemoine and Lalanne, but is silent on Halphen and Cesàro, omitting, in essence, all work on $n > 3$. His well-known survey (Czuber (1899)) of probability theory contained no citations to Cesàro. However, an immediate follow-up by Wölffing (1899, p. 79) listed the following items, giving almost exactly the sequence of papers devoted by Cesàro to geometric probability: Cesàro (1880), (1882), (1883), (1886), (1891).

Czuber's later monograph (Czuber (1928)), whose first edition appeared in 1903, only cited Cesàro (1891). It is not difficult to see how Cesàro's contribution to the problem of the broken rod, as indeed those of his predecessors, might easily be overlooked by authors writing in English. Indeed, almost the same fate befell the geometric probability work of J.-É. Barbier (Seneta *et al.* (2001)). The historical sketch, in Russian, of Peres Larigno (1985), though deferential to the contributors to geometric probablity discussed in Seneta *et al.* (2001) and even aware of Whitworth's contributions, did not mention the history of the broken rod problem.

We should mention that Cesàro also provided a complete solution to the *problem of the broken diamond,* published in Italian, in 1887. This problem also has a related and involved history, beginning with Laurent (1873). For the sake of brevity, we omit discussion of it here, as well as the solution by Cesàro (1886) of the *problem of the broken tree*.

4. Cesàro's expression

Cesàro (1883, p. 235) stated the expression (in his notation)

$$1 - Y_{r,p} = C_{n,p} \sum_{i=1}^{i=p-r+1} \pm \frac{i}{n-p+i} C_{p,i}[1-(n-p+i)X]^{n-1}, \tag{9}$$

for $r \le p$, and if $r > p$ the right-hand side is to be taken as 0. Cesàro did not say what $Y_{r,p}$ means; it is the probability $\mathrm{P}(V \ge p)$ where V is the number of segments out of the n whose length is less than X, where $X < 1/(n-r+1)$, $r = 0, 1, \ldots, p$. Lack of detail in this work is due to its abridgement by the editors P. Mansion and J. Neuberg (see our Section 3). In modern notation, $C_{p,i} = \binom{p}{i}$.

The validity of (9), starting from (1), where we shall replace x by X, is not straightforward to establish, and we feel it necessary for the record to give a verification, resulting in (12), below. Let W be the number of segments whose length is X.

From (4), for $0 \le p \le n$, we obtain

$$P(W \ge p) = \sum_{k=p}^{n}\sum_{h=0}^{n-k}(-1)^h\binom{k+h}{h}S_{k+h} = \sum_{i=p}^{n}\sum_{h=0}^{i-p}(-1)^h\binom{i}{h}S_i.$$

Now,

$$\sum_{h=0}^{t}(-1)^h\binom{i}{h} = 1, \quad \text{for } i = 0, t = 0,$$
$$= (-1)^t\binom{i}{t}\frac{i-t}{i}, \quad \text{for } 1 \le t \le i.$$

Using this, with $t = i - p$ for (9), we obtain

$$P(W \ge p) = \sum_{i=p}^{n}(-1)^{i-p}\frac{p}{i}\binom{i}{i-p}S_i,$$

interpreting this appropriately when $p = 0$. We now proceed, assuming that $p \ge 1$. We obtain, substituting $i - p + 1 = v$,

$$\begin{aligned} P(W \ge p) &= \sum_{v=1}^{n-p+1}(-1)^{v-1}\frac{p}{v+p-1}\binom{v+p-1}{v-1}S_{v+p-1}, \\ &= \sum_{v=1}^{n-p+1}(-1)^{v-1}\frac{p}{v-p+1}\binom{v+p-1}{v-1}\binom{n}{v+p-1}(1-(v+p-1))^{n-1} \end{aligned}$$

from (5) and (7). Thus, using (8),

$$P(W \ge p) = \binom{n}{p}\sum_{v=1}^{n-p+1}(-1)^{v-1}\frac{p}{v-p+1}\binom{n-p}{v-1}(1-(v+p-1)X)^{n-1}. \tag{10}$$

Then (10) can be written as

$$P(W \ge p) = \binom{n}{p}\sum_{v=1}^{r-p+1}(-1)^{v-1}\frac{p}{v-p+1}\binom{n-p}{v-1}(1-(v+p-1)X)^{n-1}, \tag{11}$$

for $n \ge r \ge p \ge 1$ and $X < 1/r$.

We proceed to show that (11) says the same as (9). In (11), set $j = v$, $s = n - r + 1$, $q = n - p + 1$ (so $q \ge s \ge 1$), and $V = n - W$ (the number of pieces of length less than or equal to X). Then (11) becomes

$$P(n-V \ge n-q+1) = \binom{n}{n-q+1}\sum_{j=1}^{q-s+1}(-1)^{j-1}\frac{n-q+1}{n-q+j}\binom{q-1}{j-1}(1-(n-q+j)X)^{n-1}.$$

Now,

$$\binom{n}{q-1}\binom{q-1}{j-1}(n-q+1) = \binom{n}{q}\binom{q}{j}j,$$

so that

$$P(V \leq q-1) = \binom{n}{q} \sum_{j=1}^{q-s+1} (-1)^{j-1} \frac{j}{n-q+j} \binom{q}{j} (1-(n-q+j)X)^{n-1}, \qquad (12)$$

which is (9) if we replace q by p, s by r, j by i, and $(-1)^{j-1}$ by $\pm$, in (12), noting moreover that $P(V \leq p-1) = 1 - P(V \geq p)$.

References

BATICLE, E. (1933). Le problème de la répartition. *C. R. Acad. Sci. Paris* **196,** 1945–1946, **197,** 632–634.

BUTZER, P. L., CARBONE, L., JONGMANS, F. AND PALLADINO, F. (1999). Les relations épistolaires entre Eugène Catalan et Ernesto Cesàro. *Acad. R. Belg. Bull. Cl. Sci. (6)* **10,** 223–271.

CESÀRO, E. (1880). Solution de Question 371, d'É. Lemoine. *Nouvelle Correspondance Math.* **6,** 74–77.

CESÀRO, E. (1882). Une question de probabilités. *Mathesis* **2,** 177–179.

CESÀRO, E. (1883). Remarques sur une question de probabilités. *Mathesis* **3,** 233–237.

CESÀRO, E. (1886). Question 233 (Artemas Martin). *Mathesis* **6,** 14–16.

CESÀRO, E. (1891). Considerazioni sul concetto di probabilità. *Periodico Mate. Inseg. Secondario* **6,** 1–13, 49–62.

CESÀRO, E. (1964). *Opere Scelte,* Vol. 1, *Algebra, Serie, Teoria dei Numeri,* Parte Prima. Edizioni Cremonese, Rome.

CESÀRO, E. (1965). *Opere Scelte,* Vol. 1, *Algebra, Serie, Teoria dei Numeri,* Parte Seconda. Edizioni Cremonese, Rome.

CZUBER, E. (1884). *Geometrische Wahrscheinlichkeiten und Mittelwerte.* Teubner, Leipzig.

CZUBER, E. (1899). Die Entwicklung der Wahrscheinlichkeitstheorie und ihrer Anwendungen. *J. Deutschen Math. Vere.* **7,** 1–279.

CZUBER, E. (1928). *Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung,* Erster Band, *Wahrscheinlichkeitstheorie - Fehlerausgleichung - Kollektivmasslehre.* Teubner, Leipzig.

DAVIS, H. T. (1941). *The Analysis of Economic Time Series.* Principia, Bloomington, IN.

FISHER, R. A. (1929). Tests of significance in harmonic analysis. *Proc. R. Soc. London A* **125,** 54–59.

FISHER, R. A. (1940). On the similarity of the distributions found for the test of significance in harmonic analysis, and in Stevens' problem in geometrical probability. *Ann. Eugenics* **10,** 14–17.

GARWOOD, F. (1940). An application of the theory of probability to the operation of vehicular-controlled traffic signals. *J. R. Statist. Soc. Suppl.* **7,** 65–77.

GUMBEL, E. (1937). Les intervalles extrêmes entre les émissions radioactives. *J. Physique Paris* **8,** 321–329, 346–452.

HALPHEN, G. (1872). Sur un problème de probabilités. *Bull. Soc. Math. France* **1,** 221–224.

JORDAN, C. (1872). Questions de probabilités. *Bull. Soc. Math. France* **1,** 256–258, 281.

KENDALL, M. G. AND MORAN, P. A. P. (1963). *Geometrical Probability.* Griffin, London.

LALANNE, L. (1878). De l'emploi de la géométrie pour résoudre certaines questions de moyennes et de probabilités. *C. R. Acad. Sci. Paris* **77,** 355–358.

LAURENT, H. (1873). *Traité du Calcul des Probabilités.* Gauthier-Villars, Paris.

LEMOINE, É. (1872). Sur une question de probabilités. *Bull. Soc. Math. France* **1,** 39–40.

LEMOINE, É. (1879). Extrait d'une lettre de M. É. Lemoine. *Nouvelle Correspondance Math.* **5,** 103.

LEMOINE, É. (1882). Quelques questions de probabilités résolues géométriquement. *Bull. Soc. Math. France* **11,** 13–24.

LÉVY, P. (1939). Sur la division d'un segment par des points choisis au hasard. *C. R. Acad. Sci. Paris* **208,** 147–149.

MANSION, P. (1878). Une question de probabilités. *Nouvelle Correspondance Mathématique* **4,** 385–386.

MORAN, P. A. P. (1968). *An Introduction to Probability Theory.* Clarendon Press, Oxford.

NOVÝ, L. AND FOLTA, J. (1971). Cesàro, Ernesto. In *Dictionary of Scientific Biography,* Vol. 3, ed. C. C. Gillispie, Scribner, New York, pp. 177–179.

PERES LARIGNO, M. T. (1985). On the history of the notion of geometric probability. *Voprosy Istor. Estestvoznan. Tekhn.* **4,** 100–103 (in Russian).

SENETA, E., PARSHALL, K. H. AND JONGMANS, F. (2001). Nineteenth-century developments in geometric probability: J. J. Sylvester, M. W. Crofton, J.-É. Barbier and J. Bertrand. *Arch. Hist. Exact Sci.* **55,** 501–524.

STEVENS, W. L. (1939). Solution to a geometrical problem in probability. *Ann. Eugenics London* **9,** 315–320.

WHITWORTH, W. L. (1886). *Choice and Chance,* 4th edn. Deighton Bell, Cambridge.

WHITWORTH, W. L. (1897). *DCC Exercises (One Thousand and Two Hundred Exercises).* Deighton Bell, Cambridge.

WHITWORTH, W. L. (1965). *Choice and Chance. With One Thousand Exercises.* Hafner, New York.

WÖLFFING, E. (1899). Erganzung des von E. Czuber in seinem Referat über Wahrscheinlichkeitsrechnung gegebenen Literaturverzeichnisses. *Math. Naturwiss. Verein. Württemberg* **3B,** 93–95.