$J = \{s, u\}$

A3

tust

tut   tus   ust   sts

ut   tu   us   ts   st

u   s   t

e

A4   s t u v

stuv

tuv   uvs   vst   stu

uv   vt   tu   vs   us   st

v   u   t   s

e

G simply connected s.s algebraic group over an alg closed field k. $B \supset T$ Borel and max'l torus. $W = N(T)/T$ be the Weyl group of G.

Bruhat decomposition

$$G = \bigcup_{w \in W} BwB$$

Let B^- be the opposite Borel subgroup to B.

e.g. $G = GL_2(\mathbb{C})$.

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $B^- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, $T = B \cap B^-$.

$N_G(T) = \{ g \in G \mid gTg^{-1} = T, \text{ ie } gT = Tg \}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{\text{fixed.}} = \begin{pmatrix} \lambda_1 a & \lambda_2 b \\ \lambda_1 c & \lambda_2 d \end{pmatrix} = [\lambda_1 C_1, \lambda_2 C_2]$$

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 a & 0 \\ 0 & \lambda_2 d \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 0 & d \\ a & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & d\lambda_2 \\ a\lambda_1 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\det = 1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & d \\ a & 0 \end{pmatrix}^{-1} = \frac{1}{-ad}\begin{bmatrix} 0 & -d \\ -a & 0 \end{bmatrix} = \frac{1}{ad}\begin{bmatrix} 0 & d \\ a & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{a} \\ \frac{1}{d} & 0 \end{bmatrix}$$

$$\begin{pmatrix} 0 & d \\ a & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \frac{1}{ad}\begin{bmatrix} 0 & d \\ a & 0 \end{bmatrix} = \frac{1}{ad}\begin{pmatrix} 0 & \lambda_2 d \\ \lambda_1 a & 0 \end{pmatrix}\begin{pmatrix} 0 & d \\ a & 0 \end{pmatrix} = \frac{1}{ad}\begin{pmatrix} \lambda_2 ad & 0 \\ 0 & \lambda_1 ad \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} \lambda_1 a & \lambda_2 b \\ \lambda_1 c & \lambda_2 d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in T$$

I need 2 conditions

1) $-\lambda_1 ab + \lambda_2 ab = 0$.
$\therefore \lambda_1 ab = \lambda_2 ab$

2) $\lambda_1 cd = \lambda_2 cd$.

$ab(\lambda_1 - \lambda_2) = 0$

$cd(\lambda_1 - \lambda_2) = 0$

If $\lambda_1 = \lambda_2$ the condition is empty (also we know it commutes with all group since $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \in Z$ "center".

Sup $\lambda_1 \neq \lambda_2 \Rightarrow ab = cd = 0$. $a, b \neq 0$.

Case 1 $b \neq 0 \Rightarrow a = 0$.
$\det = -bc \neq 0 \Rightarrow c \neq 0 \Rightarrow d = 0$. $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$

Case 2 $a \neq 0 \Rightarrow b = 0$.
$\det = ad \neq 0 \Rightarrow d \neq 0 \Rightarrow c = 0$. $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ $\in T$.

$$N(T) = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\} = \text{this is a subgroup}$$

$"S_1 \cdot (\mathbb{C}^\times)^2" \qquad (\mathbb{C}^\times)^2$

$N(T)/T \cong "S_2"$

Let w_0 be the max'l length element of W (finite Weyl group) then $\bar{B} := w_0 B w_0$.

GL_2, $B^- = sBs$.

$$= \begin{pmatrix} 0&1\\1&0 \end{pmatrix}\begin{pmatrix} a&b\\0&c \end{pmatrix}\begin{pmatrix} 0&1\\1&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&c\\a&b \end{pmatrix}\begin{pmatrix} 0&1\\1&0 \end{pmatrix} = \begin{pmatrix} c&0\\b&a \end{pmatrix} \in B^-$$

~~Lets~~ Thy:

$$GL_n(\mathbb{C}) = \underbrace{\begin{pmatrix} \times & & \times \\ & & \\ \times & & \times \end{pmatrix}}_{n} \Big\} n.$$

$T = \begin{pmatrix} \times & & 0 \\ & \times & \\ 0 & & \times \end{pmatrix}$, $B = \begin{pmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{pmatrix}$

$N_G(T) = \{ gT = Tg \}$ by sub-matrix

Fact. Conjugation preserves eigenvalues. $(Av = \lambda v)$

$(SAS^{-1})(Sv) = SAv = \lambda(Sv)$

$gT = Tg \Rightarrow gTg^{-1} = D$. T & D same eigenvalues.

then gTg^{-1} permutes the diagonal entries of T. We have a map

$$N_G(T) \longrightarrow S_n.$$

Lets fix $d = \begin{pmatrix} 1 & & 0 \\ & 2 \; 3 & \\ 0 & & \ddots \; n \end{pmatrix}$ has diff diag. entries. Should be enough!

$$gdg^{-1} = \underbrace{\begin{pmatrix} \sigma(1) & & 0 \\ & \sigma(2) & \\ 0 & & \ddots \; \sigma(n) \end{pmatrix}}_{W} = W.$$

$gd = Wg$. $\quad g = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \cdots & \end{pmatrix}$

$g = (a_{ij})$,

$$gd = \begin{pmatrix} a_{11} & 2a_{12} & 3a_{13} \\ a_{21} & 2a_{22} & 3 \\ a_{31} & 2a_{32} & 3 \\ a_{41} & \vdots & \vdots \\ & & 3 \end{pmatrix} = \begin{pmatrix} \sigma(1)a_{11} & \sigma(1)a_{12} & \cdots \\ \sigma(2)a_{21} & \sigma(2)a_{22} & \cdots \\ \sigma(3) & \cdots & \end{pmatrix}$$

~~a_{ij}~~ $= (a_{ij} \cdot j) \quad = (a_{ij}\,\sigma(i))$

$a_{ij}(j - \sigma(i)) = 0$. $\quad$ if $j \neq \sigma(i)$, $a_{ij} = 0$.

$a_{ij} \neq 0$ if $j = \sigma(i)$ $\quad a_{i,\sigma(i)} \neq 0$.

$\therefore$ g is a generalised permutation matrix.

GL_n, $N_G(T) = \left\{ \begin{pmatrix} a\, e_{\sigma(1)} \\ b\, e_{\sigma(2)} \\ \vdots \end{pmatrix} \right\}$.

$N_G(T)/_T \cong S_n$.

Let w_0 max'l in S_n.

$$w_0 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots 1 & 0 & 0 \\ 1 & \cdots & & 0 \end{pmatrix}$$

Lets see $w_0 B w_0 = B^- =$ lower Δ matrices.

$$\underbrace{\begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}\begin{pmatrix} a&b&c\\0&d&e\\0&0&f \end{pmatrix}}\begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&f\\0&d&e\\a&b&c \end{pmatrix}\begin{pmatrix} 0 & \\ & \text{column operations} \end{pmatrix} = \begin{pmatrix} f&0&0\\e&d&0\\c&b&a \end{pmatrix}$$

$w_0 B = w_0 \nabla = \Delta$

$w_0 B w_0 = \Delta w_0 = \Delta = B^-$.

Bruhat decomposition

$$G = GL_n = \bigsqcup_{w \in N(T)/_T = W} BwB = \bigsqcup_{\sigma \in S_n} B\sigma B.$$

i.e. $G = BWB$, $\quad BwB \in B \backslash^{G/B}$.

$$B \backslash^{G/B} \xrightarrow{\sim} G \backslash^{(G/B \times G/B)}$$

$$\{ B \cdot (g \cdot B) \} \qquad \cancel{\{ G \cdot (g_1 \cdot B, g_2 \cdot B) \}} \; ???$$

Understanding double cosets:

$H \backslash G / K = \{ HgK : g \in G \}$ partitions G.

Proof $f : G \longrightarrow G/K$, the pullback of the partition of the H-orbits of G/K is a partition of G //.

$A\times K \circlearrowright G$, $(h,k).g := hgk^{-1}$

with this action $\boxed{G/_{H\times K} \cong H\backslash G/K}$

Deodhar 85

On some geometric aspects of Bruhat orderings I

§1. Introduction

Let G be a simply connected semisimple algebraic group over an algebraically closed field Ω (i.e. $SL_n(\mathbb{C})$)

$G=SL_n(\mathbb{C})$ $n=2$

$=SL_2(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \middle| ad-bc=1\right\}$

$= \operatorname{Spec} \mathbb{C}[x,y,z,w]/(xz-yw-1)$

$W=N(T)/T$, $T=\left\{\begin{pmatrix} * & \\ & *\end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix}\right\}$

$\mathfrak{g} = \operatorname{Lie} G = \left\{\begin{pmatrix} a & b \\ c & -a\end{pmatrix}\right\}$ $\quad Ad: G \longrightarrow GL_{\mathfrak{g}}$ adjoint representation.

$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi(G,T)} \mathfrak{g}_\alpha$. $\quad \mathfrak{t}=\mathfrak{g}_0$.

The characters α are called roots of (G,T)

$\Phi(G,T) \subset X(T)$ finite subset.

by conjugation

$G \circlearrowright \mathfrak{g}$

hence $T \circlearrowright \mathfrak{g}$

$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x & y \\ z & -x\end{pmatrix}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix} = gg'g^{-1}$.

$\underbrace{\quad}_{\det=1}$ $\quad \underbrace{\quad}_{\in\mathfrak{g}}$ $\quad tr(AB)=tr(BA)$

$tr(ABA^{-1})=tr(B)$.

$tr=0$.

$T \circlearrowright \mathfrak{g}$

$\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix} a & b \\ c & -a\end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2}c & d\end{pmatrix}$

$X(T) = \mathbb{Z}\chi$, $\chi: T \longrightarrow \mathbb{C}^*$

$T_2 = T \cong \mathbb{C}^\times$. $\quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix} \longmapsto \lambda$

$\mathfrak{sl}_2 = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$

$ad_x(y) = [x,y]$

$Ad: G \longrightarrow Aut(\mathfrak{g})$

$g \longmapsto ad_g$

$ad_g(X) = gXg^{-1}$

the derivative of the map Ad is

$ad: \mathfrak{g} \longrightarrow Der(\mathfrak{g})$

$x \longmapsto ad_x = d(Ad)_e(x)$

$Der(\mathfrak{g}) = Lie(Aut(\mathfrak{g}))$ $\quad G \curvearrowright \mathfrak{g}$ ← vector space

$ad_x(y) = [x,y]$.

$\alpha\in X(T)$

$\mathfrak{g}_\alpha = \{x\in\mathfrak{g} \mid t\cdot x = \alpha(t)x\}$. ($x$: vector; $\cdot$: action; $t \in T$)

$\chi \in X(T)$. $\quad$ Let $\{e_1,\ldots,e_n\}$ base for $\mathfrak{g}$

$t\cdot x = txt^{-1}$.

In $\mathfrak{sl}_2$ $\quad h=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$, $e=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$, $f\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$

$\mathfrak{sl}_2 = span\{h,e,f\}$. $\quad \mathfrak{g}\cong\mathbb{C}^3 \cong \langle h\rangle \oplus \langle e\rangle \oplus \langle f\rangle$.

$t\cdot h = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} = h$. wt 0

$t\cdot e = \begin{pmatrix} 0 & t^2 \\ 0 & 0\end{pmatrix} = t^2 e$. wt $2\chi \in X(T)$

$t\cdot f = \begin{pmatrix} 0 & 0 \\ t^{-2} & 0\end{pmatrix} = t^{-2} f$. wt $-2\chi \in X(T)$

$\boxed{\begin{matrix} 2\chi = \alpha \\ -2\chi = -\alpha \end{matrix}}$

$t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & t^{-2}\end{pmatrix}$

$\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$ $\quad \alpha \neq 2,-2,0$

$\rightarrow \mathfrak{g}_\alpha = 0$.

$\mathfrak{g}_\alpha = \begin{pmatrix} 0 & * \\ 0 & 0\end{pmatrix}$ $\quad \mathfrak{g}_{-\alpha} = \begin{pmatrix} 0 & 0 \\ * & 0\end{pmatrix}$

$\mathfrak{g}_0 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda\end{pmatrix}$

Let $\lambda: \mathbb{G}_m \xrightarrow{\sim} T_2$ $\quad$ In general, is a cocharacter

$t \longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} = \operatorname{diag}(t, t^{-1})$

One parameter subgroup in SL_2

$\lambda \in X(T)^*$ e.g. $t \mapsto \operatorname{diag}(t, t^{-1})$.

Define

$U^+ := U(\lambda) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ $\operatorname{Lie} U^+ = \mathfrak{g}_\alpha$.

$U^- := U(-\lambda) = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$, $\operatorname{Lie} = \mathfrak{g}_{-\alpha}$

$u_\alpha : \mathbb{G}_a \to U^+$

$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is an isomorphism of group schemes

with the remarkable property:

$t_0 \cdot u_\alpha(a) \cdot t_0^{-1} = u_\alpha(\alpha(t_0) a)$

$\forall t_0 \in T(R)$, $a \in \mathbb{G}_a(R)$, $R = \mathbb{C}$-alg.

$$\underbrace{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}}_{t_0} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t^2 a \\ 0 & 1 \end{pmatrix} = u_\alpha(\alpha(t_0) a) = u_\alpha(t^2 a)$$

$\alpha(t_0) = t^2$

Taking $u_\alpha : \mathbb{G}_a \to U^-$

$t_0 \cdot u_{-\alpha}(a) \cdot t_0^{-1} = u_{-\alpha}((-\alpha)(t_0)(a))$

Two Borel subgr contain T, $B^+ = U^+ \cdot T$

$B^- = U^- \cdot T$.

Let $n_\alpha = u_\alpha(1)\, u_{-\alpha}(-1)\, u_\alpha(1)$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = S.$$

$N_G(T) = T_2 \sqcup T_2 n_\alpha$ $\quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$

$= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \sqcup \left\{ \begin{pmatrix} 0 & a^{-1} \\ -a & 0 \end{pmatrix} \right\}$.

$S = n_\alpha$ $\quad$ S permutes rows. $\quad S(B^+) = B^-$

$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} S^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ $\quad S(B^-) = B^+$

$N_G(T)/T = \{1, S\} =: W(G, T)$ Weyl group.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin SL_2$ $\quad \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$.

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \ & \ \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ works and has $\det = 1$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let $\alpha^\vee$ $\quad t \mapsto \operatorname{diag}(t, t^{-1})$ of $X^*(T)$

$\langle \alpha, \alpha^\vee \rangle = 2$

$$\mathbb{C}^\times \xrightarrow{\alpha^\vee} T \xrightarrow[\alpha]{2\chi} \mathbb{C}^\times$$

$$t \longmapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto t^2.$$

$\langle -\alpha, -\alpha^\vee \rangle = 2$

$s(\chi) = s_\alpha(\chi) := \chi - \langle \chi, \alpha^\vee \rangle \alpha$

$s(\alpha) = -\alpha$, $\quad s(-\alpha) = \alpha$.

Can be checked directly

$$\begin{cases} u_\alpha(t)\, u_{-\alpha}(-t^{-1})\, u_\alpha(t) = \alpha^\vee(t)\, n_\alpha \\ n_\alpha^2 = \alpha^\vee(-1) \end{cases}$$

$n_{-\alpha} := u_{-\alpha}(1)\, u_\alpha(1)\, u_{-\alpha}(1)$

Can check $n_{-\alpha} = n_\alpha^{-1}$. For each $u \in U^+ \setminus \{e\}$

$\exists!\, u' \in U^- \setminus \{e\}$ st $u u' u \in N_{SL_2}(T_2)$.

Ex $\quad SL_2 = T_2 U^+ \cup U^+ n_\alpha T_2 U^+$

in particular SL_2 is gen by T_2, U^+, n_α.

Root datum of split reductive group

X free f.rk $\mathbb{Z}$-mod $\quad X^\vee$, linear dual $= \operatorname{Hom}(X, \mathbb{Z})$.

$\langle\ \rangle : X \times X^\vee \to \mathbb{Z}$ perfect pairing

Given $\alpha \in X$ let $\alpha^\vee$ st $\langle \alpha, \alpha^\vee \rangle = 2$

$s_\alpha : X \to X$

$x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ linear map.

fixes $H = \{x \in X \mid \langle x, \alpha^\vee \rangle = 0\}$.

$s_\alpha(\alpha) = -\alpha$. let $\Phi \subset X$ finite subset of X.

and define a map

$$\Phi \longrightarrow X^\vee$$
$$\alpha \longmapsto \alpha^\vee$$

A triple $R = (X, \Phi, \alpha \mapsto \alpha^\vee)$ is root datum if

1) $\langle \alpha, \alpha^\vee \rangle = 2$
2) $s_\alpha(\Phi) \subset \Phi$ $\quad \forall \alpha \in \Phi$
3) $W(R)$ (the Weyl group of R) as a subgroup of $\mathrm{Aut}(X)$ generated by all s_α is finite
4) $\alpha \longleftrightarrow \alpha^\vee$ is bijective, call $\Phi^\vee$ the image

A root datum is reduced if $k\alpha \in \Phi$ (k rational) $\Rightarrow k = \pm 1$.

let (G,T) split reductive group / k field.

For a root α of (G,T) split reductive group, let $T_\alpha = \ker(\alpha)_{red}$ (subtorus of T of codimension 1)

e.g. SL_3 $\quad T_\alpha = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \lambda & & \\ & 1 & \\ & & \lambda^{-1} \end{pmatrix}$

or $\begin{pmatrix} 1 & 0 & 0 \\ & \lambda & 0 \\ & & \lambda^{-1} \end{pmatrix}$, $\dim T = 2$, $\dim T_\alpha = 1$.

~~SL_4 $\dim T$ 6 elements above the diagonal~~

~~and $\binom{4}{2} = 6$ nice $\dim = 1$ torus.~~

~~$\binom{4}{1} = 4$ codim 1 torus.~~

$S_n \ni w_0$

$l(w_0) = \frac{n(n-1)}{2}$ = # of positive roots

$\binom{4}{2}$ codim 1 torus of the form

$$\begin{pmatrix} x & & & \\ & x & & \\ & & x_3 & \\ & & & \ddots \; x_n \end{pmatrix} \quad \text{st } x x x_3 \cdots x_4 = 1$$

SL_n

there are $\binom{n}{2}$ = # positive roots torus T_α. $\binom{3}{2} = 3$.

SL_3 they are $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, $\lambda^2 \lambda_3 = 1$

$\begin{pmatrix} \lambda & & 0 \\ & \lambda_2 & \\ 0 & & \lambda \end{pmatrix}$, $\lambda^2 \lambda_2 = 1$ and $\begin{pmatrix} \lambda_1 & & 0 \\ 0 & \lambda & \\ & & \lambda \end{pmatrix}$, $\lambda_1 \lambda^2 = 1$

$U_\alpha = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cong \mathbb{G}_a$

The root group U_α of α is the abs subgroup $U_{G_\alpha}(\lambda)$ of G, where $\lambda \in X(T)^*$ st. $\langle \alpha, \lambda \rangle > 0$.

And $G_\alpha = C_G(T_\alpha)$.

U_α is iso to $\mathbb{G}_a$

Notation: $u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$

Big theorem (G,T) split reductive, α a root

(a) (G_α, T) is split reductive with ss rank $= 1$.

(b) $\mathrm{Lie}(G_\alpha) = t \oplus g_\alpha \oplus g_{-\alpha}$

$t = \mathrm{Lie}(T)$, $\dim g_\alpha = \dim g_{-\alpha} = 1$.

The only rational multiples of α are $\pm \alpha$

(d) $W(G_\alpha, T) = \langle s_\alpha \rangle$ involution

s_α is represented by $n_\alpha \in N_{G_\alpha}(T)(k)$

(c) U_α is normalised by T, iso to $\mathbb{G}_a$ and $\mathrm{Lie}(U_\alpha) = g_\alpha$. Any other subgroup variety of G normalised by T contains U_α iff its Lie alg contains g_α.

(d) $\exists!\ \alpha^\vee \in X_*(T)$ st $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ $\forall x \in X(T)$ and $\langle \alpha, \alpha^\vee \rangle = 2$.

(ii) G is generated by T and $\{U_\alpha\}_{\alpha\in\Phi}$.

Cor: $\dim G = \dim T + |\Phi|$.

Cor: $(X(T), \Phi(G,T), \alpha\mapsto\alpha^\vee)$ is reduced

eg (GL_n, D_n) diagonal $n\times n$.

$\alpha = \alpha_{12} = \chi_1 - \chi_2$ $\left(\chi_i \text{ maps } \mathrm{diag}(t_j)_{j=1}^{n} \mapsto t_i\right)$

$$G_\alpha = \left\{\begin{pmatrix} * & * & 0 & & \\ * & * & 0 & & 0 \\ 0 & 0 & * & 0 & \\ 0 & 0 & 0 & * & \\ & 0 & & & * \end{pmatrix}\right\}$$

$$T_\alpha = \left\{\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & * & \\ & & & \ddots \; * \end{pmatrix}\right\} = T_{\alpha_{12}}$$

$$n_\alpha = \begin{pmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & 0 & \\ & & & 1 & 0 \\ 0 & & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow s_\alpha.$$

n_α is an involution here. But it is just accident!!

$W = N(T)/T \cong S_n < GL_n$.

n_α not involution in SL_n

For $SL_{n+1}(\mathbb{C})$ $W \cong S_{n+1}$ too but $S_{n+1} \not\subset SL_{n+1}(\mathbb{C})$

Example A_n: SL_{n+1} $n\geq 1$.

$T = \{\mathrm{diag}(t_1,\dots,t_{n+1}) \mid t_1\cdots t_{n+1} = 1\}$

split maximal torus.

$W \cong S_{n+1}$. character group

$X(T) := X^*(T) = \bigoplus \mathbb{Z}\chi_i / \mathbb{Z}\chi$

cocharacter group

$X_*(T) = ?$

$\chi_i : \mathrm{diag}(t_1,\dots,t_{n+1}) \mapsto t_i$.

$\chi = \sum \chi_i$.

$X_*(T) = \left\{\sum a_i\lambda_i \in \bigoplus_i \mathbb{Z}\lambda_i \;\middle|\; \sum a_i = 0\right\}$

where $\sum a_i\lambda_i$ is the cocharacter

$t \longmapsto \mathrm{diag}(t^{a_1},\dots,t^{a_n})$

$SL_{n+1} = A_n$ (cont).

The canonical pairing

$$X^*(T)\times X_*(T) \longrightarrow \mathbb{Z}$$

$\langle \chi_j, \sum a_i\lambda_i\rangle = a_j$.

$\mathfrak{sl}_{n+1} = \mathrm{Lie}(SL_{n+1}) = \{(a_{ij})\in M_{n+1}(k) \mid \sum a_{ii} = 0\}$

$SL_{n+1} \circlearrowright \mathfrak{sl}_{n+1}$ by conj.

$\bar\chi_i :=$ class of χ_i in $X^*(T)$

(i.e. $\bar\chi_i = \chi_i \mod \mathbb{Z}\chi$)

T acts trivially on $\mathfrak{g}_0$ the set of diagonal matrices on $\mathfrak{sl}_n = \mathfrak{g}$. and acts through the character $\alpha_{ij} := \bar\chi_i - \bar\chi_j$ on kE_{ij}, $i\neq j$.

$$\mathfrak{sl}_{n+1} = \mathfrak{g}_0 \oplus \bigoplus_{i\neq j} \mathfrak{g}_{\alpha_{ij}}$$

$\mathfrak{g}_{\alpha_{ij}} = kE_{ij}$.

$\Phi = \{\alpha_{ij} \mid i\neq j\}$.

Coroots $\alpha = \alpha_{12}$

$T_\alpha = \{\mathrm{diag}(x,x,x_3,\dots,x_{n+1}),\ xxx_3\cdots = 1\}$

$$G_\alpha = \left\{\begin{pmatrix} * & * & & \\ * & * & 0 & \\ & & * & \\ & 0 & & \ddots \; * \end{pmatrix} \in SL_n\right\}$$

$W(G_\alpha, T) = \{1, s_\alpha\}$, where s_α acts by changing rows 1 & 2. it is represented by

$$n_\alpha = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ & 0 & & & \ddots \; 1 \end{pmatrix} \in N_G(T)(k)$$

$\mathrm{ord}(n_\alpha) = 4! \neq 2$.

Let $\chi = \sum_{i=1}^{n+1} a_i\bar\chi_i \in X^*(T)$.

$$s_\alpha(\chi) = a_2\bar\chi_1 + a_1\bar\chi_2 + \sum_3^{n+1} a_i\bar\chi_i$$
$$= \chi - \langle\chi, \lambda_1 - \lambda_2\rangle(\bar\chi_1 - \bar\chi_2)$$

i.e. $s_{\alpha_{12}}(\chi) = \chi - \langle \chi, \alpha_{12}^\vee\rangle \alpha_{12}$

this proves $\alpha_{12}^\vee := \lambda_1 - \lambda_2$ is coroot of α_{12}

A3

, $i \neq j$ is the coroot

$B = U^+ T$ Borel

$B \rightsquigarrow \Phi^+ = \{x_i - x_j \mid i<j\}$

base $\{x_1 - x_2, \dots, x_n - x_{n+1}\}$

The set

Φ is a root system in the v.s.

$$X^*(T) \otimes \mathbb{Q} \simeq \mathbb{Q}^{n+1} / \langle e_1 + \dots + e_{n+1} \rangle$$

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Summary A_n: let V be the hyperplane in $\mathbb{Q}^{n+1}$ of $(n+1)$-tuples (a_i) st. $\sum a_i = 0$. Let $\{\varepsilon_1, \dots, \varepsilon_{n+1}\}$ be the standard basis of $\mathbb{Q}^{n+1}$ and consider

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The pair (V, Φ) is an indecomposable root system with Dynkin diagram of type A_n. The group SL_{n+1} is split and geometrically almost-simple with root system (V, Φ)

It is simply connected because $X = P(\Phi)$ and its centre is $\mu_{(n+1)+1}$ since $P(\Phi)/Q(\Phi) \cong \mathbb{Z}/(n+1)\mathbb{Z}$.

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$$G/B = \bigsqcup_{w \in N(T)/T =: W} Bw \cdot B$$

$\Rightarrow G = \bigsqcup_{w \in W} BwB$ since $g \in G$ given, consider $g \cdot B \in G/B$, then there is w s.t. $g \cdot B \in Bw \cdot B \Rightarrow \exists b \in B$ st $g \cdot B = bw \cdot B \Rightarrow w^{-1} b^{-1} g \in B$

$\Rightarrow \exists b'$ s.t. $w^{-1} b^{-1} g = b' \Rightarrow g = b w b'$

$\therefore G = BWB$ □

⚠ Warning: $Bw \cdot B \subset G/B$ but $BwB \subset G$.

Let $\Phi^+ \subset \Phi$ set of positive roots corresponding to B and Δ simple roots in Φ^+. For $\alpha \in \Phi$, we have a one parameter subgroup U_α: $\alpha \in X(T)$

$$u_\alpha : \mathbb{G}_a \xrightarrow{\text{additive gp}} U_\alpha . \qquad \alpha(\lambda) \in \mathbb{C}^\times.$$

s.t. $\forall \lambda \in \Omega, h \in T$

$$h \cdot u_\alpha(\lambda) \cdot h^{-1} = u_\alpha(\alpha(h)\lambda)$$

eg) SL_2 $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}}_{u_\alpha(\lambda)} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 \lambda \\ 0 & 1 \end{pmatrix} = u_\alpha(t^2 \lambda) = u_\alpha(\alpha(h)\lambda)$

$\alpha : T \longrightarrow \mathbb{C}^\times$ $\quad \alpha = 2\chi = 2\varpi$.

$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \longmapsto t^2$

$h \longmapsto t^2$

Let U^+ (resp U^-) be the max'l unipotent $U^+ \subseteq B^+$ corresponding to Φ^+ (resp $\Phi^- := -\Phi^+$)

Order in Φ^+ (w.o.) then any element $u \in U^+$ can be uniquely written as $(+)$

$$\prod_{\alpha \in \Phi^+} u_\alpha(\lambda_\alpha) \quad \lambda_\alpha \in \Omega$$ w/ respect to this order.

e.g SL_3 $\quad U_{\alpha_1} = \begin{pmatrix} 1 & * & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

$U_{\alpha_2} = \begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix}$, $U_{\alpha_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$.

let $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & b \\ & 1 & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

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For $\alpha \in \Phi^+$, let $n_\alpha \in N(T)$

$n_\alpha = u_\alpha(1)\, u_{-\alpha}(-1)\, u_\alpha(1)$

e.g SL_2 $\quad \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{order } 4!!}$

Let s_α the image of n_α in $N(T)/T$.

e.g SL_2: $s_\alpha^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -Id \in T$.

$\therefore s_\alpha^2 = 1 \bmod T$.

Lemma 2.1.

For $\lambda \in \Omega^*$, $u_\alpha(\lambda^{-1}) \cdot n_\alpha \cdot u_\alpha(\lambda)$
$= h \cdot n_\alpha \cdot u_\alpha(-\lambda^{-1}) \cdot n_\alpha^{-1}$

for a suitable $h \in T$.

Proof [Deduce it from Steinberg notes lemma 19] $\square$.

Let's convince us with an SL_2 example:

e.g SL_2

$u_\alpha(\lambda^{-1}) \cdot n_\alpha \cdot u_\alpha(\lambda)$

$=$

$\begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -\lambda^{-1} & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda^{-1} & 0 \\ -1 & -\lambda \end{pmatrix}$

$= \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{\in T}\begin{pmatrix} \lambda^{-1} & 0 \\ 1 & \lambda \end{pmatrix} = -\begin{pmatrix} \lambda^{-1} & 0 \\ 1 & \lambda \end{pmatrix}$ $\quad \underbrace{\lambda \cdot (-\lambda^{-1})}_{-1} + 1 = 0$

$h \cdot n_\alpha \cdot u_\alpha(-\lambda^{-1}) \cdot n_\alpha^{-1}$

$= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\lambda^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$= h \cdot \begin{pmatrix} 0 & 1 \\ -1 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = h \cdot \begin{pmatrix} 1 & 0 \\ \lambda^{-1} & 1 \end{pmatrix}$

$= \begin{pmatrix} t & 0 \\ t^{-1}\lambda^{-1} & t^{-1} \end{pmatrix}$ set $t = -\lambda^{-1} \in \Omega^*$.

$= \begin{pmatrix} -\lambda^{-1} & 0 \\ -1 & -\lambda \end{pmatrix}$. $\quad h = \begin{pmatrix} -\lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix} \in T$.

Let $y \in W$, $y = s_1 \cdots s_k$, $s_i \in S \longleftrightarrow \Delta \subset \Phi^+$

Let $\alpha_j \in \Delta$ the root corresponding to s_i ($s_i = s_{\alpha_i}$). For $1 \leqq j \leqq k+1$ define

$U_j = U^+ \cap {}^{s_j \cdots s_k} U^-$

$U^j = U^+ \cap {}^{s_k \cdots s_j} U^+$

Notation: For a subset $A \subset G$, $g \in G$ denote ${}^g A = g A g^{-1}$.

e.g SL_2 $\quad {}^{s_\alpha}U^+ = s_\alpha U^+ s_\alpha^{-1}$.

$\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$ $\quad s_\alpha U^+ s_\alpha = U^-$

$\begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix}\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} = \begin{pmatrix} +1 & 0 \\ -\lambda & +1 \end{pmatrix}$

A_n (cont.)

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($\mathbb{G}_a$: additive gp)

s.t. $\forall \lambda \in \Omega$, $h \in T$

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eg. SL_2

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\underbrace{\begin{pmatrix} 1 & \lambda \\ 0 & 1\end{pmatrix}}_{u_\alpha(\lambda)}\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{+1}\end{pmatrix} = \begin{pmatrix} 1 & t^2\lambda \\ 0 & 1\end{pmatrix} = u_\alpha(t^2\lambda) = u_\alpha(\alpha(h)\lambda)$$

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$U_{\alpha_2} = \begin{pmatrix}1&0&*\\&1&0\\&&1\end{pmatrix}$, $U_{\alpha_3} = \begin{pmatrix}1&0&0\\0&1&*\\0&0&1\end{pmatrix}$.

let $\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix} = \begin{pmatrix}1&a&0\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&b\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&c\\0&0&1\end{pmatrix}$

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$\in T. \quad \lambda\cdot(-\lambda^{-1})+1=0$

$= \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}\lambda^{-1}&0\\1&\lambda\end{pmatrix} = -\begin{pmatrix}\lambda^{-1}&0\\1&\lambda\end{pmatrix}$

$h \cdot n_\alpha \cdot u_\alpha(-\lambda^{-1}) \cdot n_\alpha^{-1}$

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$\begin{pmatrix}0&1\\-1&-\lambda\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\-\lambda&1\end{pmatrix}$

Lemma 2.2

(i) U_j is the subgroup of U^+ gen by the 1-parameter subgroups corresponding to the set of roots $\{\phi \in \Phi^+ \mid s_k \cdots s_j(\phi) \in -\Phi^+\}$

This set consist precisely of roots
$\{\alpha_j, s_j(\alpha_{j+1}), \ldots, s_j \cdots s_{k-1}(\alpha_k)\}$

Further, any element of $u \in U_j$ can be uniquely written as
$u_{\alpha_j}(a_j) \cdot u_{s_j(\alpha_{j+1})}(a_{j+1}) \cdots u_{s_j \cdots s_{k-1}(\alpha_k)}(a_k)$

thus $U_j \simeq \mathbb{Q}^{k-j+1}$.

(ii) $U_1 \supset {}^{s_1}U_2 \supseteq {}^{s_1 s_2}U_3 \supseteq \cdots \supseteq {}^{s_1 \cdots s_k}U_{k+1} = \{id\}$

(iii) U^j = $\{\phi \in \Phi^+ \mid s_j \cdots s_1(\phi) \in \Phi^+\}$

(iv) $U^1 \subseteq U^2 \subseteq U^3 \subseteq U^4 \subseteq \cdots \subseteq U^{k+1} = U^+$

(v) For any $1 \leq j \leq k+1$, $U^+ = U_j \cdot {}^{s_j \cdots s_k}U^j$ with a uniqueness of expression

(vi) Any element $\xi \in B\dot{g} \cdot B$ can be uniquely written as $u s_1 \cdots s_k \cdot B$ with $u \in U_1$. Thus $B\dot{g} \cdot B \simeq U_1 \simeq \mathbb{Q}^k$

Pf Deduce from structure of U^+ given in $(+)$

(W, S) be any cox. system: 2

Remark subexpression def:

Is a sequence $\underline{\sigma} = (\sigma_0, \ldots, \sigma_k)$ of elts of W st.

$\sigma_0 = id$, $\sigma_{j-1}^{-1}\sigma_j \in \{id, s_j\}$ $\forall\, 1 \leq j \leq k$

$\sigma_j > \sigma_{j-1}$ or $\sigma_j = \sigma_{j-1}$ or $\sigma_j < \sigma_{j-1}$

Let $\mathcal{S}$ the set of subexpr.

For $\underline{\sigma} \in \mathcal{S}$ define $I(\underline{\sigma}) = \{j \mid \sigma_{j-1} = \sigma_j\}$

$\#0$'s $= n(\underline{\sigma}) = \# I(\underline{\sigma})$

$m(\underline{\sigma}) = \#\{j \mid \sigma_{j-1} > \sigma_j\}$

$(= \#0'\text{s})$

$\pi : \mathcal{S} \longrightarrow W$

$\underline{\sigma} \longmapsto \sigma_k$.

Lemma (Deodhar, Thm 1.1)

$\pi(\mathcal{S}) \overset{??}{=} W(\mathcal{S}) = \{x \leq g\}$

Def $\mathcal{D} \subset \mathcal{S}$ with the additional cond:
$\sigma_j \leq \sigma_{j-1} s_j$ ("Distinguished subexpr")

Rmk $\pi(\mathcal{D}) = W(\mathcal{S})$ non trivial (Prop 5.2)

Theorem 1.1

(i) $B\dot{g} \cdot B = \bigsqcup_{\underline{\sigma} \in \mathcal{D}} D_{\underline{\sigma}}$.

(ii)
$D_{\underline{\sigma}} \simeq \mathbb{Q}^{m(\underline{\sigma})} \times (\mathbb{Q}^*)^{n(\underline{\sigma})}$

$(\mathbb{Q}^* := \mathbb{Q} - \{0\})$

(iii) For $\underline{\sigma} \in \mathcal{D}$ $\exists!\, x \in W$ st.

$D_{\underline{\sigma}} \subseteq B^- x \cdot B$. This x is $\pi(\underline{\sigma})$

Corollary 1.2

$B\dot{g} \cdot B \cap B^- x \cdot B = \bigsqcup_{\substack{\underline{\sigma} \in \mathcal{D} \\ \pi(\underline{\sigma}) = x}} D_{\underline{\sigma}}$.

(In particular, $B\dot{g} \cdot B \cap B^- x \cdot B \neq \phi$ iff $x \in W(\mathcal{S})$)

§3. Proof of theorem 1.1

Proposition 3.1

Let $u_1 \in U_1$ fixed. For $0 \leq j \leq k$, let $\sigma_j \in W$ be the unique s.t
$u_1 s_1 \cdots s_j \in B^- \sigma_j B$. (Birkhoff decomposition)
Then $(\sigma_0, \ldots, \sigma_k)$ is distinguished.

Pf: $w \in W$, $s \in S$ then
$wBs \subseteq B^- ws B \cup B^- w B$

This is a variant of 'T3' of def. of a Tits system (G, B, N, S) (Use lemma 2.1).
(Also, $wBs \cap B^- wB \neq \emptyset \Rightarrow ws > w$).

e.g. SL_2

$$eBs \subset B^- esB \sqcup B^- B$$

$$\| $$

$$B \cdot n_\alpha = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -c & 0 \end{pmatrix}.$$

$\sigma_0 = id$ (its the unique?) $\quad u_1 \in U_1$.

$$U_1 = U^+ \cap {}^{s_1 \cdots s_k} U^- = U^+ \cap yU^- y^{-1}$$

$u_1 \in B^- \sigma_0 B$ unique st. this happens?

$$U_1 = U^+ \cap {}^y U^- = U^+ \cap yU^- y^{-1}$$

$$wBs \subseteq B^- wsB \cup B^- wB.$$

e.g. let SL_2 and $y = s = n_\alpha$, $U_\alpha = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Remember $\xi \in By \cdot B$ can be written uniquely as $us_1 \cdots s_k \cdot B$ st $u \in U_1$

then $U_1 \cong By \cdot B \cong \mathbb{A}^{l(y)} = \mathbb{A}^k$.

$u \longmapsto uy \cdot B$

Since $U_1 \subseteq U^+ \subseteq B$

$Bs \cdot B \cong U^+ \cap {}^y U^- \qquad n_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$${}^{n_\alpha} U^- = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \right\} = U^+.$$

$$Bs \cdot B = U^+ \cong U_1$$

lets understand the map

$$U^+ \xleftrightarrow{\sim} Bs \cdot B$$

key point: $\quad u \longmapsto us \cdot B$.

$*$ U-orbits = B orbits on G/B.

$u_1 \in \bigsqcup B^- w \cdot B \quad \exists!\, w \in W$

st $u_1 \in B^- \bar{w} B$. $\quad \bar{w}$ lift of w in $N(T)$.

$u_1 = b^- \bar{w} b$. $\forall b, b^-$, $b \in B$, $b^- \in B^-$.

$G = B^- W B$.

$u_1 s \cdot B \in Bs \cdot B$.

e.g SL_2. $y = s = n_\alpha$, $U_\alpha = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset B$.

$U_1 = U^+ \cap {}^s U^- = U^+ \cap U^+$.

let $u = \begin{pmatrix} 1 & \lambda_0 \\ 0 & 1 \end{pmatrix}$.

$$G = B^- eB \sqcup B^- n_\alpha B$$

$\cancel{\begin{pmatrix} \lambda & b \\ 0 & \lambda^{-1} \end{pmatrix}} \qquad B^- eB = B^- B$

$$\|$$

$$\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}\begin{pmatrix} \lambda & c \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda a & ac \\ \lambda b & bc + \frac{1}{a\lambda} \end{pmatrix}$$

$$B^- B = \left\{ \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \cdot \begin{pmatrix} e & f \\ 0 & d \end{pmatrix} \;\middle|\; \begin{matrix} ab - c = 1 \\ ed - f = 1 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} ae & af \\ ce & cf + bd \end{pmatrix} \;\middle|\; \begin{matrix} ab - c = 1 \\ ed - f = 1 \end{matrix} \right\}$$

$$\det = ae(cf + bd) - acef$$

$$= ae(cf + bd - cf) = 1.$$

$$\Rightarrow aebd = 1$$

$\dim B^- \cdot B = 2$: in G/B, $\dim B = ?$.

$\dim B^- B = ?$

$$B = \left\{ \begin{pmatrix} a & c \\ & b \end{pmatrix} \;\middle|\; \begin{matrix} ab - c = 1 \\ \Rightarrow c = -1 + ab \end{matrix} \right\} = \left\{ \begin{pmatrix} a & -1+ab \\ & b \end{pmatrix} \right\}$$

$B = \left\{ \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \right\} \cong \mathbb{A}^2$, $\dim B = 2$.

$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}$ $\quad \dim G = 3$.

$G/B \cong \mathbb{P}^1_{\mathbb{C}} \cong \mathbb{A}^1 \sqcup *$.

$SL_2 = \bigsqcup B^- w B$ dim $B = 2$. dim $G = 3$.

$B = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right\}$, $B^- = \left\{\begin{pmatrix} c & 0 \\ d & c^{-1}\end{pmatrix}\right\}$ dim $B^- = 2$

$$B^-B = \left\{\begin{pmatrix} c & 0 \\ d & c^{-1}\end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right\} \quad (\neq)$$

$$= \left\{\begin{pmatrix} ac & bc \\ ad & bd + \frac{1}{ac}\end{pmatrix} \;\middle|\; \begin{matrix} a \neq 0 \\ c \neq 0 \end{matrix}\right\}$$

dim $G/B = 1$, SL_2 "=" $\bigsqcup$ "C^w"
$w \in W$.

Easier to understand:

$$SL_2/B = \bigsqcup_{w\in W} B^- w \cdot B = \bigsqcup_{w \in W} C^w$$

$$C^w = w_0 C_{w_0 w} = w_0 B w_0 w \cdot B.$$

$w_0 B w_0 = B^-$

$B^- \cdot B = w_0 B w_0 \cdot B \xleftrightarrow{\sim} B w_0 \cdot B = C_e$

$B^- s \cdot B = sBss \cdot B = sB \cdot B \longleftrightarrow B \cdot B$. dim $=0$

$B^- \cdot B = sBs \cdot B \xleftrightarrow{\sim} Bs \cdot B = C_s$. dim $=1$.

$B^- s \cdot B = s \cdot B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot B$. dim $=0$

$B^- \cdot B = sBs \cdot B$ dim $=1$

then $B^- s B$ dim $= 2 = 0 + 2$ $B \cong \mathbb{C} \times \mathbb{C}^*$

$B^- B$ dim $= 3 = 1 + 2$.

$SL_2 \cong B^- s B \sqcup B^- B$ } not completely same dimsh.

$wBs \subseteq B^- wsB \sqcup B^- wB$.

$U_1 \cong By \cdot B$.

$u_1 \in U_1$

$U_1 = U^+ \cap {}^y U^-$

More that Birkhoff decomposition!

SL_2 | $y = s$.

$U_1 = U^+ \cap {}^s U^-$.

${}^s U^- = s U^- s^{-1}$ $s = s_\alpha$.

$= U^+$.

then $U_1 = U_\alpha = U^+$.

$u_1 \in U_1 \subset G = \bigsqcup B^- w B$

$u_1 \in B^- s \cdot B \sqcup B^- \cdot B$, $u_1 B \in B^- s \cdot B \sqcup B^- \cdot B$

$u_1 \cdot B \in \underset{B^- \cdot B}{\underset{\parallel}{sBs}} \cdot B$ or $u_1 \in s \cdot B = B^- s \cdot B$

$sBs \cdot B \sqcup s \cdot B = G/B$

$sBsB \sqcup sB = G$.

$u_1 = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

$s \cdot B$ (0-cell)

$\parallel$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 & a^{-1} \\ -a & -b \end{pmatrix}$

$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -a^{-1} & 0 \end{pmatrix}$

Caution ⚠

$s \cdot B$ correspond to the B-orbit of $e \cdot B$

$B \cdot B$ and then apply s.

$u_1 \in U_1 \xleftrightarrow{\sim} Bs \cdot B$

$u_1 \longmapsto u_1 s \cdot B$.

$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$G = \underbrace{sBsB}_{\dim = 3} \sqcup \underbrace{sB}_{\dim = 2}.$$

Birkhoff

$sB = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \to \dim = 2$

$\overset{B^-B}{\shortparallel}$

$sBsB = ? = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e & f \\ 0 & g \end{pmatrix}$

$= \begin{pmatrix} 0 & c \\ -a & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e & f \\ 0 & g \end{pmatrix}$

$= \left\{\begin{pmatrix} c & 0 \\ b & a \end{pmatrix}\begin{pmatrix} e & f \\ 0 & g \end{pmatrix}\right\} = B^-B.$

$= \left\{\begin{pmatrix} ac & bc \\ ad & bd + \frac{1}{ac} \end{pmatrix} \middle| \; ac \neq 0 \right\}$ (*)

$ac = \lambda \in \mathbb{C}^*$, $a = \frac{\lambda}{c}$, $c = \frac{\lambda}{a}$.

$= \left\{\begin{pmatrix} \lambda & \frac{b\lambda}{a} \\ a, d & bd + \frac{1}{\lambda} \end{pmatrix}\right\}$

$= \cancel{\{ \ldots \}}$ $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ua \\ 0 & a^{-1} \end{pmatrix}$

$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix}$

$B = TU$, $B^- \overset{?}{=} U^- T$.

$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & (ab)^{-1} \end{pmatrix}.$

$B^-B = ?$ $B^- = U^-T$?

$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ au & a^{-1} \end{pmatrix}$ ✓.

$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ ua^{-1} & a^{-1} \end{pmatrix}$ ✓.

$B^- = U^-T = TU^-$.

$B^-B = U^- T U^+$.

$= \left\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}$

$= \left\{\begin{pmatrix} \lambda & 0 \\ \lambda a & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\} = \left\{\begin{pmatrix} \lambda & \lambda b \\ \lambda a & \lambda ab + \lambda^{-1} \end{pmatrix}\right\}$

$\lambda \neq 0.$

$sB = \left\{\begin{pmatrix} 0 & a^{-1} \\ -a & b \end{pmatrix}\right\} = B^- sB.$

$sB \cong \mathbb{C}^* \times \mathbb{C}$

$sBsB = B^-B \cong \mathbb{C}^* \times \mathbb{C}^2.$

$SL_2 \cong (\mathbb{C} \times \mathbb{C}^*) \sqcup (\mathbb{C}^2 \times \mathbb{C}^*).$

- $\underset{U_1}{\overset{\shortparallel}{U^+}} \subseteq B^-B$. $u_1 \in B^- e B$.
  $U_1 \subseteq B^-B$. $\sigma_0 = id$.

$B^-B = sBsB$, $B^-sB = sBssB = sB$.

$u_1 s_1 \in B^- \sigma_j B$

$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -1 & 0 \end{pmatrix} \in B^- sB.$

$(0, s)$ dist subexpression.

$u_1 \in U_1$ fixed. $\sigma_0 = id$.

$u_1 s_1 \in B^- e B s_1 \subset B^- s_1 B \sqcup B^- B$

$wBs \subseteq B^- ws B \cup B^- w B$, $wBs \cap B^- wB \underset{\neq \emptyset}{=} \Rightarrow ws > w$

Prop 3.1 let $u_1 \in U_1$ fixed, for $0 \leq j \leq k$ let $\sigma_j \in W$ the unique s.t. $u_1 s_1 \cdots s_j \in B^- \sigma_j B$. Then $\underline{\sigma} = (\sigma_0, \ldots, \sigma_k) \in \mathcal{D}$.

Proof $\sigma_0 = id$ derv from above.

$u_1 s_1 \ldots s_j \in B^- \sigma_{j-1} B s_j$

$\therefore u_1 s_1 \ldots s_j \in B^- \sigma_{j-1} s_j B \cup B^- \sigma_{j-1} B$

(1) $u_1 s_1 \ldots s_j \in B^- \sigma_{j-1} s_j B$ or

(2) $u_1 s_1 \ldots s_j \in B^- \sigma_{j-1} B$

(2) $\Rightarrow \sigma_{j-1} s_j > \sigma_{j-1}$ and $\sigma_j = \sigma_{j-1}$. (this is a U0.)

(1) $\Rightarrow \sigma_{j-1} s_j = \sigma_j$ (this is a 1 $\left\{\begin{matrix} U1 \text{ or} \\ D1 \end{matrix}\right.$)

In any case $\sigma_j \in \left\{\begin{matrix} \sigma_{j-1} \\ \sigma_j \end{matrix}\right.$ and no D0

then it is distinguished $\square$. $\underline{\sigma} \in \mathcal{D}$

We got a map $\eta : U_1 \longrightarrow \mathcal{D}$.
we will see η is surjective and look fibers.

Prop 3.2 Fix $\underline{\sigma} \in \mathcal{D}$ and $j \in \{1, \ldots, k\}$
Define a subset $\Omega(\underline{\sigma}, j)$ of Ω as follows.

$$\Omega(\underline{\sigma}, j) = \begin{cases} \Omega & \text{if } \sigma_{j-1} > \sigma_j \\ \{0\} & \sigma_{j-1} < \sigma_j \\ \Omega^* & \sigma_{j-1} = \sigma_j \ \cup 0 \end{cases}$$

Then we have;
(i) There is a unique surjective morphism
$f_j : \Omega(\underline{\sigma}, j) \times U_{j+1} \longrightarrow U_j$ s.t.

$\sigma_{j-1} f_j(t, u_{j+1}) s_j \cdots s_k = b_j^- \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$\exists\, b_j^- \in B^-$ and $v_{j+1} \in U^{j+1}$.

(ii) The image of f_j is a locally closed subset of U_j and f_j is iso onto its image.

Proof
Case (a) $\sigma_{j-1} > \sigma_j$, $\sigma_j = \sigma_{j-1} s_j$
$\sigma_j(\alpha_j) \in \Phi^+$, $\Omega(\sigma, j) = \Omega$. Define
$f_j : \Omega \times U_{j+1} \to U_j$ by
$f_j(t, u_{j+1}) = u_{\alpha_j}(t) \cdot s_j u_{j+1} s_j^{-1}$

$U_j = U^+ \cap {}^{s_j \cdots s_k} U^-$, ${}^{s_1 \cdots s_{j-1}} U_j \supseteq U_{j+1}$.

$U_1 \cong \Omega^{\ell(y)}$, $U_j \cong \Omega^{k-j+1}$

$U_2 \cong \Omega^{\ell(y)-1}$, etc....

$U_j = \langle U_\alpha \mid \alpha \in \{\alpha_j, s_j(\alpha_{j+1}), \ldots, s_j \cdots s_{k-1}(\alpha_k)\} \rangle$

e.g. SL_3

$$\begin{pmatrix} 1 & \alpha & \alpha+\beta \\ & 1 & \beta \\ & & 1 \end{pmatrix}$$

3 positive roots.
$\alpha, \beta, \alpha+\beta$.
$\Delta = \{\alpha = \alpha_1, \beta = \alpha_2\}$.

$U_1 = U^+ \cap {}^y U^-$ $\quad s_1 = s \quad s_3 = s$
$\quad s_2 = t \quad k = 3$.

$y = sts$. $\quad s = s_\alpha$, $t = s_\beta$

$U_1 = U^+ \cap {}^{w_0} U^-$ $\quad U_1 = \{U_\alpha \mid \alpha \in \{\alpha_1, s(\alpha_2), st(\alpha_3)\}$

${}^{w_0} U^- = sts\, U^- sts = w_0 U^- w_0 = U^+$.

$U_1 = U^+$. $\quad U_1 = \langle U_\alpha \mid \alpha \in \{\alpha, s(\alpha_2), st(\alpha_3)\} \rangle$

$U_2 = U^+ \cap {}^{ts} U^-$ $\quad \alpha_3 = \alpha$.

$= U^+ \cap ts U^- st$. $\quad t(\alpha) = \alpha + \beta$.

$U_2 = \langle U_\alpha \mid \alpha \in \{\beta, s_2(\alpha_3)\} \rangle$

α, β. $\quad s(\alpha) = -\alpha$. $\quad U_3 =$
$s(\beta) = \alpha + \beta$
$t(\alpha) = \alpha + \beta$
$t(\beta) = -\beta$.

$U_1 = \langle U_\alpha \rangle_{\alpha \in I}$ $\quad I = \{\alpha, \alpha+\beta, \beta\} \subset \Phi^+$.

$st(\alpha) = s(\alpha+\beta) = -\alpha + \alpha + \beta = \beta$.

$\therefore U_1 = U^+$.

$U_2 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ $\quad U_3 = \{\alpha_3, \cancel{s(\alpha_4)}\}$ ($\alpha_3 = \alpha$), no α_4.

${}^{t} U_1 \supseteq U_2$.

$U_3 = U_{\alpha_3} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

${}^{s} U_2 \supseteq U_3$? $\times$ $\quad U_2$

$s = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$s u_2 s^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & b \\ -1 & 0 & -a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$

$U_2 = \{\alpha_2, s_2(\alpha_3)\}$.

$\alpha_2 = \beta$, $s_2(\alpha_3) = t(\alpha) = (\alpha+\beta)$

$U_1 \supseteq {}^sU_2 \supseteq {}^{st}U_3$. Only the non obvious inclusion is ${}^sU_2 \supseteq {}^{st}U_3$.

${}^sU_2 = U_2$.

$st\, U_3\, ts$. $s = \begin{pmatrix} 0&1&0\\ -1&0&0\\ 0&0&1 \end{pmatrix}$, $t = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&-1&0 \end{pmatrix}$

$$st = \begin{pmatrix} 0&1&0\\ -1&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&-1&0 \end{pmatrix} = \begin{pmatrix} 0&0&1\\ -1&0&0\\ 0&-1&0 \end{pmatrix} = st.$$

$$\cancel{ts = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}\begin{pmatrix} 0&1&0\\ -1&0&0\\ 0&0&1 \end{pmatrix} = \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix} = ts.}$$

$$\cancel{st\, U_3\, t^{-1}s^{-1} = \begin{pmatrix} 0&0&1\\ -1&0&0\\ 0&-1&0 \end{pmatrix}\begin{pmatrix} 1&a&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}}$$

$$\cancel{= \begin{pmatrix} 0&0&1\\ -1&-a&0\\ 0&-1&0 \end{pmatrix}\begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}}$$

$$\cancel{= \begin{pmatrix} 1&0&0\\ 0&-1&-a\\ 0&0&-1 \end{pmatrix}}$$

$t^{-1} = \begin{pmatrix} 1&0&0\\ 0&0&-1\\ 0&1&0 \end{pmatrix}$, $s^{-1} = \begin{pmatrix} 0&-1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}$.

$$t^{-1}s^{-1} = \begin{pmatrix} 0&-1&0\\ 0&0&-1\\ 1&0&0 \end{pmatrix}$$

$$st\, U_3\, t^{-1}s^{-1} = \begin{pmatrix} 0&0&1\\ -1&0&0\\ 0&-1&0 \end{pmatrix}\begin{pmatrix} 1&a&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&-1&0\\ 0&0&-1\\ 1&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&1\\ -1&-a&0\\ 0&-1&0 \end{pmatrix}\begin{pmatrix} 0&-1&0\\ 0&0&-1\\ 1&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0\\ 0&1&a\\ 0&0&1 \end{pmatrix}$$

then we checked ${}^{st}U_3 \subset {}^sU_2 = U_2$!!

$U_2 \overset{?}{=} U^+ \cap {}^{ts}U^-$. [We are checking!]

$ts = \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}$, $\cancel{(ts)^{-1} = s^{-1}t^{-1} = \begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix}}$

$$s^{-1}t^{-1} = \begin{pmatrix} 0&-1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0\\ 0&0&-1\\ 0&1&0 \end{pmatrix} = \begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{pmatrix}.$$

$$\left\{\begin{pmatrix} 1&0&a\\ 0&1&b\\ 0&0&1 \end{pmatrix}\right\} = U_2$$

$$ts\, U^-\, s^{-1}t^{-1} = \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}\begin{pmatrix} 1&0&0\\ a&1&0\\ b&c&1 \end{pmatrix}\begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{pmatrix}$$

$$= \begin{pmatrix} a&1&0\\ b&c&1\\ 1&0&0 \end{pmatrix}\begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{pmatrix} = \begin{pmatrix} 1&0&a\\ c&1&b\\ 0&0&1 \end{pmatrix}$$

$${}^{ts}U^- \cap U^+ = \begin{pmatrix} 1&0&a\\ 0&1&b\\ 0&0&1 \end{pmatrix} \checkmark.$$

we lose it intersecting with U^+

$U_1 \cong U^+ \cong Bsts \circ B$. $U_1 \cong \Omega^{\ell(\sigma)}$

$U^+ \cap {}^{sts}U^- = U^+ \cap (w_0 U^- w_0) = U^+ \cap U^+ = U^+$.

$n: U_1 \to \mathcal{D}$ (to prove). $U_j \cong \Omega^{k-j+1}$

Prop 3.2 $\sigma \in \mathcal{D}$ $j \in \{1, \dots, k\}$ fixed.

$$\Omega(\Sigma, j) = \begin{cases} \Omega & \sigma_{j-1} > \sigma_j \quad (D1) \\ \{0\} & \sigma_{j-1} < \sigma_j \quad (U1) \\ \Omega^* & \sigma_{j-1} = \sigma \quad (U0). \end{cases}$$

Thm;

(i) $\exists$ injective morphism

$$f_j : \Omega(\Sigma, j) \times U_{j+1} \longrightarrow U_j \text{ st.}$$

$$\sigma_{j-1} f_j(t, u_{j+1}) s_j \cdots s_k = b_j^- \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$$

$\exists\, b_j^- \in B^-$ and $v_{j+1} \in U_{j+1}$.

(ii) f_j locally closed image $\subset U_j$ and f_j is iso onto its image.

(a) Case $\sigma_{j-1} > \sigma_j$ (D1).

$\sigma_j = \sigma_{j-1} s_j$, $\sigma_j(\alpha_j) \in \Phi^+$. $\Omega(\underline{\sigma}, j) = \Omega$.

$f_j : \Omega \times U_{j+1} \longrightarrow U_j$

$(t, u_{j+1}) \longmapsto u_{\alpha_j}(t) s_j u_{j+1} s_j^{-1}$.

U_{j+1} "=" $\{\alpha_{j+1}, s_{j+1}(\alpha_{j+2}), \dots, s_{j+1} \cdots s_{k-1}(\alpha_k)\}$.

eg SL_3. 101 DUU 101 sts

$y = sts$

$U^+ = U_1 = \left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ 0 & & 1 \end{pmatrix}\right\}$ $U_2 = \left\{\begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\}$

$U_3 = \left\{\begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$ $U_1 \supset {}^s U_2 \supset {}^{st} U_3 \supset {}^{sts} U_4$ $\overset{\text{id.}}{=}$

$U_4 = id.$

$y = s_1 s_2 s_3 = sts$. $\alpha_1 = \alpha$ ($\alpha_3 = \alpha$).
$= s\,t\,s$. $\alpha_2 = \beta$

$\sigma_0 = id$, $\sigma_1 = s$, $\sigma_2 = st$, $\sigma_3 = e$
U1 U0 D1.

$\underline{\sigma} = (e, s, s, e)$ $U_1 \cong \Omega^3$

j=3

$f_3 : \Omega \times U_4 \longrightarrow U_3 \cong \Omega$ $s_3 = s$.

$(t, u_{j+1}) \longmapsto u_{\alpha_3}(t) s_3 u_{j+1} s_3^{-1}$.

$(t, u_4) \longmapsto \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cancel{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{\text{id}}{\overset{}{u_4}} \cancel{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$
$= \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\sigma_{j-1}(u_{\alpha_j}(t) s_j u_{j+1} \cancel{s_j^{-1}}) \cancel{s_j} \cdots s_k$ $\quad \bar{B} \ni$, $U_{j+1} \ni$
$\overset{?}{\cong}$
$= b_j \sigma_j \underbrace{u_{j+1} s_{j+1} \cdots s_k}_{a} v_{j+1}$. $\quad \not\ni b_j^- \quad v_{j+1}$
$\overset{?}{\cong}$
$\sigma_{j-1} u_{\alpha_j}(t) s_j \underbrace{u_{j+1} s_{j+1} \cdots s_k}_{a}$.
||

$\sigma_{j-1} \underbrace{u_{\alpha_j}(t)}_{\in U_j} s_j a \overset{?}{=} b_j \sigma_j a v_{j+1}$ $\quad \not\ni v_j \in U_{j+1}$.

$U_{j+1} = \langle U_\alpha \mid \alpha \in \{\alpha_{j+1}, s_{j+1}(\alpha_{j+2}), \dots, s_{j+1} \cdots s_{k-1}(\alpha_k)\}\rangle$

$U_{j+1} = U^+ \cap {}^{s_{j+1} \cdots s_k} U^-$

$U_1 \supset {}^{s_1} U_2$

${}^{s_1 \cdots s_{j-1}} U_j \supset {}^{s_1 \cdots s_j} U_{j+1}$

$(s_1 \cdots s_{j-1}) = x$.

$x U_j x^{-1} \supset x s_j U_{j+1} s_j^{-1} x^{-1}$

$U_j \supset s_j U_{j+1} s_j^{-1}$

$s_{j+1} \cdots s_k = z$. $\quad s_j U_{j+1} \subset U_j s_j$

LHS $=$ $\sigma_{j-1} u_{\alpha_j}(t) s_j \overset{s_j^{-1} s_j}{u_{j+1}} z = b_j^- \sigma_j \overline{u_{j+1} z v_{j+1}}$ (A)

$\sigma_{j-1} u_{\alpha_j}(t) s_j u_{j+1} z$.

wBs $\subset$ W

① Why $\underbrace{u_{\alpha_j}(t)}_{\in U_j} \underbrace{s_j u_{j+1} s_j^{-1}}_{\in U_j} \in U_j$? ✓

well defined! Now prove (A)

LHS $= \sigma_{j-1} \underbrace{f_j(t, u_{j+1})}_{\in U_j \in U^+ \subseteq B} s_j z = b_j^- \sigma_j u_{j+1} z v_{j+1}$.

$\sigma_j s_j^{-1} f_j(t, u_{j+1}) s_j z$

$\sigma_{j-1} B s_j \subseteq \bar{B} \sigma_{j-1} s_j B \cup \bar{B} \sigma_{j-1} B$

$\overbrace{\sigma_{j-1} \boxed{u_{\alpha_j}(t)} s_j}^{p} u_{j+1} z = LHS$
$\in U^+ \subset B$

$\in \sigma_{j-1} B s_j$ $\quad$ but $p \in \bar{B} \sigma_{j-1} B$
$\in \sigma_{j-1} U^+ s_j$ $\quad \Rightarrow \sigma_{j-1} s_j > \sigma_{j-1}$ —X—
$U^+ \quad U^+ \quad = \sigma_j$

then $p \in \bar{B} \sigma_{j-1} s_j B = \bar{B} \sigma_j B$

$p = b_j^- \sigma_j b$ $\quad b_j^- \in \bar{B}$, $b \in B$. maybe $b \in U^+$.

LHS $= b_j^- \sigma_j \underbrace{b\, u_{j+1} z}_{\psi}$. $\quad \psi \in b u_j s_{j+1} \cdots s_k$.

$= b_j^- \sigma_j b\, u_{j+1} s_j^{-1} s_j z$

pf $b\, w_{j+1} \underbrace{s_j^{-1} s_j s_{j+1} \cdots s_k}_{z} = u_{j+1} z v_{j+1}$ $\forall v_{j+1} \in U^{j+1}$.

iff $\underline{z^{-1} w_{j+1}^{-1} b\, w_{j+1} z \in U^{j+1}}$ (#)

$U^+ = U_j^{s_j \cdots s_k} U^j$, $U^j = U^+ \cap {}^{s_k \cdots s_j} U^+$.

$(s_k \cdots s_j)(w_{j+1}^{-1} b\, w_{j+1})(s_k \cdots s_j)^{-1}$.

$\overbrace{(s_k \cdots s_j)}^{z^{-1} s_j} (u^+) \overbrace{(s_j \cdots s_k)}^{s_j z} \in U^j \subset U^{j+1}$

$U^+ = U_{j+1}^{s_{j+1} \cdots s_k} \cdot U^{j+1}$

$u^+ s_j u_{j+1} \overbrace{s_{j+1} \cdots s_k}^{z}$.

$\sigma_{j-1} u_{\alpha_j}(t) s_j u_{j+1} z = (z^{-1} u_{j+1}^{-1} \sigma_j^{-1} b_j^{-1})^{-1} v_{j+1}$

$\Leftrightarrow \underbrace{s_k \cdots s_{j+1}}_{z^{-1}} \underbrace{\left(u_{j+1}^{-1} \sigma_j^{-1} b_j \sigma_{j-1} u_{\alpha_j}(t) s_j u_{j+1}\right)}_{\in U^+ ?} z \in U^{j+1} = U^+ \cap z^{-1} U^+ z$.

$z^{-1} U^+ z \in U^{j+1}$.

only need to see

$* = \cancel{u_{j+1}^{-1}} \sigma_j^{-1} b_j \sigma_{j-1} \underbrace{u_{\alpha_j}(t)}_{} s_j \cancel{u_{j+1}} \in U^+$.

$\boxed{\in U^+ \subset B}$

$p \in B^- \sigma_{j-1} s_j B$.

$* \in \sigma_j^{-1} B^- \sigma_j B$. this is more complicated.

Go back to (#)

$z^{-1} w_{j+1}^{-1} b\, u_{j+1} z \in U^{j+1} = U^+ \cap z^{-1} U^+ z$.

pf (a) $z^{-1} w_{j+1}^{-1} \overset{U^+}{b}\, w_{j+1} z \in U^+$ and

$u_{j+1}^{-1} b\, u_{j+1} \in U^+$ ✓.

$z^{-1} \underbrace{w_{j+1}^{-1} u^+ u_{j+1}}_{\in U^+ \text{ but too strong.}} z \in U^+$?

$U_{j+1} = U^+ \cap z U^- z^{-1}$

$u_{j+1} s_{j+1} \cdots s_k \in U_{j+1} s_{j+1} \cdots s_k$.

$\sigma_{j-1} u_j s_j z$

$z^{-1} w_{j+1}^{-1} u^+ w_{j+1} z = \tilde{u}^+ \Leftarrow$

$w_{j+1}^{-1} u^+ w_{j+1} z \in z U^+$

$u^+ w_{j+1} z \in u_{j+1} z U^+$.

$s_j U_{j+1} s_j^{-1} \subset U_j$.

$z^{-1} = s_k \cdots s_{j+1}$.

$s_k \cdots s_{j+1} s_j \underbrace{s_j^{-1} \overbrace{u_{j+1}^{-1} u^+ u_{j+1}}^{\text{if } \in U_{j+1}} s_j^{-1}}_{\in U_j ?} s_j s_{j+1} \cdots s_k$

I will come back to this later. (solved)

$\sigma_{j-1} < \sigma_j$ (U1) $\sigma_j = \sigma_{j-1} s_j$

$\Omega(\sigma_j, j) = \{0\}$,

$f_j : \{0\} \times U_{j+1} \longrightarrow U_j$

$(0, w_{j+1}) \longmapsto s_j u_{j+1} s_j^{-1} \in U_j$

since $s_j U_{j+1} s_j^{-1} \subset U_j$.

$\sigma_{j-1} f_j(0, w_{j+1}) s_j \cdots s_k = b_j^{-1} \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$.

LHS $=$ $\underline{\sigma_{j-1} s_j} u_{j+1} s_j^{-1} s_j \cdots s_k$.

$= \sigma_j u_{j+1} s_j^{-1} s_j \cdots s_k$

$= \sigma_j w_{j-1} s_{j+1} \cdots s_k = b_j^{-1} \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$U^1 = U^+ \cap {}^{s_k \cdots s_1} U^+$.

$1 \in U^1$? $1 \in U^+$ ✓. $T \subset U^j \; \forall j$

$1 \in {}^{s_k \cdots s_1} U^+$? $T \subset U_j \; \forall j$.

$1 \in s_k \cdots s_1 U^+ s_1^{-1} s_2^{-1} \cdots s_k^{-1}$.

$s_k \cdots s_1 T s_1^{-1} s_2^{-1} \cdots s_k^{-1} = T$

$v_{j+1} = 1$ in this case!

$\sigma_{j-1} > \sigma_j$ $\boxed{D1}$ $\sigma_j = \sigma_{j-1} s_j$ $\Omega(\sigma, j) = \Omega_j$

$f_j : \Omega_j \times U_{j+1} \longrightarrow U_j$

$(t, w_{j+1}) \longmapsto w_{\alpha_j}(t)\, s_j\, w_{j+1}\, s_j^{-1}$

$\sigma_{j-1}\, \underbrace{w_{\alpha_j}(t)\, s_j u_{j+1} s_j^{-1}}_{\in U_j \subset U^+}\, s_j \cdots s_k = b_j^- \sigma_j w_{j+1} s_{j+1} \cdots s_k \nu_{j+1}$

$= \sigma_j s_j^{-1} w_{\alpha_j}(t) s_j w_{j+1} s_j^{-1} s_j \cdots s_k.$

$U_j = \langle \alpha_j, s_j(\alpha_{j+1}), \ldots, s_j \cdots s_{k-1}(\alpha_k) \rangle.$

$\sigma_{j-1} U^+ s_j \supset \sigma_{j-1} U_j s_j$

$\cap \quad \subset B^- \sigma_j U^+ ?$

$\sigma_{j-1} B s_j \subset B^- \sigma_j B$?

$s_j = s_j^{-1}$ since $s_j \in N(T)/T$.

$= b_j^- \sigma_j u^+ w_{j+1} s_j^{-1} s_j \cdots s_k$

$= b_j^- \sigma_j u^+ w_{j+1} s_{j+1} \cdots s_k.$

$U_1 \supset {}^{s_1}U_2 \supset \cdots \supset {}^{s_1 \cdots s_{j-1}} U_j \supset {}^{s_1 \cdots s_j} U_{j+1}$

${}^{s_1 \cdots s_{j+1}} U_{j+1}$

$U^+ = U_j (s_j \cdots s_k U^j s_k \cdots s_j)$

also $U^{j+1} = U^+ \cap {}^{s_k \cdots s_{j+1}} U^+$

pf. $u^+ w_{j+1} s_{j+1} \cdots s_k \in w_{j+1} s_{j+1} \cdots s_k U^{j+1}$.

$\boxed{1)}$ $u^+ w_{j+1} s_{j+1} \cdots s_k \in w_{j+1} s_{j+1} \cdots s_k U^+$

2) $u^+ w_{j+1} \underbrace{s_{j+1} \cdots s_k}_{z} \in w_{j+1} s_{j+1} \cdots s_k s_k \cdots s_{j+1} U^+ s_{j+1} \cdots s_k$

$\quad \| \quad w_{j+1} U^+ s_{j+1} \cdots s_k.$

2) $u^+ w_{j+1} \in w_{j+1} U^+$.

$\Leftrightarrow w_{j+1}^{-1} u^+ w_{j+1} \in U^+$ true!

1)

$\underbrace{w_{j+1}^{-1}}_{\in U^+} u^+ \underbrace{w_{j+1}}_{\in U^+} \in {}^{s_{j+1} \cdots s_k} U^+.$

$s_{j+1} \cdots s_k U^+ s_k \cdots s_{j+1}$

$\boxed{sU^+s}$

$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&a\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$

$= \begin{pmatrix}0&1\\-1&-a\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\-a&1\end{pmatrix} \in U^+$.

$U_j = U^+ \cap {}^{s_j \cdots s_k} U^-$

$w_{j+1} \in s_{j+1} \cdots s_k U^-_{s_k \cdots s_{j+1}}$ $\exists\, u^- \in U^-$ st.

$w_{j+1} = s_{j+1} \cdots s_k u_1^- s_k \cdots s_{j+1}$

$(w_{j+1})^{-1} = s_{j+1} \cdots s_k u_2^- s_k \cdots s_{j+1}.$

$\underbrace{w_{j+1}^{-1}}_{\in U^+} u^+ w_{j+1} =$

$\|$

$\tilde{u}^+ w_{j+1} = \tilde{u}^+ s_{j+1} \cdots s_k u^- s_k \cdots s_{j+1}.$

for some $\tilde{u}^+ \in U^+$.

$w_{j+1}^{-1} u^+ w_{j+1}$

$= s_{j+1} \cdots s_k (u_2^-)^{-1} s_k \cdots s_{j+1} u^+ s_{j+1} \cdots s_k u_1^- s_k \cdots s_{j+1}$

$\in {}^{s_{j+1} \cdots s_k} U^+$

$\Leftrightarrow (u_1^-)^{-1} s_k \cdots s_{j+1} u^+ s_{j+1} \cdots s_k (u_1^-) \in U^+$

$\|$

$\tilde{u}^+ w_{j+1} = \tilde{u}^+ s_{j+1} \cdots s_k u^- s_k \cdots s_{j+1} \in {}^{s_{j+1} \cdots s_k} U^+$

$w_{j+1}^{-1} u^+ w_{j+1}$ $/\!/$

$\Leftrightarrow \tilde{u}^+ s_{j+1} \cdots s_k u^- \in s_{j+1} \cdots s_k U^+$

$U^- = w_0 U^+ w_0$. $\quad U^- = w_0 U^+ w_0$.

$w_0 = s_1 \cdots s_k s_{k+1} \cdots s_n$.

$l(y w_0) = l(w_0) - l(y)$

$\to w_0 =$

$\tilde{u}^+ \in U^+$ (notation)

$u^+ s_{j+1} \cdots s_k \underbrace{s_k \cdots s_{j+1} s_{k+1} \cdots s_m}_{= w_0}$

$\Leftrightarrow s_k \cdots s_{j+1} u^+ s_{j+1} \cdots s_k u^- \in U^+$

$= s_k \cdots s_{j+1} u^+ (s_{k+1} \cdots s_m \tilde{u}^+ s_m \cdots s_{k+1}) s_{j+1} \cdots s_k \in U^+$

$\sigma_{j-1} U_j s_j \subset B^- \sigma_j B$

$\Rightarrow w U_j s_j \subset B^- w s_j B \qquad w s_j < w.$

$U_j \supset s_j U_{j+1} s_j.$

$U_j = U^+ \cap {}^{s_j \cdots s_k} U^-.$

$w U_j s_j \subset w U^+ s_j \cap w\, {}^{s_j \cdots s_k} U^- s_j$

$\cap ? \qquad \cap ?$

$B^- w s_j U^+ \qquad B^- w s_j\, {}^{s_j \cdots s_k} U^-$

$w U_j s_j \overset{?}{\subset} B^- w s_j U_j.$

need to prove

$w_{j+1}^{-1} u_j w_{j+1} \in {}^{s_{j+1} \cdots s_k} U^+.$

$(s_{j+1} \cdots s_k)(U^-)^{-1} s_j \cdots s_k \tilde{U}^- s_{j+1} \cdots s_k U^- \in {}^{s_{j+1} \cdots s_k} U^+$

$\Leftrightarrow (U^-)^{-1} s_j \cdots s_k U^- s_{j+1} \cdots s_k U$

$w_{j+1}^{-1} w_j u_{j+1} = s_{j+1} \cdots s_k U^+ s_k \cdots s_{j+1} \; \forall u^+$

$\Leftrightarrow \underbrace{s_k \cdots s_{j+1} u_{j+1}^{-1}}\, w_j\, \underbrace{u_{j+1} s_{j+1} \cdots s_k} \in U^+$

$s_k \cdots s_{j+1} \quad (U^-)^{-1} s_k \cdots s_{j+1} (u_j) s_{j+1} \cdots s_k U^- \in U^+.$

$u_{j+1} = s_{j+1} \cdots s_k U^- s_k \cdots s_{j+1}$

$(u_{j+1})^{-1} = s_{j+1} \cdots s_k (U^-)^{-1} s_k \cdots s_{j+1}.$

$u_j = s_j s_{j+1} \cdots s_k U^- s_k \cdots s_{j+1} s_j.$

$(U_j^-)^{-1} s_k \cdots s_{j+1} s_j s_{j+1} \cdots s_k U^- s_k \cdots s_{j+1} s_j s_{j+1} \cdots s_k U^-$

$\boxed{1)}\; U^+ U_{j+1} s_{j+1} \cdots s_k \in \tilde{U}_{j+1} s_{j+1} \cdots s_k U^{j+1}$

$$\boxed{U^+ U_{j+1} \in U_{j+1}\, {}^{s_{j+1} \cdots s_k} U^{j+1} = U^+ ?}$$

But $U^+ = U_{j+1}\, {}^{s_{j+1} \cdots s_k} U^{j+1}.$

$U^+ U_{j+1} \in U^+.$

$$\boxed{U^+ = U_j\, {}^{s_j \cdots s_k} U^j} \quad (*)$$ very important decomposition!

1) is true! for U_{j+1}

Should be not hard to prove!

$\boxed{\text{If } \quad w U^+ s \subset B^- w s U^+ \quad ws < w}$

$\boxed{U^+ U_{j+1} = U_{j+1} U^{j+1} \; ? \quad \text{should be true.}}$

Case $\sigma_{j-1} = \sigma_j \qquad UO$

$\Omega(\underline{\sigma}, j) = \Omega^* \qquad \sigma_{j-1} s_j > \sigma_{j-1} = \sigma_j.$

$(\sigma_{j-1}(\alpha_j) \in \Phi^+ \Leftrightarrow \sigma_{j-1} s_j > \sigma_{j-1})$

$(w(\alpha_j) \in \Phi^+ \Leftrightarrow w s_j > w) \to$ Prop 5.7 Humphreys.

let $t \in \Omega^*$, $u_{j+1} \in U_{j+1}$

$u_{\alpha_j}(t) \cdot u_{j+1} = \tilde{u}_{j+1} s_{j+1} \cdots s_k \cdot v_{j+1}^{-1} s_k \cdots s_{j+1} \quad (**)$

for unique $\tilde{u}_{j+1} \in U_{j+1}$ and $v_{j+1} \in U^{j+1}$ (2.2.(v).)

Define

$f_j : \Omega^* \times U_{j+1} \longrightarrow U_j$

$(t, u_{j+1}) \mapsto \overbrace{u_{\alpha_j}(t^{-1}) s_j \tilde{u}_{j+1} s_j}^{\in U_j} \in U_j$

pd

$\operatorname{Im} f_j = U_j \setminus {}^{s_j} U_{j+1}$

Consider $u_{\alpha_j}(d) \cdot s_j \tilde{u}_{j+1} s_j \qquad d \neq 0. \quad \tilde{u}_{j+1} \in U_{j+1}$

check u_{j+1}

$(*)$

$(*)\; u_{\alpha_j}(-d^{-1}) \cdot \tilde{u}_{j+1} = u_{j+1} s_{j+1} \cdots s_k v_{j+1} s_k \cdots s_{j+1}$ why? follows

with $u_{j+1} \in U_{j+1}$,

from $u_j(-t) = (u_j(t))^{-1}$

$(*)\; \Omega^* \longrightarrow \Omega^*$

$t \longmapsto t^{-1}$ is a morphism.

$(tg)^{-1} = t^{-1} g^{-1}.$

$f_j(d^{-1}, u_{j+1}) = u_{\alpha_j}(d) s_j \tilde{u}_{j+1} s_j^{-1}$

I cannot conclude from $(*)$ that $u_{\alpha_j}(d) s_j \tilde{u}_{j+1} s_j^{-1} \in U_j \setminus {}^{s_j} U_{j+1}.$

Check this! solved!

$f_j(-d^{-1}, \tilde{u}_{j+1}) = u_{\alpha_j}(-d) s_j u_{j+1} s_j$

solved!!

$\sigma_{j-1} u_{\alpha_j}(t^{-1}) s_j \tilde{u}_{j+1} s_{j+1} \cdots s_k$

$= \sigma_{j-1} u_{\alpha_j}(t^{-1}) s_j u_{\alpha_j}(t) u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$= \sigma_{j-1} h s_j u_{\alpha_j}(-t^{-1}) s_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$\exists h \in T \qquad = b_j^- \sigma_{j-1} u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$b_j^- = \sigma_{j-1} h s_j u_{\alpha_j}(-t^{-1}) s_j^{-1} \sigma_{j-1}^{-1} \in B^-$

$\square$

check this $\sigma_{j-1}(\alpha_j) \in \Phi^+$

Proof of Theorem 1.1

$\eta : U_1 \longrightarrow \mathcal{D}$

$U_1 \cong B y_0 B$

$u_1 \mapsto u_1 y_0 B.$

Define for $\underline{\sigma} \in \mathcal{D}$ fixed.

$$D_{\underline{\sigma}} = \{ u_1 y_0 B \in B y_0 B \mid \eta(u_1) = \underline{\sigma} \}$$

It is clear that

(i) $B y_0 B = \bigcup_{\underline{\sigma} \in \mathcal{D}} D_{\underline{\sigma}}$ is a disjoint union

(ii) Let $\underline{\sigma}$ fixed element of $\mathcal{D}$. Define $A_j \subseteq U_j$ $(1 \leq j \leq k+1)$ as follows.

$A_{k+1} = id.$

$A_j = f_j(\Omega(\underline{\sigma}, j) \times A_{j+1})$

$A_j \subset U_j$ is locally closed and iso to

$\Omega^{m_j(\underline{\sigma})} \times (\Omega^*)^{n_j(\underline{\sigma})}$ where

#DI's

$m_j(\underline{\sigma}) = \#\{ p \mid j \leq p \leq k \ \& \ \Omega(\underline{\sigma}, p) = \Omega \}$

$n_j(\underline{\sigma}) = \#\{ \underline{\qquad\qquad} = \Omega^* \}$

#0's = #00's $\quad m_1(\underline{\sigma}) = m(\underline{\sigma}), \ n_1(\underline{\sigma}) = n(\underline{\sigma}).$

$$A_1 \cong \Omega^{m(\underline{\sigma})} \times (\Omega^*)^{n(\underline{\sigma})}$$

claim $A_1 = \eta^{-1}(\underline{\sigma}) = D_{\underline{\sigma}}.$

Let $u_1 \in A_1$ then $\exists$ sequences $\{u_j\}_1^{k+1}$ and $\{t_j\}_1^k$ s.t. $\forall j$.

(a) $u_j \in A_j$

(b) $t_j \in \Omega(\underline{\sigma}, j)$

(c) $u_j = f_j(t_j, u_{j+1})$

This proves η is surjective!

Prop 3.2 $\Rightarrow$ $u_1 s_1 \cdots s_j \in B^- \sigma_j \circ B \ \forall j$

then by definition $\eta(u_1) = \underline{\sigma}$, or $A_1 \subseteq \eta^{-1}(\underline{\sigma})$

Conversely, let $u_1 \in \eta^{-1}(\underline{\sigma})$. We prove by induction on j that there exist sequences $\{u_j\}_1^{k+1}$ (starting with the given u_1) and $\{t_j\}_1^k$ st.

(a) $t_j \in \Omega(\underline{\sigma}, j)$

(b) $u_j = f_j(t_j, u_{j+1})$

To see this observe that the image of f_j is U_j (resp. ${}^{s_j}U_{j+1}, U_j \setminus {}^{s_j}U_{j+1}$) if $\sigma_{j-1} > \sigma_j$ (resp. $\sigma_{j-1} < \sigma_j$, $\sigma_{j-1} = \sigma_j$).

$u_1 \in \eta^{-1}(\underline{\sigma}) \Rightarrow u_1 \in \operatorname{Im} f_1$

$\eta(u_1) = \underline{\sigma} = (\sigma_0, \dots, \sigma_k).$

$u_1 \in B^- B.$

$u_1 s_1 \in B^- \sigma_1 B.$

U1 or U0

$\sigma_0 = id.$

$\sigma_1 \geq \sigma_0.$

$f_1 : \Omega(\underline{\sigma}, 1) \times U_2 \longrightarrow U_1$

$\{0\}$ or Ω^*

Case 1 (b) $\quad \sigma_{j-1} < \sigma_j \quad j = 1.$

$\sigma_0 < \sigma_1 = s_1.$

$f_1 : \{0\} \times U_2 \longrightarrow U_1$

$(0, u_2) \longmapsto s_1 u_2 s_1$

but the image in this case is ${}^{s_1}U_2.$

pf) $\eta(u_1) = \underline{\sigma} = (e, s_1, \dots)$

$\Rightarrow u_1 = s_1 u_2 s_1 \not\equiv u_2.$

but $u_1 s_1 \in B^- s_1 B$

$(U_2 = U^+ \cap {}^{s_2 \cdots s_k} U^-)$

$U_2 = \langle \alpha_2, s_2(\alpha_3), \dots, s_2 s_3 \cdots s_{k-1}(\alpha_k) \rangle$

$u_1 s_1 = b^- s_1 b$

$u_1 = b^- s_1 b s_1 \qquad U^2 = U^+ \cap {}^{s_k \cdots s_2} U^+$

By prop 3.2 (i)

$(s_1 u_2 s_1) s_1 \cdots s_k$

$= s_1 u_2 s_2 \cdots s_k = b_1^- s_1 u_2 s_2 \cdots s_k v_2$

for suitable $b_1^- \in B^-$ & $v_2 \in U^2$

$\forall u_2 \quad s_1 u_2 s_1 \in B^- s_1 u_2 s_2 \cdots s_k U^2 s_k \cdots s_1$

U1.

p.d] $s_1 U_2 = U_1$.

p.d] $s_1 U_2 s_1^{-1} \supseteq U_1$.

$\sigma_0 = e$.

$\sigma_1 = s$.

$\sigma_1 > \sigma_0$.

$e(\alpha_s) = \alpha_s \in \Phi^+$.

$U_1 \subset {}^{s_1}U_2$ iff

$\underbrace{s_1\left(U^+ \cap {}^{s_2 \cdots s_k} U^-\right) s_1}_{U_2} \supset {}^{s_1 \cdots s_k} U^- \cap U^+$

Let $u^+ \in U_1 = U^+ \cap {}^{s_1 \cdots s_k} U^-$.

$u^+ = s_1 \cdots s_k u^- s_k \cdots s_1 \in U^+$.

$= s_1 (s_2 \cdots s_k) u^- (s_k \cdots s_2) s_1 \in {}^{s_1}U^+$.

p.d] $(s_2 \cdots s_k) u^- (s_k \cdots s_2) \in U^+$.

~~$\forall u \in U^- / \exists$~~ $\cancel{U^- \subseteq {}^{s_k \cdots s_2} U^+}$

$\cancel{s_1(\alpha_s) = -\alpha_s}$.

$s_1(U_{\alpha_1}) = -U_{\alpha_1}$

$\underbrace{U^+ \cap {}^{s_1 \cdots s_k} U^-}_{U_1} \subseteq s_1 {}^{s_2 \cdots s_k} U^- s_1 = {}^{s_1 \cdots s_k} U^-$

$= s_1 (s_2 \cdots s_k) U^- (s_k \cdots s_2) s_1$ //

Suppose $\underbrace{s_1 (s_2 \cdots s_k) u^- (s_k \cdots s_2) s_1}_{= u^+} \in U^+$

$u^- \to u^+$.

$U^- \to U^+$. then prove that

$(s_2 \cdots s_k) u^- (s_k \cdots s_2) \in U^+$.

$u^- \in {}^{s_2 \cdots s_k} U^+$.

This is false but if $u_1 \in \eta^{-1}(\Sigma)$ then it is true.

$u_1 \in B^- B = w_0 B w_0 B$. $\quad U_1 \subseteq B y_0 B$

$u_1 s_1 \in B^- s_1 B = w_0 B w_0 s_1 B$

$U_2 = U^+ \cap {}^{s_2 \cdots s_k} U^-$.

SOLVED

$s_1 U_2 s_1 = s_1 U^+ s_1 \cap {}^{s_1 \cdots s_k} U^-$.

$u_1 \in B^- B$

$u_1 s_1 \in B^- s_1 B$

$u_1 s_1 \cdots s_k = u_1 y \in B^- \sigma_k B$.

$u_1 s_1 \notin B^- B = w_0 B w_0 B$.

$u_1 s_1 = b^- s_1 b$.

eg] SL_3. $\quad y = sts$. $\quad \alpha = \alpha_s \quad \alpha_2 = \alpha_3$

$s_1 = s, \; s_2 = t, \; s_3 = s$.

sts

$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ (e, & s, & s, & e) \end{array}$.

$1 \le j \le 4$. $\quad A_4 = \{id\} = U_4$.

$A_3 = f_3(\Omega(\Sigma, 3) \times A_4)$.

$U_1 \supset {}^{s}U_2 \supset {}^{st}U_3 \supset {}^{sts}U_4 = id$.

$U_1 = U^+ \cap {}^{sts}U^- = U^+ = \left\{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}\right\}$.

$U_2 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} = \langle \underset{\beta}{\alpha_2}, \underset{\alpha+\beta}{t(\alpha_3)} \rangle$.

$\begin{pmatrix} 1 & \alpha & \alpha+\beta \\ -\alpha & 1 & \beta \\ -\alpha-\beta & -\beta & 1 \end{pmatrix} \quad U_3 = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

U1 U0 D1

$\sigma_0 = e$, $\sigma_1 = s$, $\sigma_2 = s$, $\sigma_3 = e$.

$\Omega(\Sigma, 1) = \{0\}$. U1

$\Omega(\Sigma, 2) = \Omega^*$ U0.

$\Omega(\Sigma, 3) = \Omega$ D1.

$f_1: \{0\} \times U_2 \longrightarrow U_1$.

$u_2 \longmapsto s_1 u_2 s_1^{-1}$.

$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$.

who is $u_1 \in U_1$ here?

$u_1 \in U_1$ is s.t.

$u_1 \in B^- B = w_0 B w_0 B$

$Bw_0 \cdot B$ = B-orbit of F_{w_0}.

$F_{w_0} = w_0 \{\text{standard flag}\}$.

$w_0 \{ 0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \mathbb{C}^3 \}$.

$w_0 = sts$. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $w_0 e_1 = e_3$, $w_0 e_2 = e_2$, $w_0 e_3 = e_1$

(12)(23)(12)

$(12)\,id = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$(23)(12)\,id = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$F_{w_0} = \{ \langle e_3 \rangle \subset \langle e_3, e_2 \rangle \subset \mathbb{C}^3 \}$

$\dim(Bw_0 \cdot B) = 3$

$Bw_0 \cdot B \cong \mathbb{C}^3$

SL_2

$\begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$.

$B^- B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

$sBsB = \begin{pmatrix} \lambda & 0 \\ \lambda^{-1} a & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & \lambda b \\ \lambda^{-1} a & \lambda^{-1} ab + \lambda^{-1} \end{pmatrix}$

$BsB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & \lambda b \\ \lambda^{-1} a & \lambda^{-1} ab + \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} a & \lambda^{-1} ab + \lambda^{-1} \\ \lambda & \lambda b \end{pmatrix}$

$sBsB \cong \mathbb{C}^* \times \mathbb{C}^2$. $B = \mathbb{C}^* \times \mathbb{C}$, $\dim B = 2$.

$sBs \cdot B \cong \mathbb{C}$

$sBsB = \begin{pmatrix} \lambda \neq 0 & \lambda b \\ \lambda^{-1} a & \text{---} \end{pmatrix}$ dim = 3.

$sB = \begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & a \end{pmatrix}$ dim = 2

$U^+ = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

SL_3

$u_1 \in B^- B$ always.

$\underline{i} = (t, s, s, t)$

$B^- B \supset U^+ \cdot U U^-$.

$u_1 \in U^+$ ✓.

$u_1 s_1 \in B^- s_1 B$. $\Leftrightarrow$ $u_1 s_1 \notin B^- B$.

$u_1 s_1 t \in B^- s_1 B = sts B st B$.

$u_1 st \in B^- s_1 B$

$u_1 sts \in B^- B$. Difficult. lets follow Deodhar's idea.

lets build $A_1 \subset U_1$, $A_2 \subset U_2$, $A_3 \subset U_3$, $A_4 \subset U_4$ then any $u_1 \in A_1$ satisfy all conditions!

$\mathbb{C}(\underline{i}, 1) = \{0\}$

$\mathbb{C}(\underline{i}, 2) = \mathbb{C}^*$

$\mathbb{C}(\underline{i}, 3) = \mathbb{C}$

$\alpha_3 = \alpha_{s_3} = \alpha_1$

$f_1 : \{0\} \times U_2 \longrightarrow U_1 = U^+$

$(0, u_2) \longmapsto s_1 u_2 s_1^{-1}$

$f_2 : \mathbb{C}^* \times U_3 \longrightarrow U_2$

$(t, u_3) \longmapsto u_{\alpha_2}(t^{-1}) s_2 \tilde{u}_3 s_2^{-1}$

$f_3 : \mathbb{C} \times U_4 \longrightarrow U_3$

$(t, u_4) \longmapsto u_{\alpha_3}(t) s_3 u_4 s_3^{-1}$

$A_4 = id$.

$A_3 = f_3(\mathbb{C} \times A_4) = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = U_3$

$A_2 = f_2(\omega\Omega^* \times A_3)$

$s t(\alpha_1)$

$s(\alpha+\beta)$

$= -\alpha+\alpha+\beta = \beta$.

$\tilde{u}_3 \in U_3 = ?$

$\cancel{u_{\alpha_3} = u_{\alpha_3}(t) \cdot u_4}$

$u_{\alpha_2}(t) \cdot u_3 = \tilde{u}_3 \, s \, v_3^{-1} s$.

for unique $\tilde{u}_3 \in U_3$, $v_3 \in U^3$.

$U_1 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $U_2 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$

$U_3 = \left\{ \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ $U_4 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

$U^4 = \langle \alpha \mid \alpha \in \Phi^+ \rangle = U^+$.

$U^3 = \langle \overset{\Phi^+}{\alpha} \mid s(\alpha) \in \Phi^+ \rangle \subset \Phi^+$.

$s(\alpha_3) = -\alpha_3$, $s(\alpha+\beta) = -\alpha+\alpha+\beta = \beta$.

$s(\beta) = \alpha+\beta \in \Phi^+$.

$U^3 = \{\alpha+\beta, \beta\} = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

$U^2 = \{\alpha \mid ts(\alpha) > 0\}$.

$ts(\beta) = t(\alpha+\beta) = \alpha + \cancel{\beta} - \cancel{\beta}$
$= \alpha \in \Phi^+$.

$ts(\alpha+\beta) = t(\beta) = -\beta$ ×.

$ts(\alpha) = t(-\alpha) = -\alpha-\beta$ ×.

$U^2 = \{\beta\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

$\cup$

$U^1 = \{\alpha \mid sts\alpha > 0\} \subset U^2$.

U^1, $(sts)\beta = s(\alpha) \in \Phi^-$ ×.

$U^1 = \{id\}$.

$\uparrow$

$\boxed{\text{Computation of } U^1 \text{ for } SL_3, \; y = sts}$

$u_{\alpha_2}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$ $t \neq 0$. $t \in \omega\Omega^*$.

$u_{\alpha_2}(t) \cdot u_3 = \tilde{u}_3 \, s \, v_3^{-1} s^{-1}$.

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\in u_3} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$.

$v_3^{-1} = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

$s v_3 s^{-1} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}}$ $\begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & +1 & +d \\ -1 & 0 & -c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\parallel$

$\begin{pmatrix} 1 & 0 & d \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$

$\tilde{u}_3 s v_3^{-1} s^{-1} = \begin{pmatrix} 1 & \tilde{a} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & \tilde{a} & d - c\tilde{a} \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ & 1 & t \\ & & 1 \end{pmatrix}$

$\Rightarrow \tilde{a} = a$

$c = -t$.

$d = c\tilde{a} = -ta$. $\therefore v_3^{-1} = \begin{pmatrix} 1 & 0 & -t \\ 0 & 1 & -ta \\ 0 & 0 & 1 \end{pmatrix}$

$\tilde{u}_3 = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = u_3$.

$\left\{ \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \right\}$

$\parallel$

$A_2 = f_2(\omega\Omega^* \times A_3)$

$A_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right.$

$t \neq 0$

$\therefore A_2 \cong \omega\Omega \times \omega\Omega^*$.

$$A_2 = \left\{ \begin{pmatrix}1&0&a\\0&1&b\\0&0&1\end{pmatrix} \middle| \begin{matrix} a\in\Omega \\ b\in\Omega^* \end{matrix} \right\}$$

$$f_1 : \{0\}\times U_2 \longrightarrow U_1$$
$$(0, w_2) \longmapsto s_1 u_2 s_1^{-1}$$

$$A_1 = f_1(\{0\}\times A_2).$$

$$s_1 a_2 s_1^{-1} = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&a\\0&1&b\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$$

$$\cancel{\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&a\\0&1&b\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}}$$

$$\cancel{\begin{pmatrix}1&0&a\\0&0&1\\0&-1&-b\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix} = \begin{pmatrix}1&a&0\\0&1&0\\0&-b&1\end{pmatrix}}$$

$$= \begin{pmatrix}0&1&b\\-1&0&-a\\0&0&1\end{pmatrix}\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&b\\0&1&-a\\0&0&1\end{pmatrix}$$

$b \neq 0.$

$$A_1 = \left\{ \begin{pmatrix}1&0&b\\0&1&a\\0&0&1\end{pmatrix} \middle| \begin{matrix} b\in\Omega^* \\ a\in\Omega \end{matrix} \right\} = D_\Gamma.$$

$u_1 = \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$ does the work?

pf $u_1 = s_1 u_2 s_1^{-1}$ $\nexists\, u_2 \in U_2 = \begin{pmatrix}1&0&*\\0&1&*\\0&0&1\end{pmatrix}.$

I only know "a priori" that

$u_1 \in \bar{B}B$

$u_1 = \begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}$, $u_1 s \in B^- s B$ $\quad \bar{B}sB$

$$u_1 s = \begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix} = \cancel{\begin{pmatrix}0&1&c\\-1&-a&-b\\0&0&1\end{pmatrix}}$$
$$= \begin{pmatrix}-a&1&b\\1&0&c\\0&0&1\end{pmatrix}$$

$$\bar{B}sB = sts\, Bsts\cancel{s}B = sts\,[BstB].$$

$BstB$. $\quad st = \begin{pmatrix}0&0&1\\-1&0&0\\0&-1&0\end{pmatrix}$

$B = TU^+$

$$BstB = TU^+stU^+.$$

$$h = \begin{pmatrix}\lambda&0&0\\0&\mu&0\\0&0&\frac{1}{\lambda\mu}\end{pmatrix} := \begin{pmatrix}a_1&0&0\\0&a_2&0\\0&0&a_3\end{pmatrix}$$

$$BstB = h\cdot\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\-1&0&0\\0&-1&0\end{pmatrix}\begin{pmatrix}1&x&y\\0&1&z\\0&0&1\end{pmatrix}$$

$$= h\cdot\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\-1&-x&-y\\0&-1&-z\end{pmatrix}$$

$$= h\cdot\begin{pmatrix}-a&-xa-b&1-ya-zb\\-1&-x-c&-y-zc\\0&-1&-z\end{pmatrix}$$

$$= \begin{pmatrix}-\lambda a&-\lambda xa-\lambda b&\lambda-\lambda ya-\lambda zb\\-\mu&-\mu x-\mu c&-\mu y-\mu zc\\0&-\frac{1}{\mu\lambda}&-\frac{z}{\mu\lambda}\end{pmatrix}$$

$BstB$.

$\|$

$$B^- sB = sts\, BstB = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} BstB$$

$$= \begin{pmatrix}0&-\frac{1}{\mu\lambda}&-\frac{z}{\mu\lambda}\\-\mu&-\mu x-\mu c&-\mu y-\mu zc\\-\lambda a&-\lambda xa-\lambda b&\lambda-\lambda ya-\lambda zb\end{pmatrix}$$

$\|$

$$\begin{pmatrix}-\tilde{a}&1&\tilde{b}\\+1&0&\tilde{c}\\0&0&1\end{pmatrix} \quad \cancel{\begin{pmatrix}0&1&\tilde{c}\\-1&-\tilde{a}&-\tilde{b}\\0&0&1\end{pmatrix}}$$

$+1 = -\mu$, $\boxed{\mu = -1}$, $-\lambda a = 0$.

$\lambda \neq 0$ $\boxed{a = 0}$

$\boxed{\lambda = 1}$ $\frac{1}{\lambda\mu} = -1$ $\boxed{b = 0}$

$-\mu x = \mu c$.
$x = -c$.

$$= \begin{pmatrix} 0 & +1 & z \\ 1 & 0 & y - z^2 \\ 0 & 0 & 1 \end{pmatrix}$$

$\tilde{a} = 0$.
$\tilde{b} = z$.
$\tilde{c} = -\mu y - \mu z c$
$= y + zc = y - z^2$

I was hoping for the condition $\tilde{b} \neq 0$ but did not appear $u_1 s \in B^- s B$ not enough info.

At least I got $\tilde{a} = 0$.
$\tilde{b} \neq 0$ should come from $u_1 st \in B^- s B$, let's see.

$$u_1 = \begin{pmatrix} 1 & \tilde{a} & \tilde{b} \\ 0 & 1 & \tilde{c} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -\tilde{a} & -\tilde{b} & 1 \\ -1 & -\tilde{c} & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$\tilde{a} = 0$

$a = 0$.

I conclude here the non-trivial observation that $u_1 s_1 \in B^- s B$ & $u_1 st \in B^- s B$ forces to $u_1 \in A_1$ as defined before

$A_1 = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{matrix} b \in \Omega^* \\ a \in \Omega \end{matrix} \right\}$

$\boxed{a = 0}$ $\lambda \neq 0$. $\lambda \neq 0$.

$\boxed{\tilde{a} = 0}$ $\lambda = \lambda z b$ $z \neq 0$
$b \neq 0$

$\lambda(1 - zb) = 0$.
$1 = zb$.

$\mu = 1$. $+\lambda b = +1$, $b = \frac{1}{\lambda}$.

$z = \lambda$.

$\frac{-z}{\lambda}$. $c = 1 + y$

$-\tilde{b} = \frac{-1}{\mu\lambda}$ then $\tilde{b} \neq 0$.
Got it!

However, to determine $\mathrm{Im}\, f_1 \ni u_1$ it suffices with $u_1 s \in B^- s B$ in this example.

$\mathrm{Im}\, f_1 = {}^{s_1}U_2 = s_1 (U^+ \cap {}^{ts}U^-) s_1^{-1}$

$= sU^+s^{-1} \cap {}^{sts}U^-$.

$u_1 s \in B^- s B = sts\, B s^{-1} t^{-1} B$.
$u_1 \in sts\, B s^{-1} t^{-1} B s^{-1}$. $u_1 \in U^+$ clearly.

$B^- s B = U^- s B$, $u_1 s \in U^- s B$

pd) $u_1 = s_1 u_2 s_1^{-1}$ $u_2 \in U_2 = U^+ \cap {}^{ts}U^-$.
$u_1 \in s_1 U_2 s_1^{-1}$
$\Rightarrow s_1^{-1} u_1 s_1 \in U_2 = U^+ \cap {}^{ts}U^-$.

1) $s_1^{-1} u_1 s_1 \in U^+$
2) $s_1^{-1} u_1 s_1 \in ts\, U^- s^{-1} t^{-1}$

$\underbrace{u_1 s_1 \in B^- s B}_{\text{use this somewhat}}$.

$u_1 \in U^+ \cap {}^{sts}U^-$.

$u_1 = sts\, u^- s^{-1} t^{-1} s^{-1}$ $\exists u^-$.

then $s^{-1} u_1 s = ts u^- s^{-1} t^{-1} \in ts U^- s^{-1} t^{-1}$

we proved 2).

1) seems false in general, but let us use our extra data $u_1 s_1 \in B^- s B = U^- s B$

$s_1^{-1} u_1 s_1 = ts\, u^- s^{-1} t^{-1}$

$s^{-1} u_1 s_1 \in s^{-1} B^- s B = s^{-1} sts\, B s^{-1} t^{-1} \cancel{s}\, \cancel{s} B$
$= ts B s^{-1} t^{-1} B$.
$= ts B s t B$.

$u_1 = b^- s_1 b s_1^{-1}$

$s_1^{-1} u_1 s_1 = s_1^{-1} b^- s_1 b$

$(\times)$ ~~I'll come back to this later.~~ Solved!

$D_{\underline{\sigma}} = \{ u_1 y_\circ B \in B y_\circ B \mid \eta(u_1) = \underline{\sigma} \}$

$A_1 \subseteq D_{\underline{\sigma}}$. Since for $u_1 \in A_1 = f_1(\Omega(\underline{\sigma}, 1) \times A_2)$
$\exists u_2 \in A_2$ $\exists t_2 \in \Omega(\underline{\sigma}, 1)$ st. $u_1 = f_1(t_2, u_2)$
$u_2 \in A_2 = f_2(\Omega(\underline{\sigma}, 2), A_3)$ etc...
then $\exists \{u_j\}_1^{k+1}, \{t_i\}_1^k$ st.
(a) $u_j \in A_j$, (b) $t_j \in \Omega(\underline{\sigma}, j)$, (c) $u_j = f_j(t_j, u_{j+1})$.

$\boxed{\text{Prop 3.1}}$ & $\eta(u_1)=\sigma$ $\Rightarrow$ $u_1 s_1 \cdots s_j \in B^- \sigma_j B \quad \forall j$.

$u_1 \in U^+ \subset B^- B$, $u_1 s_1 \overset{?}{\in} B^- \sigma_1 B$.

(*) $\sigma_{j-1} f_j(t_j, u_{j+1}) s_j \cdots s_k = b_j^- \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$\forall b^- \in B^-$, $v_{j+1} \in U^{j+1} \subset U^+$.

$u_j = f_j(t_j, u_{j+1})$, $t_j \in \Omega(\sigma, j)$

$j=1$

$\sigma_0 \cdot f_1(t_1, u_2) s_j \cdots s_k = b_1^- \sigma_1 u_2 s_{j+1} \cdots s_k v_2$.

$u_1 \in B^- B^+$ since $u_1 \in U^+ \subset B^- B$

$\sigma_0 = id$ works!

$u_1 s_1 \overset{?}{\in} B^- \sigma_1 B$. lets use (*).

$u_1 s_1 \cdots s_k = b_1^- \sigma_1 u_2 s_2 \cdots s_k v_2 \quad \forall v_2 \in U^2$.

$u_1 y \overset{?}{\in} B^- \sigma_k B$

$u_1 s_1 \cdots s_j \overset{?}{\in} B^- \sigma_j B$.

$\sigma_{j-1} u_j s_j \cdots s_k = b_j^- \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$\forall b_j \in B^-, \forall v_{j+1} \in U^{j+1}$

eg $y = sts$.

$u_1 s_1 s_2 s_3 = u_1 sts \overset{?}{\in} B^- \sigma_3 B$

$u_1 sts = ?$, $\sigma_0 f_1(t_1, u_2) sts =$

$= b_1^- \sigma_1 u_2 ts$.

$\sigma_0 = id$.

$u_1 sts = b_1^- \sigma_1 u_2 ts v_2 \overset{?}{\in} B^- \sigma_3 B$

$\boxed{j=3}$ $\sigma_2 u_3 s = b_3^- \sigma_3 u_4 v_4 \quad \forall b_3, v_4$

$f_k(t_k, u_{k+1})$

$\sigma_{k-1} \cdot \overset{\shortparallel}{u_k} s_k = \cancel{b_k^- \sigma_k u_{k+1} v_{k+1}}$.

~~[illegible]~~ $v_{k+1} \in U^+ = U^{k+1}$

$\boxed{y = sts}$

$u_1 = b_1^- \sigma_1 u_2 ts v_2 sts$

$u_1 s = b_1^- \sigma_1 u_2^{ts} v_2$.

$U^2 = U^+ \cap {}^{st} U^+$.

$\boxed{id}$

$u_1 \in B^- B$. ✓

$u_1 s_1 \in B^- \sigma_1 B$

$u_1 st \in B^- \sigma_2 B$

$u_1 sts \in B^- \sigma_3 B$ ✓

$\boxed{\text{we have}}$ $u_1 sts = b_1^- \sigma_1 u_2 ts v_2$.

$\sigma_1 u_2 ts = b_2^- \sigma_2 u_3 s v_3$

$\sigma_2 u_3 s = b_3^- \sigma_3 u_4 v_4$.

$u_4 = u_4$

$U^4 = U^+$ $\quad u_4 = u^+$.

$u_1 sts = b_1^-$

$u_1 sts = b_1^- b_2^- b_3^- \sigma_3 \underbrace{u^+ v_4 v_3 v_2}_{\in U^+} \in B^- \sigma_3 U^+$ ✓

$u_1 st = b_1^- \sigma_1 u_2 ts v_2 s^{-1}$

$= b_2^- \sigma_2 \underset{\in U^+}{u_3} s v_3 v_2 s^{-1} \in B^- \sigma_2 U^+$

$\underbrace{\in U^+ \cap {}^s U^+}_{\in U^+}$ $\subset B^- \sigma_2 B$

Lets prove using (*) that $u_1 \in A_1$ is in D_σ (i.e. $\eta(u_1) = \sigma$).

i.e. we prove $u_1 s_1 \cdots s_j \in B^- \sigma_j B \quad \forall j \in \{0, \dots, k\}$.

First we prove $j = k$.

by (*) $u_1 s_1 \cdots s_k = b_1^- \sigma_1 u_2 s_2 \cdots s_k v_2$

$\sigma_1 u_2 s_2 \cdots s_k = b_2^- \sigma_2 u_3 s_3 \cdots s_k v_3$.

$\sigma_2 u_3 s_3 \cdots s_k = b_3^- \sigma_3 u_4 s_4 \cdots s_k v_4$

$\Rightarrow$

$u_1 s_1 \cdots s_k = b_1^- b_2^- b_3^- \cdots b_k^- \sigma_k u_{k+1} v_{k+1} \cdots v_4 v_3 v_2$.

But $U^{k+1} \supset U^k \supset U^{k-1}$ then $v_{k+1} \cdots v_2 \in U^{k+1} = U^+$

$u_{k+1} \in U^+$

$b_1^- \cdots b_k^- \in B^-$

then $u_1 s_1 \cdots s_k \in B^- \sigma_k B$.

$(\in B^- \sigma_k U^+)$.

In the same way we get

(LHS=) $u_1 s_1 \cdots s_{k-1} = b_1^- \sigma_1 u_2 s_2 \cdots s_{k-1} (s_k v_2 s_k^{-1})$

$\sigma_1 u_2 s_1 \cdots s_{k-1} = b_2^- \sigma_2 u_3 s_3 \cdots s_{k-1} (s_k v_3 s_k^{-1})$

etc.... then

$$\text{LHS} = b_1^- b_2^- \cdots b_{k-1}^- \sigma_{k-1} u_k \underbrace{(s_k v_k s_k^{-1}) \cdots (s_k v_2 s_k^{-1})}$$

$= s_k v_k' s_k^{-1}$ abuse of notation

$= s_k v_k s_k^{-1}$

$$u_1 \in B^- \sigma_{k-1} u_k \overbrace{s_k v_k s_k^{-1}}^{\in U^+}$$

$s_k v_k s_k^{-1} \in s_k U^k s_k^{-1} = s_k U^+ s_k^{-1} \cap s_k (s_k U^+ s_k^{-1}) s_k^{-1}$

$v_k \in U^k = U^+ \cap {}^{s_k}U^+ \subseteq {}^{s_k}U^+ = s_k U^+ s_k^{-1}$

$s_k^2 = id \bmod T$, $s_k^2 \in T$.

$s_k v_k s_k^{-1} \in s_k^2 U^+ s_k^{-2} \in T U^+ T = U^+$.

then $u_1 s_1 \cdots s_{k-1} \in B^- \sigma_{k-1} B$.

$U^j = U^+ \cap {}^{s_k \cdots s_j} U^+$

$s_j \cdots s_k U^j s_k^{-1} \cdots s_j^{-1} \subseteq \overbrace{(s_j \cdots s_k)(s_k \cdots s_j)}^{\in T} U^+ \underbrace{(s_j^{-1} \cdots s_k^{-1})(s_k^{-1} \cdots s_j^{-1})}_{\in T}$

then

$s_j \cdots s_k U^j s_k^{-1} \cdots s_j^{-1} \subseteq T U^+ T = U^+$.

$\therefore u_1 \cdots u_{j-1} \in B^- \sigma_{j-1} B$ $\square$.

then $\eta(u_1) = \underline{\sigma}$. $A_1 \subseteq \eta^{-1}(\underline{\sigma})$

Conversely, let $u_1 \in \eta^{-1}(\underline{\sigma})$.

We'll prove by induction on j that

$\exists \{u_j\}_1^{k+1}$ (starting with given u_1)

and $\{t_j\}_1^k$ s.t

(a) $t_j \in \mathbb{C}\Omega_3(\underline{\sigma}, j)$

(b) $u_j = f_j(t_j, u_{j+1})$

To see this, observe we have 3 cases

(1) $\mathrm{Im}\, f_j = U_j$ D1

(2) $\mathrm{Im}\, f_j = {}^{s_j}U_{j+1}$ U1

(3) $\mathrm{Im}\, f_j = U_j \setminus {}^{s_j}U_{j+1}$. U0.

$u_1 \in \eta^{-1}(\underline{\sigma}) \overset{?}{\Rightarrow} u_1 \in \mathrm{Im}\, f_1$.

codim $B^- s B s \subset B^- B$ difference 1.

Case $i=1$. $\sigma_{i-1} > \sigma_i$, $\sigma_i = s \in S$.

$A = U_1 \cap B^- s B s$, $A \subset U_1$ diff $= 1$??

$$U_1 \supseteq {}^{s_1}U_2, \quad U_1 \supseteq A$$

SAME ROOTS

pf) ${}^{s_1}U_2 \supseteq A$. ~~(*) Pending~~ solved).

then $\exists!$ $t_1 \in \mathbb{C}\Omega_3(\underline{\sigma}, 1)$, $u_2 \in U_2$ s.t.

$u_1 = f_1(t_1, u_2)$. Having defined $u_1, \ldots, u_j$

and $t_1, \ldots, t_{j-1}$ in the same (Pending) way

$u_j \in \mathrm{Image}\, f_j$ (or else $u_1 \notin \eta^{-1}(\underline{\sigma})$).

$\underbrace{\sigma_{k-1} f_k(t_k, u_{k+1})}_{u_k} s_k = b_k^- \sigma_k u_{k+1} v_{j+1}$

$A_{k+1} = id$ since $A_{k+1} \subset U_{k+1} = id$.

$A_{k+1} = id$, $f_k(\mathbb{C}\Omega(\underline{\sigma}, k) \times A_{k+1}) = A_k$.

$u_k = f_k(t_k, u_{k+1})$

eg SL_3 $\sigma = (e, s, s, e) \in \mathcal{D}$. $y = sts$

$u_1 \in \eta^{-1}(\sigma)$, $u_1 =$

$$= \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, \begin{matrix} b \in \mathbb{C}^* \\ a \in \mathbb{C} \end{matrix} \right\} \qquad A_1 = \mathbb{C}^* \times \mathbb{C}$$

$U_1 = U^+$, $U_2 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $a_2 = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ $a_2 \in U_2$.

$s_1 a_2 s_1^{-1} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$ $u_1 = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

$$u_1 = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \quad b \in \Omega^* \quad a \in \Omega \qquad u_2 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

$$s u_2 s^{-1} = u_1 = f_1(0, u_2) \qquad t_1 = 0.$$

$u_3 = ?$

$$f_2 : \Omega^* \times U_3 \longrightarrow U_2$$

$$(t, u_3) \longmapsto u_{\alpha_2}(t^{-1})\, s_2 \tilde{u}_3 s_2^{-1}$$

$f_2(t_2, u_3) = u_2$. Let's see how

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t^{-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tilde{u}_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$j = 2$

$$u_{\alpha_2}(t) \cdot u_3 = \tilde{u}_3\, s_3\, v_3^{-1} s_3^{-1}$$

for unique $\tilde{u}_3 \in U_3$, $v_3^{-1} \in U^3$.

$$U_3 = \left\{ \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad U^3 = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$u_{\alpha_2}(t) \cdot u_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \tilde{a} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \tilde{a} & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \tilde{U}_3 \cdot \underbrace{\begin{pmatrix} 1 & * & * \\ * & * & * \\ * & * & 1 \end{pmatrix}}_{s U^3 s^{-1}} \quad \text{etc.}$$

$s_1 U^3 s_1^{-1} = ?$

$$\cancel{\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} 0 & 1 & b \\ -1 & 0 & -a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} =$$

$$\cancel{\begin{pmatrix} 1 & 0 & a \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a & \\ 0 & 0 & \\ 0 & & \end{pmatrix}}$$

$$s_3 U^3 s_3^{-1} = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } u_3 = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = u_{\alpha_1}(a)$$

$$u_{\alpha_2}(t) \cdot u_3 = \begin{pmatrix} 1 & \tilde{a} & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & \tilde{a} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{u}_3} \cdot \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \tilde{a} & x + y\tilde{a} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$s U^3 s^{-1} = U^3$$

$$\tilde{u}_3 = u_3$$

$$f_2 : \Omega^* \times U_3 \longrightarrow U_2$$

$$(t, u_3) \longmapsto u_{\alpha_2}(t^{-1})\, s_2 \tilde{u}_3 s_2^{-1} \qquad \tilde{u}_3 = u_3$$

$$s_2 u_3 s_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$u_{\alpha_2}(t^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

$$f_2(t, u_3) = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & t^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq a.$$

$$u_1 = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \qquad b \in \Omega^* \Rightarrow b^{-1} \in \Omega^* \qquad a \in \Omega.$$

$$u_2 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \qquad u_1 = f_1(0, u_2)$$

$$f_2(b^{-1}, u_{\alpha_1}(a)) = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = u_2$$

$$u_2 = f_2(t_2, u_3)$$

$$t_2 = b^{-1}, \quad u_3 = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$u_1 = f_1(0, u_2)$$
$$= f_1(0, f_2(t_2, u_3))$$

$$f_3 : \Omega \times U_4 \longrightarrow U_3$$
$$(t, u_4) \longmapsto u_{\alpha_1}(t)\, s\, u_4\, s^{-1}$$

$$\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad f_3(a, id) = u_3.$$

$$U_4 = id. \qquad \boxed{\begin{array}{c|c|c} t_1 & t_2 & t_3 \\ \shortparallel & \shortparallel & \shortparallel \\ 0 & b^{-1} & a \end{array}}$$

$$u_1 = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

$$u_1 = u_{\alpha_1+\alpha_2}(b) \cdot u_{\alpha_2}(a)$$

$$u_2 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = u_{\alpha_1+\alpha_2}(-a)\, u_{\alpha_2}(b)$$

$$u_3 = u_{\alpha_1}(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

~~(*) Pending~~ other try: $\quad u_1 s \in B^- s B \Rightarrow u_1 \in B^- s B s^{-1}$

$${}^{s}U_2 \subset U_1 \qquad E = U_1 \cap B^- s B s^{-1}.$$

$$E \subset U_1 \qquad \text{DONE} \quad \overset{\shortparallel}{{}^{s}U_2}\,! \quad \text{(same roots)}$$

$$B^- s B s^{-1} = sts\, B\, s^{-1} t^{-1} \cancel{s}\, \cancel{s}\, B s^{-1}$$
$$= T\, sts\, U^+ s^{-1} t^{-1}\, U^+ s^{-1}$$

$$s^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$sts = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$sts = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}}_{st} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$s^{-1} t^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(sts\, U^+ s^{-1} t^{-1}\, U^+ s^{-1} \cap U_1) = ?$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{sts} \underbrace{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}}_{U^+} \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{s^{-1}t^{-1}} \underbrace{\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{U}^+} \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{s^{-1}}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -c \\ 1 & a & b \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & x & y \\ 0 & 1 & z \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & z \\ -1 & -x-c & -y-cz \\ a & xa+b & 1+ay+bz \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & z \\ -x-c & 1 & -y-cz \\ xa+b & -a & 1+ay+bz \end{pmatrix} \cap U^+$$

$$\Rightarrow \begin{array}{l} a = 0. \\ b = 0. \\ -x = +c. \\ -c = x \end{array} \qquad \begin{pmatrix} 1 & \overset{\shortparallel}{0} & z \\ 0 & 1 & -y+xz \\ 0 & 0 & 1 \end{pmatrix} \in U_2.$$

general case $y = s_1 \cdots s_k$

$$U_1 \supseteq {}^{s_1}U_2 \supseteq U_1 \cap B^- s_1 B s_1^{-1}$$

$U_2 = U^+ \cap {}^{s_2 \cdots s_k}U^-$

$U_1 = U^+ \cap {}^{s_1 \cdots s_k}U^-$

$U_1 \cap B^- s B s^{-1} = U^+ \cap (s_1 \cdots s_k U^- s_k^{-1} \cdots s_1^{-1}) \cap U^- s U^+ s^{-1}$

Better: $U_1 = \langle U_\alpha \mid \alpha \in \{\alpha_1, s_1(\alpha_2), \dots, s_1 \cdots s_{k-1}(\alpha_k)\} \rangle$.

$U_1 = \prod U_\alpha$. $u_1 \in U_1$ $u_1 = u_{\alpha_1}(t_1)\, u_{s(\alpha_2)}(t_2) \cdots$

$u_1 = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{pmatrix}$ $U_1 \cong \mathbb{A}^k$. $U_j \cong \mathbb{A}^{k-j+1}$.

$u_1 \in B^- s_1 B s_1^{-1} \Rightarrow u_1 \in B^- B$. So no info?

$\boxed{\begin{array}{l} \text{(c)}\ u_1 s_1 \in B^- e B \ (\text{ie } \sigma_1 = e) \quad u_1 \in B^- B \ (\sigma_0 = id) \quad \sigma_0 = \sigma_1 \\ u_1 \in B^- e B s_1 \quad e s_1 > \\ \quad \Rightarrow u_1 \in B^- B \text{ no contradiction } \checkmark \end{array}}$

SL_3

$B^- B = T \begin{pmatrix} 0&0&1 \\ 0&-1&0 \\ 1&0&0 \end{pmatrix} \begin{pmatrix} 1&a&b \\ 0&1&c \\ 0&0&1 \end{pmatrix} s^{-1} t^{-1} s^{-1} \begin{pmatrix} 1&x&y \\ 0&1&z \\ 0&0&1 \end{pmatrix}$

$= sts\, B\, s^{-1} t^{-1} s^{-1} B$

$s^{-1} t^{-1} = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$, $s^{-1} t^{-1} s^{-1} = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix} \begin{pmatrix} 0&-1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$

$= \begin{pmatrix} 0&0&1 \\ 0&-1&0 \\ 1&0&0 \end{pmatrix}$

$s^{-1} t^{-1} s^{-1} = sts = (sts)^{-1}$

$w_0 = w_0^{-1}$?

$BB = T \cdot \begin{pmatrix} 0&0&1 \\ 0&-1&0 \\ 1&0&0 \end{pmatrix}$

$U^- U^+ = \begin{pmatrix} 0&0&1 \\ 0&-1&-c \\ 1&a&b \end{pmatrix} \begin{pmatrix} 0&0&1 \\ 0&-1&-z \\ 1&-x&-y \end{pmatrix} = \begin{pmatrix} 1&-x&-y \\ -c&1+xc&z+yc \\ b&-a-xb&1-za-yb \end{pmatrix}$

$U^+ \cap U^- U^+ = \begin{pmatrix} 1&-x&-y \\ 0&1&z \\ 0&0&1 \end{pmatrix} = U^+$.

$b = 0$
$a = 0$
$c = 0$.

$U^+ \subset U^- U^+$
but that is obvious

$U^+ \subset B^- B$, $U^+ \subset B U^+ \subset B^- B$.

$u_1 \in B^- B$ no info

$u_1 \in B^- s_1 B s_1$ lot of info

and $\Rightarrow B^- B$. in particular.

pf) $U^+ \cap (s_1 \cdots s_k U^- s_k^{-1} \cdots s_1^{-1}) \cap U^- s U^+ s^{-1}$
$\subseteq {}^{s_1}U_2$.

let $u \in$

1) $u = s_1 \cdots s_k U^- s_k^{-1} \cdots s_1^{-1}$ $\forall U^-$

2) $u = \tilde{U}^- s U^+ s^{-1}$ $\forall \tilde{U}^-, U^+$.

3) $u \in U^+$

${}^{s_2}U_2 = (s_1 U^+ s_1^{-1}) \cap {}^{s_1 \cdots s_k}U^-$

Key computation:

$s_1 U^+ s_1^{-1}$

$\begin{pmatrix} 0&1&0 \\ -1&0&0 \\ 0&0&1 \end{pmatrix} \begin{pmatrix} 1&x&y \\ 0&1&z \\ 0&0&1 \end{pmatrix} \begin{pmatrix} 0&-1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$

$\begin{pmatrix} 0&1&0 \\ -1&0&0 \\ 0&0&1 \end{pmatrix} \begin{pmatrix} x&-1&y \\ 1&0&z \\ 0&0&1 \end{pmatrix}$

$= \begin{pmatrix} 1&0&z \\ -x&1&-y \\ 0&0&1 \end{pmatrix}$

$s_1 U^+ s_1^{-1} \not\subseteq U^+$.

for group conjugation

$s(A \cap B)s^{-1} \overset{?}{=} sAs^{-1} \cap sBs^{-1}$

$sas^{-1} = sbs^{-1} \Rightarrow a = b \Rightarrow a \in A \cap B$

$\Downarrow$

$sas^{-1} s b^{-1} s^{-1} = e$.

$sab^{-1}s^{-1} = e$

$ab^{-1} = e$ ✓.

$A \cap B \subset D \cap C$. $B \subset C$

$\Leftrightarrow$ $A \cap B \subset D$

& $A \cap B \subset C$.

$s_2 U_2 \subset U_1$ — This is non trivial for me!

1) $s_1 U^+ s_1^{-1} \cap {}^{s_1 \cdots s_k}U^- \subset {}^{s_1 \cdots s_k}U^-$

trivially.

2) $s_1 U^+ s_1^{-1} \cap {}^{s_1 \cdots s_k} U^- \subset U^+$.

$x = s_1 U^+ s_1^{-1} = s_1 \cdots s_k U^- s_k^{-1} \cdots s_1^{-1}$

$\Rightarrow U^+ = s_2 \cdots s_k U^- s_k^{-1} \cdots s_2^{-1}$

this U^+ is a very special type of element of U^+. $U^+ \in U_2$.

$U^+ \in s_2 \cdots s_k U^- s_k^{-1} \cdots s_2^{-1}$

Remember 3 important things

1) $G \curvearrowright \mathrm{Lie}\, G =: \mathfrak{g}$ $\quad g \cdot x := g x g^{-1}$ $\quad g \in G, x \in \mathfrak{g}$

$$T \curvearrowright V \Rightarrow V = \bigoplus_{\lambda \in X(T)} V_\lambda.$$

$$V_\lambda = \{ v \in V \mid \forall t \in T \;\; t \cdot v = \lambda(t) v \}$$

The character lattice is $X(T)$

$$T \curvearrowright \mathfrak{g} \Rightarrow \mathfrak{g} = \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha.$$

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid \forall h \in T \;\; h x h^{-1} = \alpha(h) x \}.$$

In $G = SL_{n+1}$ the roots are ?
(type A_n)

$$X(T) = \bigoplus_i \mathbb{Z} x_i / \mathbb{Z} x.$$

$$x_i : \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{n+1} \end{pmatrix} \mapsto t_i$$

$x_i \in X(T)$, $x_i : T \longrightarrow \mathbb{C}^\times$.

$T \cong (\mathbb{C}^\times)^n$. $\quad x = \sum_{i=1}^{n+1} x_i$

$\overline{x}_i \in X(T)$, $\overline{x}_i = x_i \bmod x$.

$$\alpha_{ij} := \overline{x}_i - \overline{x}_j \quad \begin{cases} i \neq j . \\ 1 \leq i, j \leq n+1 \end{cases}$$

$\Phi(G, T) = \{ \alpha_{ij} \mid i \neq j \;\; 1 \leq i, j \leq n+1 \}$.

$\# \Phi = \frac{n(n+1)}{2}$.

2) $h \cdot u_\alpha(t) \cdot h^{-1} = u_\alpha(\alpha(h) \cdot t)$

3) $s_i := u_{\alpha_i}(1) u_{-\alpha_i}(-1) \cdot u_{\alpha_i}(1)$

4) [Very important]

$$w_\alpha(t) U_\beta w_\alpha(t)^{-1} = \pm 1 t^{\langle \beta, \alpha \rangle} U_{w_\alpha \beta}$$

1-param subgp.

$t = 1$, $w_\alpha(t) = u_\alpha(t) u_{-\alpha}(\tilde{t}) u_\alpha(t)$

$w_\alpha(t) = s_\alpha$

Then $\quad s_\alpha U_\beta s_\alpha^{-1} = U_{s_\alpha \beta}$

In particular

$s_\alpha U_\beta s_\alpha \subset U^+$ if $\beta \neq \alpha$ positive roots

$U_{-\alpha} \ni s_\alpha U_\alpha s_\alpha \subset U^-$

---

5) $w s_i > w$ iff $w(\alpha_{s_i}) > 0$.

e.g. $s > e \quad \alpha_s > 0$

$st > s$ iff $s(\alpha_t) > 0$.

$sts > st$ iff $st(\alpha_s) > 0$

$sts(\tilde{\alpha}) < 0$. $\tilde{\alpha} = ?$ $\quad \tilde{\alpha} = \alpha_s$

of course $stss \not> sts$ then $sts(\alpha_s) < 0$.

$st(s(\tilde{\alpha})) < 0 \quad s(\tilde{\alpha}) = ?$

of course $stt \not> st$ then $st(\alpha_t) < 0$

need $s(\tilde{\alpha}) = \alpha_t \quad \tilde{\alpha} = s(\alpha_t)$.

$\tilde{\alpha} = \{ \alpha_s, s(\alpha_t), st(\alpha_s) \}$.

$ts(\tilde{\alpha}) = \alpha_s$

$\tilde{\alpha} = st(\alpha_s)$ etc... finally can prove

$s_1 U_2 \subset U_1$.

With all of this is clear

$U_j := \prod_{\alpha \in A_j} U_\alpha \quad A_j = \{\phi \in \Phi^+ \mid s_k \cdots s_j(\phi) < 0\}$

then $A_j = \{\alpha_j, s_j(\alpha_{j+1}), s_j s_{j+1}(\alpha_{j+2}), \ldots$

(☆☆) $\ldots, s_j s_{j+1} \cdots s_{k-1}(\alpha_k)\}$.

Since ${}^{s_\alpha}U_\beta = U_{s_\alpha \beta}$. then

$${}^{s_j}U_{j+1} = {}^{s_j}\left[\prod_{\alpha \in A_{j+1}} U_\alpha\right] = \prod_{\alpha \in A_{j+1}} U_{s_j(\alpha)}$$

$$= \prod_{\alpha \in s_j A_{j+1}} U_\alpha.$$

But $s_j A_{j+1} \subseteq A_{j+1}$ since (☆☆).

Then

(☒) $\boxed{U_j = U_{\alpha_j} \times {}^{s_j}U_{j+1}}$ in particular

${}^{s_j}U_{j+1}$. Since $U^+ = \prod_{\alpha \in \Phi^+} U_\alpha$

decomposition in unique way.

Pending (Try after knowing the structure of U^+)

$u_1 \in \pi^{-1}(\mathbb{I})$ $\quad u_1 \in B^- B$ no info

$u_1 s_1 \in B^- s_1 B$ a lot of info

$u_1 \in U^+ \cap B^- s_1 B s_1^{-1}$

$U_1 \cap B^- s_1 B s_1^{-1} \subseteq {}^{s_1}U_2$.

$u_1 \in B^- s_1 B s_1^{-1} \cap U_1$

$u_1 \in U^- s_1 U^+ s_1^{-1} \cap U_1$

$u_1 = U^- s_1 U^+ s_1^{-1} \in U_1$

$u_1 = (U^- \, {}^{s_1}U^+) \cap U_1 \subset U_1 \subset U^+$

$s_1 \cdots s_k = y$. $\quad A_2$

$U_2 \rightsquigarrow \alpha \in \{\alpha_2, s_2(\alpha_3), \ldots, s_2 \cdots s_{k-1}(\alpha_k)\}$

$U_1 \rightsquigarrow \alpha \in \{\alpha_1, s_1(\alpha_2), \ldots, s_1 \cdots s_{k-1}(\alpha_k)\}$, $\quad \overset{\shortparallel}{A_1}$

Lets see components of u_1

Define

$\mathcal{B} =$ roots appearing non-trivially on u_1.

$\mathcal{B} \subset A_1$ of course.

Suppose $\alpha_1 \in \mathcal{B}$.

then $U^- \cdot {}^{s_1}U^+$ intersect U_2,

$U^+ \rightsquigarrow \Phi^+$

$U^- \rightsquigarrow \Phi^-$

${}^{s_1}U^+ \rightsquigarrow s_1\Phi^+$.

$U^- \rightsquigarrow \Phi^-$

$U^- \cdot {}^{s_1}U^+ \rightsquigarrow s_1\Phi^+ \cup \Phi^-$

but $s_1\Phi^+ \cup \Phi^- = \Phi \setminus \{\alpha_{s_1}\}$

$U^- \cdot {}^{s_1}U^+ \cap U_1 \rightsquigarrow \underbrace{(s_1\Phi^+ \cup \Phi)}_{\Phi \setminus \{\alpha_{s_1}\}} \cap A_1 \overset{\shortparallel}{\mathcal{B}}$.

$\mathcal{B} = (s_1\Phi^+ \cup \Phi) \cap A_1 \not\ni \alpha_1$.

$\mathcal{B} = A_1 - \{\alpha_1\} = s_1 A_2$

then $u_1 \in {}^{s_1}U_2$ or.

$U_1 \cap B^- s_1 B s_1 \subseteq {}^{s_1}U_2$ //

they equal!

The other thing pending was proving case (a) of prop 3.2 satisfies condition (i).

$\sigma_{j-1} > \sigma_j$ $\boxed{D1}$ $\sigma_j = \sigma_{j-1} s_j$.

$\sigma_j(\alpha_j) > 0$.

$f_j(t, u_{j+1}) = x_{\alpha_j}(t)\, s_j\, u_{j+1} s_j^{-1}$

using $(\square)$ we can see

$\mathrm{Im}\, f_j = U_j$.

$\sigma_{j-1} f_j(t, u_{j+1}) s_j \cdots s_k =$

$b_j^- \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$\nexists\ b_j^- \in B^-$ & $v_{j+1} \in U^{j+1}$.

$U^j = \prod_{\alpha \in \Phi^j} U_\alpha$, $\Phi^j := \left\{ \phi \in \Phi^+ \;\middle|\; s_j \cdots s_k(\phi) > 0 \right\}$

$U^+ = U_j \cdot {}^{s_j \cdots s_k} U^j \mid U_j := U^+ \cap {}^{s_k \cdots s_j} U^+$.

Cont

$\sigma_{j-1} \underbrace{\left[x_{\alpha_j}(t) s_j u_{j+1} \cancel{s_j^{-1}} \right] \cancel{s_j}}_{\in U^j} \cdots s_k$

Attention!

$= b_j^- \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$.

pd see case (c)

$\sigma_{j-1} x_{\alpha_j}(t) s_j u_{j+1} \in B^- \sigma_j u_{j+1}$ ${}^{s_{j+1} \cdots s_k} U^{j+1}$ "Complement" of U_{j+1} in U^+

$\sigma_j s_j$ ‖

$\sigma_{j-1} U_j s_j \in$

$\sigma_{j-1} U_j s_j$

$\cap|$??

$B^- \sigma_j U_{j+1}$

or I'll prove w/ a bit different approach

just had to choose here id element.

$\sigma_{j-1} U_j s_j = \sigma_1 s_j^{-1}$

$\sigma_{j-1} U_{\alpha_j} s_j \subsetneq_? B^- \sigma_j U_\beta\ \forall \beta$?

For the moment I know

$\sigma_{j-1} U_{\alpha_j} s_j \subset B^- \sigma_j U^+$.

Since $\sigma_{j-1} s_j < \sigma_{j-1}$ Done!!

σ_j. ~~Still pending (*)~~

Let $u_1 \in \eta^{-1}(\Sigma)$ Case $\sigma_{j-1} < \sigma_j$ $U.1$

$f_j : \{0\} \times U_{j+1} \to U_j$

$(0, u_{j+1}) \mapsto s_j u_{j+1} s_j^{-1}$

Sup $\Sigma = (e, s_1, \ldots)$, $u_1 \in \eta^{-1}(\Sigma)$

$u_1 \in B^- B$

$u_1 \in B^- s_1 B s_1^{-1}$

$u_1 \in \mathrm{Im}\, f_1$?

$f_1 : \{0\} \times U_2 \hookrightarrow U_1$

$(0, u_2) \mapsto s_1 u_2 s_1^{-1}$.

$\mathrm{Im}\, f_1 = {}^{s_1}U_2 \subset U_1$. $y = s_1 \cdots s_k$.

pd ${}^{s_1}U_2 \supset U_1 \cap B^- s_1 B s_1^{-1}$.

$U_1 \cap B^- s_1 B s_1^{-1} = U_1 \cap U^- s_1 U^+ s_1^{-1}$

$U^- {}^{s_1}U^+ \rightsquigarrow \Phi^- \cup s_1 \Phi = \Phi - \{\alpha_{s_1}\}$.

$U_1 \rightsquigarrow \{\alpha_{s_1}, s_1(\alpha_{s_2}), s_1 s_2(\alpha_{s_3}), \ldots, s_1 \cdots s_{k-1}(\alpha_k)\}$

Φ_1.

$U_2 \rightsquigarrow \{\alpha_{s_2}, s_2(\alpha_{s_3}), \ldots\}$

Φ_2.

$s_1 \Phi_2 = \Phi_1 - \{\alpha_{s_1}\}$

${}^{s_1}U_2 \rightsquigarrow s_1 \cdot \Phi_2$.

$U_1 \cap U^- {}^{s_1}U^+ \rightsquigarrow \Phi - \{\alpha_{s_1}\} \cap \Phi_1$.

‖ $\cap$ ‖

${}^{s_1}U_2 \rightsquigarrow \Phi_1 - \{\alpha_{s_1}\}$

Then $u_1 \in \mathrm{Im}\, f_1$

Case $\sigma_{j-1} = \sigma_j$.

Case $\sigma_{j-1} = \sigma_j$ $\rightarrow$ (UO) Use $U^+ = U_j \cdot {}^{s_j \cdots s_k} U^j$

let $t \in \Omega^*$ and $u_{j+1} \in U_{j+1}$ fixed.

(**) $\underbrace{u_{\alpha_j}(t)}_{\in U_{\alpha_j}} \cdot \underbrace{u_{j+1}}_{\in U_{j+1}} = \tilde{u}_{j+1} \, \underbrace{s_{j+1} \cdots s_k \, v_{j+1}^{-1} s_k^{-1} \cdots s_{j+1}^{-1}}$

$\in U^+$ $\quad {}^{s_j}U_{j+1} \subset U_j$. $\quad \exists!\, \tilde{u}_{j+1} \in U_{j+1}$, $v_{j+1} \in U^{j+1}$ unique

define
$f_j : \Omega^* \times U_{j+1} \longrightarrow U_j$

$(t, u_{j+1}) \longmapsto \underbrace{u_{\alpha_j}(t^{-1})}_{\in U_j} \cdot \underbrace{s_j \tilde{u}_{j+1} s_j^{-1}}_{\in U_j}$. $\quad \in U_j$.

f_j injective by (**).

f_j is mor since $f_{\Omega^*} : \Omega^* \to \Omega^*$, $t \mapsto t^{-1}$ is morph.

$\operatorname{Im} f_j \overset{?}{=} U_j \setminus {}^{s_j}U_{j+1}$. let $d \neq 0$ and $\tilde{u}_{j+1} \in U_{j+1}$

A general element of RHS is: $u_{\alpha_j}(d) \cdot s_j \tilde{u}_{j+1} s_j^{-1} \in ?$ (by (**))

but
$u_{\alpha_j}(-d^{-1}) \cdot \tilde{w}_{j+1} = u_{j+1} s_{j+1} \cdots s_k v_{j+1} s_k^{-1} \cdots s_{j+1}^{-1}$

$-d^{-1} \in \Omega^*$ $\quad \tilde{w}_{j+1} \in U_{j+1}$ $\quad \exists!\, u_{j+1} \in U_{j+1}$ and $v_{j+1} \in U^{j+1}$.

$(\tilde{w}_{j+1} = u_{j+1})$

we want to check $u_{\alpha_j}(d) s_j \tilde{u}_{j+1} s_j^{-1} \in \operatorname{Im} f$

but $f_j(-d^{-1}, \tilde{w}_{j+1}) = u_{\alpha_j}(-d) s_j u_{j+1} s_j^{-1}$.

$f_j(d^{-1}, u_{j+1}) = ?$ find if $\tilde{U}_{j+1}$ is "U_{j+1}"

$\Rightarrow u_{\alpha_j}(d^{-1}) u_{j+1} = \tilde{w}_{j+1} s_{j+1} \cdots s_k v_{j+1}^{-1} s_k^{-1} \cdots s_{j+1}^{-1}$.

$f_j(d^{-1}, w_{j+1}) = u_{\alpha_j}(d) \cdot s_j \tilde{u}_{j+1} s_j^{-1}$

$\therefore U_j \setminus {}^{s_j}U_{j+1} \subseteq \operatorname{Im} f_j$.

$\operatorname{Im} f_j \subseteq U_j \setminus {}^{s_j}U_{j+1}$ by construction.

$\therefore \Omega^* \times U_{j+1} \cong U_j \setminus {}^{s_j}U_{j+1} = \operatorname{Im} f_j$.

lets check (i)

$\sigma_{j-1} f_j(t, u_{j+1}) s_j \cdots s_k$

$= b_j^- \sigma_j u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$\exists!\, v_{j+1} \in U^{j+1}$, $b_j^- \in B^-$.

$\sigma_{j-1} f_j(t, u_{j+1}) s_j \cdots s_k$

$= \sigma_{j-1} u_{\alpha_j}(t^{-1}) s_j \tilde{u}_{j+1} s_j^{-1} s_j \cdots s_k$

$= \sigma_{j-1} u_{\alpha_j}(t^{-1}) s_j \tilde{u}_{j+1} s_{j+1} \cdots s_k$

$= \sigma_{j-1} \underbrace{u_{\alpha_j}(t^{-1}) s_j u_{\alpha_j}(t)}\, u_{j+1} s_{j+1} \cdots s_k v_{j+1}$

$= \sigma_{j-1} h s_{\alpha_j} u_{\alpha_j}(-t^{-1}) s_{\alpha_j}^{-1} u_{j+1} s_{j+1} \cdots s_k v_{j+1}$.

(c.f. lemma 2.1)

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & t^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -t \end{pmatrix}$

$= \begin{pmatrix} -t^{-1} & 0 \\ -1 & -t \end{pmatrix} = \begin{pmatrix} -t^{-1} & 0 \\ 0 & -t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix}$

$= b_j^- \sigma_{j-1} \underline{u_{j+1} s_{j+1} \cdots s_k v_{j+1}}$, $\exists!\, b_j$

$= \sigma_{j-1} h s_j u_{\alpha_j}(-t^{-1}) s_j^{-1} \underline{u_{j+1} s_{j+1} \cdots s_k v_{j+1}}$

$b_j^- = \sigma_{j-1} h s_j u_{\alpha_j}(-t^{-1}) s_j^{-1} \sigma_{j-1}^{-1} \in B^-$?

$\sigma_{j-1} = \sigma_j$ (UO) then $\sigma_{j-1} s_j > \sigma_{j-1}$

$\Rightarrow \sigma_j s_j > \sigma_j$.

then $\sigma_j(\alpha_{s_j}) > 0$, $\sigma_{j-1}(\alpha_{s_j}) > 0$

$\Downarrow$

$\sigma_{j-1}(-\alpha_{s_j}) < 0$

$\sigma_{j-1} h \underbrace{s_j u_{\alpha_j}(-t^{-1}) s_j^{-1}}_{\in {}^{s_j}U_{\alpha_j} = U_{-\alpha_j}} \sigma_{j-1}^{-1}$

$\therefore b_j^- \in T \, {}^{\sigma_{j-1}}U_{-\alpha_j} = T U_{\sigma_{j-1}(-\alpha_j)} \subset T U^- = B^-$

$\boxed{B^- = TU^-}$

then $b_j^- \in B^-$ $\square$.

Try of case (a) $\sigma_{j-1} > \sigma_j$ D1.
after understanding case (b) and (c).

$\sigma_{j-1} > \sigma_j$. $\sigma_j = \sigma_{j-1} s_j$.

then $\sigma_j s_j > \sigma_j$. $\sigma_j(\alpha_{s_j}) > 0$.

$\Rightarrow \sigma_j(-\alpha_{s_j}) < 0$

$f_j : \Omega_j \times U_{j+1} \longrightarrow U_j$

$(t, u_{j+1}) \longmapsto u_{\alpha_j}(t) s_j u_{j+1} s_j^{-1}$

Obviously by $(\boxtimes)$ $\mathrm{Im}\, f_j = U_j$.
clearly injective too. See if satisfies (ii).

$\sigma_{j-1}\left[u_{\alpha_j}(t) s_j u_{j+1} s_j^{-1}\right] s_j \_\_\_ s_k$.

$= \sigma_j h s_j u_{\alpha_j}(t) s_j^{-1} \underline{u_{j+1} s_{j+1} \_\_\_ s_k}$.

to make this the right power we had to add $h \in T$

(we picking $u_{j+1} = 1$!!!)

$= b_j^- \sigma_j \underbrace{u_{j+1} s_{j+1} \_\_\_ s_k}$

$b_j^- = h \sigma_j s_j u_{\alpha_j}(t) s_j^{-1} \sigma_j^{-1}$

$b_j^- \in T\, {}^{\sigma_j}\!\left({}^{s_j}U_{\alpha_j}\right)$.

$T\, {}^{\sigma_j}\!\left(U_{-\alpha_j}\right)$

$T\, U_{\underbrace{\sigma_j(-\alpha_j)}_{<0}} \subset T U^- = B^-$ □.

Finally!! This was extremely hard for me! Deodhar method for case (c) is very enlightening.

Let $u_1 \in \eta^{-1}(\underline{\sigma})$ we trying to define $\{t_i\}_{i=1}^{k}$, $\{u_i\}_{i=1}^{k+1}$

$\underline{\sigma} = (e, s_1, \_\_\_)$ done before (U1)

$\exists\, t_1, u_2$ s.t. $u_1 = f_1(0, u_2)$

$u_1 = s_1 u_2 s_1^{-1}$ we used

$u_1 \in \eta^{-1}(\underline{\sigma}) \Rightarrow u_1 \in B^- s_1 B s_1^{-1}$.

and $B^- s_1 B s_1^{-1} \cap U_1 = {}^{s_1}U_2 = \mathrm{Im}\, f_1$.

(U0) [D1 cannot happen at start]

$\underline{\sigma} = (e, e, \_\_\_)$. $\sigma_j(\alpha_j) > 0$

$\sigma_{j-1} = \sigma$. $\sigma_0 = \sigma_1$. $\sigma_1(\alpha_1) > 0$.

$u_1 \in \eta^{-1}(\sigma) \Rightarrow u_1 \in B^- B$.
$u_1 s_1 \in B^- B$

then $u_1 \in B^- B s_1^{-1}$.

$U_1 \cap B^- B s_1^{-1} = ?$ (we hope $U_1 \setminus {}^{s_1}U_2$)

$U_1 \cap B^- B s_1^{-1} = U_1 \cap U^- U^+ s_1^{-1}$.

$U_1 \rightsquigarrow A_1 = \{\alpha_1, s_1(\alpha_2), s_1 s_2(\alpha_3), etc.\}$

$U_1 \setminus {}^{s_1}U_2 \rightsquigarrow A_1 - s_1 A_2 = \{\alpha_1\}$

Then $U_1 \setminus {}^{s_1}U_2 = U_{\alpha_1}$ (also can obtain this by $(\boxtimes)$)

substract

$U^- U^+ s_1^{-1} = ?$

$T s_1^{-1} = s_1 T$

$U^- \rightsquigarrow \Phi^- \rightsquigarrow$ ignore then

$U^+ \rightsquigarrow \Phi^+$

$\boxed{U_1 \cap U^+ s_1^{-1}} \rightsquigarrow ?$

$U_\alpha \cdot s_1^{-1} = ?$ $s_1 U_\alpha s_1^{-1} = U_{s_1(\alpha)}$.

$U_\alpha s_1^{-1} = s_1^{-1} U_{s_1(\alpha)}$.

~~e.g. $U_1 = U^+$ SL_3.~~

$\cancel{\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & & b \\ 1 & & c \\ 0 & & 1 \end{pmatrix}}$ yikes.

$\underbrace{U^- U^+ s_1^{-1}} \cap U_1$.

Φ? No!

$\cancel{\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

?

$\cancel{\begin{pmatrix} 1+ab \end{pmatrix}}$?

~~$U_\alpha s_1 T \ni U_\beta$ $\forall \beta$.~~

~~when~~? $\underline{Q}$: $U_\alpha \cdot s_1 \subset T U_\beta$ $\forall \beta$ when?
$\alpha > 0$. $\beta > 0$.

Only when $U_\alpha s_1 = U_\alpha$.

$U_\alpha \cdot S = U_{\alpha'}$ when it happens?.

$\alpha, \alpha' > 0$.

SL_3

$$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} S$$

$\sigma_1 = \sigma_0$.

To prove

$$U_1 \cap B^- B s_1^{-1} \overset{\text{maybe equal}}{\subseteq} U_1 - {}^{s_1}U_2 \; (= \mathrm{Im} f_1)$$

$$\cong (U_{\alpha_1} - \{0\}) \times {}^{s_1}U_2$$

$\pi_1 : U_1 \longrightarrow U_{\alpha_1}$ in an obvious way.

$\tilde{\pi} : U_1 \longrightarrow \Omega$.

$\pi_1 : U_1 \cap B^- B s_1^{-1} \longrightarrow U_{\alpha_1} - \{u_{\alpha_1}(0)\}$.

$\tilde{\pi}_1 : U_1 \cap B^- B s_1^{-1} \longrightarrow \Omega^*$

I will skip this but seems true and we should use the $h_\alpha(t)$ (and w_α) functions.

$w_\alpha(t) = u_\alpha(t)\, u_{-\alpha}(-t^{-1})\, u_\alpha(t)$

$w_\alpha(1) = s_\alpha$.

$h_\alpha(t) = w_\alpha(t)\, w_\alpha(1)^{-1} = w_\alpha(t) \cdot s_\alpha^{-1}$

SL_2 $\quad w_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & t \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$

$w_\alpha(t=1) = s_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

~~$w_\alpha(t)$~~

$h_\alpha(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -t & 0 \\ 0 & -t^{-1} \end{pmatrix}$

for SL_3 I guess $\begin{pmatrix} 0 & t & 0 \\ -t^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ & $\begin{pmatrix} -t & 0 & 0 \\ 0 & -t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

G semisimple simply connected alg gp.

$\Rightarrow$ "G is made by SL_2 parts".

e.g. $G_\alpha \hookrightarrow SL_3$

$U_\alpha \subseteq G_\alpha = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset SL_3$.

$A_1 = \eta^{-1}(\Sigma)$

$\Rightarrow D_\Sigma = \{u_1 y \cdot B \in By \cdot B \mid u_1 \in A_1\}$

$D_\Sigma = A_1 y \cdot B \simeq A_1 \simeq$

$A_1 \cong \Omega^{m(\Sigma)} \times (\Omega^*)^{n(\Sigma)}$

In particular $D_\Sigma \neq \emptyset \;\; \forall \Sigma \in \mathcal{D}$ (very non-trivial result!)

(iii) Thm 1.1

Fix $\Sigma \in \mathcal{D}$ Let $u_1 y \cdot B \in D_\Sigma$

we know that

$u_1 s_1 \cdots s_k \in B^- \sigma_k B$ then $u_1 y \cdot B \in B^- \sigma_k \cdot B$. Thus

$D_\Sigma \subseteq B^- \sigma_k \cdot B$

$\sigma_k \in W(y)$ of course.

Corollary 1.2.

$$B\dot{y} \cdot B \cap B\dot{x}^{-1} \cdot B =$$

$$= \bigsqcup_{\substack{\sigma \in \mathcal{D} \\ \pi(\sigma) = x}} D_{\sigma} \left(= C_y \cap C^x \right)$$

In particular, $C_y \cap C^x \neq \phi$ iff $x \in W(y)$ (i.e. $x \leq y$).

Proof

$$B\dot{y} \cdot B \cap B\dot{x}^{-1} \cdot B = \left(\bigsqcup_{\sigma \in \mathcal{D}} D_{\sigma} \right) \cap B\dot{x}^{-1} \cdot B$$

$$= \bigsqcup_{\sigma \in \mathcal{D}, \pi(\sigma) = x} D_{\sigma},$$

by Theorem 1.1 (iii).