What is modular representation theory?

§1. Introduction and motivation

- Modular rep. theory = rep. theory over a field k of char $p > 0$.
- Hugely different (often more difficult) than over fields of char $= 0$ (e.g. $\mathbb{C}$)

(1) Semi-simplicity fails: Maschke's theorem for reps of finite groups $/\mathbb{C}$ fails in char p (proof involves dividing by $|G|$)

e.g. $G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle \curvearrowright V = kx \oplus ky$

$$g \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad g_V^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad g_V^p = 0.$$

V is reducible and indecomposable. order of g_V is p in char $k = p$

such reps abound for general fg gps $/k$

(2) New symmetry: funny arithmetic

$$(a+b)^p = a^p + b^p, \quad a, b \in k.$$

this often gives rise to new submodules which don't exist in char 0.

Underlying mechanism: Frobenius endo

$$Fr : G \longrightarrow G$$

and associated Frobenius twist on Rep G.

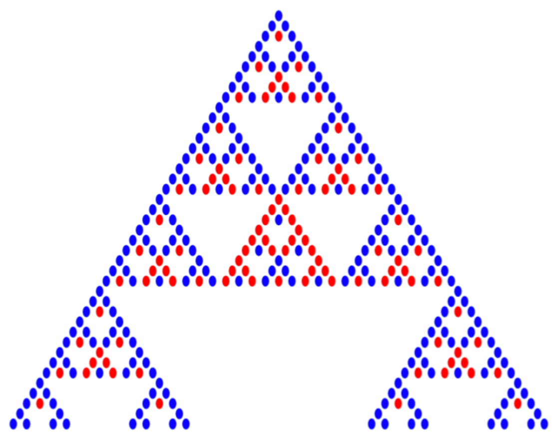
$$\text{Rep } G \rightsquigarrow \text{"Fractal" structure on Rep } G.$$

(3) Subtler geometric connections: finding character formulas for simple modules has been a guiding problem in rep theory over the last 50 years

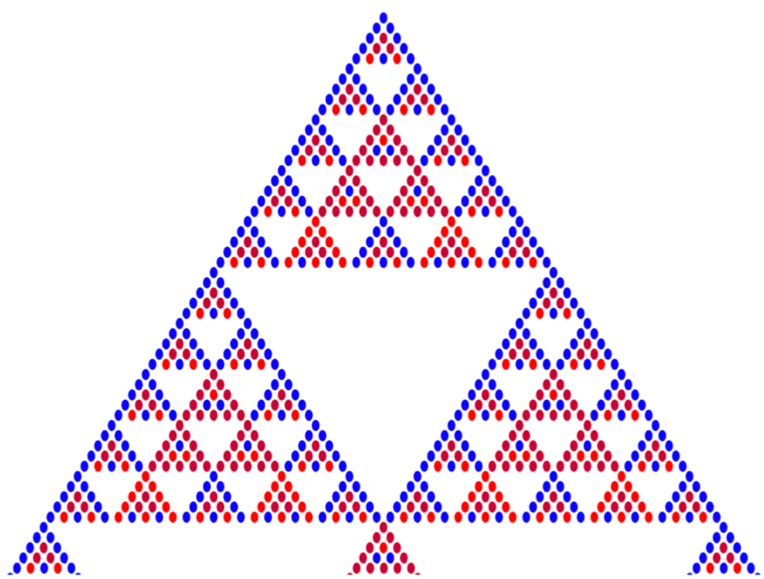
characteristic zero: deep geometric proofs of K-L conj. by Brylinski-Kashiwara and Beilinson - Bernstein (1980's)

char $p > 0$: still in process of being solved (Lusztig's conj. has led the way.)

- Goal today: Explore some of these features, emphasising SL_2.
- Along the way we will encounter + explain the following pictures:



$p=3$



$p=5$

- These images are from:

G. Williamson. "Modular representations and reflection subgroups" arxiv, 2019.

§2. Foundations

- We fix $k = \bar{k}$ field of char. $p > 0$ $(k = \overline{\mathbb{F}_p})$
- For us, an algebraic group over k is a group G w/ the structure of k-variety

$$\underbrace{\begin{aligned} m &: G \times G \longrightarrow G \, , \quad (g,h) \longmapsto gh \\ \iota &: G \longrightarrow G \, , \quad g \longmapsto g^{-1} \end{aligned}}_{\text{regular functions}}$$

- A homomorphism of alg. groups $G \longrightarrow H$ should respect both structures: a regular group homomorphism.
- Examples: $\mathbb{G}_a = (k, +) = \mathbb{A}^1_k$

$\mathbb{G}_m = (k^\times, \times) = \mathbb{A}^1_k - \{0\}$ and more generally

$\mathbb{G}_m^r =$ tori, $r \geqslant 1$.

$GL_n(k) = D(\det) \underset{\text{open}}{\subseteq} \mathbb{A}^{n^2}_k$.

- Scheme X defined over $\mathbb{F}_p$, have a Frobenius endomorphism

$$Fr : X \longrightarrow X$$

In the case of affine varieties, Fr is given by the p-th power on coordinates.

e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$,

on $Mat_{2\times 2}(k) = \mathbb{A}^4_k$. (Not always the identity map!)

Special case: alg. group hom. $Fr : G \longrightarrow G$.

- An algebraic representation of G is an alg. group homomorphism

$$\varphi : G \longrightarrow GL_n(k) = GL(V)$$

for some f.d. k-vector space V. This amounts to a gp. homomorphism.

$$\varphi(g) = \left(z_{ij}(g)\right)_{i,j=1,\dots,n} \quad \text{s.t. for each}$$

$z_{ij} : G \longrightarrow k$ is a regular function.

§3. Examples for SL_2, Chevalley's Thm

- Let $G = SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_{2\times 2}(k),\ ad-bc = 1 \right\}$.

Examples of algebraic reps:

(1) $SL_2 \longrightarrow GL_1(k)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto 1$, the trivial rep.

(2) $SL_2 \hookrightarrow GL_2(k)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

the natural representation,

(3) $SL_2 \longrightarrow GL_3(k)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

- In fact, (1) and (3) comes from (2).

Let
$$V = kx \oplus ky = \text{nat}.$$

Then $\Lambda^2(V) = k(x\wedge y) = \det$ has

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x \wedge y) &= (ax+cy)\wedge(bx+dy) \\ &= (ad-bc)(x\wedge y) \\ &= x \wedge y, \end{aligned}$$

so we recover the trivial rep.

- We also $S^2(V) = kx^2 \oplus kxy \oplus ky^2$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^2 = (ax+cy)^2 = a^2x^2 + 2acxy + c^2y^2$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot xy = \cdots$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot y^2 = \cdots$$

calculations show we recover (3).

- More generally, we have

$$S^n(V) =: \nabla_n = kx^n \oplus kx^{n-1}y \oplus \cdots \oplus ky^n. \text{ of dim } n+1,$$

for all $n \geq 0$.

Then (1), (2), (3) are precisely $\nabla_0, \nabla_1, \nabla_2$, respectively

- Let $M = SL_2$-module. By restriction, M is a module for the maximal torus

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in k \right\} \simeq G_m.$$

$$\rightsquigarrow M = \bigoplus_{\underset{\text{weight}}{n \in \mathbb{Z}}} M_n \qquad \text{where } \underset{\text{weight space}}{M_n} = \left\{ m \in M : \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} m = z^n m \text{ for all } z \in k^\times \right\}$$

- <u>key fact</u>: ∇_n has a unique simple submodule

$$L_n = \operatorname{soc} \nabla_n \xhookrightarrow{i} \nabla_n$$

of highest weight n, where $\operatorname{coker}(i)$ has a composition series by L_m for $m < n$.

- If $n! \neq 0$ in k, then $\nabla_n \cong \nabla_n^*$ (because $S^n(V)^* \cong S^n(V^*)$, (see Brian Conrad notes)) and we can deduce then that $\nabla_n = L_n$ is simple.

In fact, in characteristic zero,

$$\{\text{simple } Sl_2\text{-modules}\}/_{\cong} \overset{(*)}{\longleftrightarrow} \{\nabla_n\}_{n\geq 0}.$$

- But in characteristic p, the story is not so simple (pun intended!).

- <u>Example</u>: $k = \overline{\mathbb{F}_3}$. The following formulas

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^3 = (ax+cy)^3 = a^3x^3 + c^3y^3$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot y^3 = (bx+dy)^3 = b^3x^3 + d^3y^3$$

$\Rightarrow$ $kx^3 + ky^3$ is a proper submodule of $\nabla_3 = kx^3 \oplus \cdots \oplus ky^3$

In fact is simple, $L_3 = kx^3 \oplus ky^3$.

- How did L_3 arise? Given Sl_2-rep

$$Sl_2 \xrightarrow{\varphi} GL(M)$$

Consider the rep corresponding to

$$Sl_2 \xrightarrow{Fr} Sl_2 \xrightarrow{\varphi} GL(M)$$ we obtain the Frobenius twist $M^{(1)}$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting as $Fr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$.

- <u>check</u>: $L_3 = \nabla_1^{(1)} = L_1^{(1)}$.

- Correct reformulation of (*):

$$\{\text{Simple } Sl_2\text{-modules}\}/_{\simeq} \longleftrightarrow \{L_n\}_{n\geq 0}$$

- <u>General theorem [Chevalley]</u>; let $T \subseteq G$ be a max'l torus, $X = \mathrm{Hom}(T, G_m)$. Then iso classes of simple G-modules are classified by an explicit set $X_+ \subset X$ of dominant weights.

$$(\mathbb{Z}_{\geq 0} \subset \mathbb{Z})$$

- Caution: while the parameter set X_+ does not vary with p, the structure of the L_λ, $\lambda \in X_+$ certainly does (as we have seen).

§4. Characters and Pascal's Δ

- Generalising the above: if $T \subseteq G$ is a max'l torus, then any G-module M admits a decomp.

$$M = \bigoplus_{\lambda \in X} M_\lambda$$ — λ-weight space

where $M_\lambda = \{ m \in M \mid tm = \lambda(t) m \quad \forall t \in T \}$.

- The character of M is

$$\mathrm{ch}\, M = \sum_{\lambda \in X} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[X]$$

- After dimension, character is the most basic attribute of M.
- ch is additive on exact sequences,

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

$$\rightsquigarrow \mathrm{ch}(N) = \mathrm{ch}(M) + \mathrm{ch}(P)$$

$$\mathrm{ch}(M \otimes N) = (\mathrm{ch} M)(\mathrm{ch} N) \quad \text{if we let } e^\lambda \cdot e^\mu = e^{\lambda+\mu}.$$

- Example: Recall $\nabla_n = kx^n \oplus kx^{n-1}y \oplus \cdots \oplus ky^n$.
  $G = SL_2$

Then $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot x^i y^{n-i} = (zx)^i (z^{-1}y)^{n-i}$

$$= z^i x^i z^{i-n} y^{n-i}$$
$$= z^{2i-n} x^i y^{n-i}$$

so $x^i y^{n-i} \in (\nabla_n)_{2i-n}$ and hence we see that ∇_n has

1-dim'l non-zero weight spaces

$$-n, -n+2, \dots, n-2, n \in \mathbb{Z}$$

i.e. $$\mathrm{ch}\,\nabla_n = e^{-n} + e^{-n+2} + \dots + e^{n} = \frac{e^{n} - e^{-n-2}}{1-e^{-2}}.$$

- There exist analogues of the ∇_n for all reductive G, and $\mathrm{ch}\,\nabla_n$ is given by Weyl's character formula.
- Assume $p=3$. The following depicts submodules $L_n \subseteq \nabla_n = kx^n \oplus \dots \oplus ky^n$ for $0 \leq n \leq 6$.

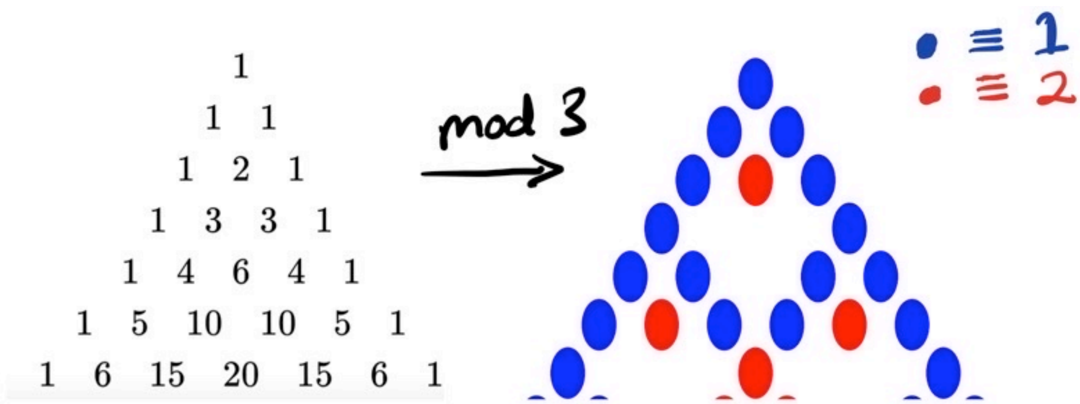
See picture in notes!

$$k$$
$$kx \qquad ky$$
$$kx^2 \qquad kxy \qquad ky^2$$
$$kx^3 \qquad ky^3$$
$$kx^4 \qquad kx^3y \qquad ky^3x \qquad ky^4$$
$$kx^5 \qquad kx^4y \qquad kx^3y^2 \qquad kx^2y^3 \qquad kxy^4 \qquad ky^5$$
$$kx^6 \qquad k \qquad ky^6$$

- From this we can read off the characters directly:

$$e^0$$
$$e^{-1} \qquad e^{1}$$
$$e^{-2} \qquad e^{0} \qquad e^{2}$$
$$e^{-3} \qquad e^{3}$$
$$e^{-4} \qquad e^{-2} \qquad e^{2} \qquad e^{4}$$
$$e^{-5} \qquad e^{-3} \qquad e^{-1} \qquad e^{1} \qquad e^{3} \qquad e^{5}$$
$$e^{-6} \qquad e^{0} \qquad e^{6}$$

This should recall our previous picture: (reduction of Pascal's triangle mod 3)



- This diagram is obtained by reducing mod 3 and coding residue classes.
- How does modular Pascal's Δ connect to characters?

§5. Frobenius kernels, Steinberg $\otimes$-theorem

- Motivation: suppose $N \trianglelefteq H$ are finite groups, and let us assume:

  All simple N-modules extend to H-modules (+)

- Clifford: If V is a simple H-module then $V|_N$ is semisimple with G-conjugate simple summands, all isomorphic (by (+)):

$$V|_N \cong V' \oplus \cdots \oplus V'$$

(Conjugation: for $g \in H$, $W \longrightarrow W^g$ where in W^g the action is

$$h \cdot w = ghg^{-1} w$$

• Then
$$\mathrm{Hom}_N(V',V)\otimes V' \xrightarrow{\cong} V,$$
$$f\otimes v' \longmapsto f(v')$$
is an iso. of H-modules.

• <u>Upshot</u>: simple H-module $\simeq$ (simple H/N-module) $\otimes$ (simple N-module)

• <u>Back to algebraic groups</u>: Assume technical conditions on G: semisimple, simply connected

• Can consider an exact seq.
$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{Fr} G \longrightarrow 1$$

See Jantzen's book. Frobenius kernel = "N". p-restricted weights

• <u>Curtis</u>: There is an explicit set $X_1 \subseteq X_+$ such that
$$\{\text{simple } G\text{-modules}\}/\cong \;\longleftrightarrow\; \{L_\lambda|_{G_1}\}_{\lambda\in X_1}$$

• <u>Theorem [analogue to upshot]</u>: If $\lambda=\mu+\nu\in X_+$ with $\mu\in X_1$, $\nu=p\nu'\in X_+$. Then
$$L_\lambda \cong L_\mu \otimes L_{\nu'}^{(1)}$$

simple for $G_1 = N$

simple for $G\cong G/G_1$ $=$ "G/N"

• In our setting, every $\lambda\in X_+$ can be written (sort of)
$$\lambda = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_r, \quad \lambda_i\in X_1$$
"expansion in base p"

• <u>Corollary [Steinberg]</u>:
$$L_\lambda \cong L_{\lambda_0}\otimes L_{\lambda_1}^{(1)}\otimes\cdots\otimes L_{\lambda_r}^{(r)}$$

<u>Example:</u> $G = SL_2$ has

$$X = \mathbb{Z} \supseteq X_+ = \mathbb{Z}_{\geq 0} \supseteq X_1 = \{0, \dots, p-1\}.$$

So (*) is the base p expansion of $\lambda \in \mathbb{Z}_{\geq 0}$, $\lambda = n = n_0 + pn_1 + \cdots + p^r n_r$, $0 \leq n_i \leq p-1$.

- Since $L_m = \nabla_m$ for $0 \leq m \leq p-1$, now have

$$\mathrm{ch}\, L_n = (\mathrm{ch}\, \nabla_{n_0})(\mathrm{ch}\, \nabla_{n_1})^{(1)} \cdots (\mathrm{ch}\, \nabla_{n_r})^{(r)}$$
$$\underset{(*)}{=} \prod_{i=0}^{r} \left(e^{-n_i} + e^{-n_i+2} + \cdots + e^{n_i-2} + e^{n_i}\right)^{(i)}$$

where $(e^m)^{(i)} = e^{p^i m}$ is extended linearly.

- We can ask when $(L_n)_{n-2j} \neq 0$ i.e. when does e^{n-2j} appear in $\mathrm{ch}\, L_n$?
- Write $j = j_0 + j_1 p + \cdots + j_r p^r$ in base p.
  It is visible from the product (*) that to get e^{n-2j} we need

$$j_i \leq n_i \quad \text{for all} \quad 0 \leq i \leq r.$$

- Another way of phrasing that: no p-adic carries when adding j to $n-j$.
- <u>Kummer:</u> $v_p\binom{n}{j} = \#$ p-adic carries when adding j to $n-j$.

- So $(L_n)_{n-2j} \neq 0 \iff v_p\binom{n}{j} = 0$
  $\iff \binom{n}{j} \not\equiv 0 \pmod{p}$,

  solving the modular Pascal mystery!
- Other explanations possible, using Shapovalov form + Jantzen filtration.

Books:

J. C. Jantzen. *Representations of Algebraic Groups*. American Mathematical Society, Providence, RI, 2003.

T. A. Springer. *Linear Algebraic Groups,* volume 9 of *Progress in mathematics*. Birkhaüser, 1981.

Online notes:

I. Losev. Lectures on Representation Theory. https://gauss.math.yale.edu/~il282/RT/

Articles:

J. Ciappara and G. Williamson. Lectures on the geometry and modular representation theory of algebraic groups. To appear in *Journal of the Australian Mathematical Society*, 2020.

G. Williamson. Algebraic representations and constructible sheaves. *Japanese Journal of Mathematics*, 12(2):211–259, 2017.