A Huygens principle for diffusion and anomalous diffusion in spatially extended systems

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We present a universal view on diffusive behavior in chaotic spatially extended systems for anisotropic and isotropic media. For anisotropic systems, strong chaos leads to diffusive behavior (Brownian motion with drift) and weak chaos leads to superdiffusive behavior (Lévy processes with drift). For isotropic systems, the drift term vanishes and strong chaos again leads to Brownian motion. We establish the existence of a nonlinear Huygens principle for weakly chaotic systems in isotropic media whereby the dynamics behaves diffusively in even space dimension and exhibits superdiffusive behavior in odd space dimensions.

dynamical systems | pattern formation | lattice models | symmetry

In this article, we study the dynamics of spatially extended systems with symmetry. In particular, we consider anisotropic systems that are invariant under translations but not rotations, and isotropic systems that are invariant under translations and rotations. Examples of anisotropic systems are those with a preferred spatial direction, such as fluid systems advected by a directed mean flow or reaction–advection diffusion systems; examples of isotropic systems are those without any preferred direction, e.g., reaction–diffusion systems. This paper is concerned with the following general question: Given a certain type of dynamics (regular or chaotic), can we say anything about the diffusive behavior of the solution?

The answer to this question is different for isotropic and nonisotropic systems and is summarized in Table 1. As we explain below, many of these results exist in some form in the literature, and our main contribution here is to bring them together in a unified way. Furthermore, we establish a dichotomy for weakly chaotic dynamics in isotropic systems whereby generically there is diffusion in even dimensions but superdiffusion in odd dimensions, which does not seem to have been noticed previously.

In the remainder of this section, we describe the results summarized in Table 1. Throughout, we let d denote the number of space dimensions. In anisotropic systems in which there is translation symmetry but no rotation symmetry, the simplest kind of solution is a traveling wave propagating with linear speed $c\in\mathbb{R}^d$. By regular dynamics, we mean traveling waves and periodic modulations of traveling waves; such solutions are bounded in a frame of reference moving with constant speed. Coullet and Emilsson (1) considered a family of modified Ginzburg–Landau equations on the line ($d=1$) and observed that chaotically modulated traveling waves exhibit Brownian motion-like diffusive behavior in a frame of reference moving with constant wave speed. The situation may be generalized to arbitrary dimensions and is made rigorous in ref. 2 for strongly chaotic systems. The term "strong chaos" is defined below and includes many classical examples of chaotic systems. For example, systems satisfying the Gallavotti–Cohen chaotic hypothesis are strongly chaotic, as is the classical Lorenz attractor. Recently, interest has grown in so-called weakly chaotic dynamics, exemplified by Pomeau–Manneville intermittency (3). In the anisotropic case, this leads to superdiffusive behavior (4–6).

The isotropic case, in which the dynamical system is invariant under the Euclidean group of rotations and translations of d-dimensional space, $d\geq 2$, exhibits a richer variety of possible behaviors. The additional invariance with respect to rotations may lead to cancellations of the linear drift present in the anisotropic cases. For regular dynamics, a dichotomy exists between even and odd spatial dimensions. Analogous to the Huygens principle stating that one can hear only in an ambient odd dimensional space, here a nonlinear Huygens principle is operating: in even dimensions the solutions are bounded, whereas in odd dimensions solutions propagate linearly. A manifestation of this is the behavior of spiral waves in 2D excitable media (7, 8); the spiral tip moves around in circles or flower-petal meanders. In the strongly chaotic case, the dichotomy disappears and the behavior is like a Brownian motion without drift. [In the situation of spiral waves, such behavior is called "hypermeander" (7, 9–11). Although there now is good theoretical understanding (12–15), a conclusive demonstration of the existence of hypermeander in physical or numerical investigations of spiral wave dynamics remains an open problem.] The weakly chaotic case in the isotropic case was not studied previously. Again, there is no linear drift. We establish here another instance of a nonlinear Huygens principle: generically superdiffusive behavior prevails in odd dimensions, but the superdiffusion is suppressed in even dimensions and replaced by Brownian motion.

APPLIED MATHEMATICS

Spatially Extended Systems with Symmetry

Here, we adopt the standard perspective of decomposing the dynamics into the dynamics on the symmetry group and the dynamics orthogonal to it. Systems with symmetry, or "equivariant dynamical systems," thus are cast into a so-called skew product of the form

$$x=f(x),\quad \dot{g}=g\xi(x), \qquad [1]$$

on $X\times G$, where the dynamics on the symmetry group G is driven by the shape dynamics on a cross-section X transverse to the group directions. Here, $g\xi(x)$ denotes the action of the group element $g\in G$ on $\xi(x)\in T_eG$ (the Lie algebra of G). Substituting the solution $x(t)$ for the shape dynamics into the $\dot{g}$ equation yields the nonautonomous equation $\dot{g}=g\xi(x(t))$ to be solved for the group dynamics. The simplest example is the case of a traveling wave in a 1D system with translation symmetry, where the shape dynamics is an equilibrium solution $x(t)\equiv x_0$ in the frame of reference moving with constant wave speed, and the dynamics on the translation group orbit describes the linear drift of the reference frame in physical space. A more interesting example is provided by spiral waves in excitable media in which periodic shape dynamics leads to "meandering" of the spiral tip (16). Here, the shape dynamics $x(t)$ is periodic and the group dynamics $g(t)$ evolves quasiperiodically.

The skew product formulation has proved successful in describing both local bifurcations and global dynamics in pattern formation

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Table 1. Rates of propagation for given dynamics in a d-dimensional anisotropic or isotropic medium

	Anisotropic medium	Isotropic medium	
Dynamics	$d\geq 1$	$d\geq 3$ odd	$d\geq 2$ even
Regular	$ct+$bounded	$ct+$bounded	Bounded
Strongly chaotic	$ct+$diffusive	Diffusive	Diffusive
Weakly chaotic	$ct+$superdiffusive	Superdiffusive	Diffusive

$c\in\mathbb{R}^d$ denotes a general vector in $\mathbb{R}^d$.

(8, 10, 11, 17–19), in constructing efficient numerical methods for equivariant systems (20–22), and in studying Hamiltonian systems, for example to explain dynamics of periodic orbits relevant to planetary dynamics (23) and observed spectra of CO_2 molecules (24).

Adopting the decomposition Eq. **1**, we can rephrase our main question: Given a certain shape dynamics for $\dot{x}=f(x)$ (regular, strongly chaotic or weakly chaotic), can we say anything about the growth rate of solutions $g(t)$ on the group? More specifically, what is the expected diffusive behavior of $g(t)$? In the following, we first recall results for the anisotropic case and then treat isotropic systems.

Anisotropic Case. In the anisotropic case, the symmetry group is the group of translations $G=\mathbb{R}^d$ of d-dimensional space. The skew product **1** reduces to

$$\dot{x}=f(x) \quad [2]$$

$$\dot{p}=\phi(x), \quad [3]$$

where ϕ takes values in $\mathbb{R}^d$ and $p\in\mathbb{R}^d$ represents the translation variable. Without loss of generality, we assume initial conditions $x(0)=x_0$ and $p(0)=0$. Eq. **3** may be integrated to yield

$$p(t)=\int_0^t \phi(x(s))\,ds. \quad [4]$$

If the shape dynamics Eq. **2** consists of an equilibrium $x(t)\equiv x_0$, we obtain $p(t)=ct$ with $c=\phi(x_0)$. This includes the case of a traveling wave moving with constant speed c mentioned above. For a periodic solution $x(t+T)=x(t)$, we obtain $p(t)=ct+O(1)$ with $c=\int_0^T \phi(x(s))\ ds$.

Next, suppose there is a chaotic attractor $\Lambda\subset X$ for the shape dynamics with ergodic invariant measure μ. Using Eq. **4**, it follows from the Birkhoff ergodic theorem that for typical initial conditions x_0,

$$\frac{1}{t}p(t)=\frac{1}{t}\int_0^t \phi(x(s))\,ds\to c,$$

where $c=\int_\Lambda \phi\,d\mu$ is the time-average of ϕ. Typically $c\neq 0$, in which case there is linear drift as for the regular case. For strongly chaotic shape dynamics, it follows from refs. 2, 25, and 26 that there exists $\lambda>0$ such that for typical initial conditions x_0,

$$p(t)=ct+W(t)+O\left(t^{\frac{1}{2}-\lambda}\right)\quad \text{a.e.}, \quad [5]$$

where W is a d-dimensional Brownian motion with covariance matrix Σ. This implies the central limit theorem: $\mu(x_0:(p(t)-ct)/\sqrt{t}\in I)\to \Pr(Y\in I)$ for each rectangle $I\subset\mathbb{R}^d$, where $Y\sim N(0,\Sigma)$ is a normally distributed d-dimensional random variable with mean 0 and covariance matrix Σ. Another consequence is that the sequence $(p(nt)-cnt)/\sqrt{n}$ converges weakly to W in the space of continuous sample paths. [This is called weak convergence of $p(t)-ct$ to Brownian motion, whereas Eq. **5** is strong convergence.] For $d=1$, this describes the case of chaotically modulated traveling waves as observed in ref. 1.

Weakly chaotic dynamical systems are characterized by "sticky" equilibria, periodic solutions, and so on, in which the dynamics exhibits laminar behavior interspersed with intermittent chaotic bursts (3). It is well-known (4) that for such intermittent systems, the usual central limit theorem may break down, leading to fluctuations of the Lévy type rather than of the Gaussian type. In those situations, it was established (5) that solutions propagate superdiffusively as t^γ for some $\gamma\in(\frac{1}{2},1)$. More precisely, $t^{-\gamma}(p(t)-ct)$ converges in distribution to an α-stable law where $\alpha=1/\gamma$. Let W_α denote the corresponding Lévy process (possessing increments that are independent, stationary, and with distributions proportional to this stable law). Then, $p(t)-ct$ converges weakly to $W_\alpha(t)$ by ref. 6. This concludes the discussion of the anisotropic case in Table 1.

Isotropic Case. In the isotropic case, the symmetry group is the Euclidean group $\mathbf{E}(d)=\mathbf{SO}(d)\ltimes\mathbb{R}^d$ consisting of rotations and translations of d-dimensional space, and the skew product equations are given by

$$\dot{x}=f(x),\quad \dot{A}=Ah(x),\quad \dot{p}=Av(x), \quad [6]$$

where $A\in\mathbf{SO}(d)$ represents the rotation variables and $p\in\mathbb{R}^d$ represents the translation variables. Without loss of generality, we choose as initial conditions $x(0)=x_0$, $A(0)=\mathbf{I}$, and $p(0)=0$. Note that $\mathbf{SO}(d)$ consists of $d\times d$ orthogonal matrices with determinant 1, and that h, being an element of the Lie algebra of $\mathbf{SO}(d)$, is a skew-symmetric matrix. (We suppose throughout that $d\geq 2$, because otherwise we would be in the anisotropic situation.)

If the shape dynamics consists of an equilibrium $x(t)\equiv x_0$, then the dynamics on the rotation group may be integrated to yield $A(t)=\exp(th(x_0))$. We choose coordinates so that the skew-symmetric matrix $h(x_0)$ is diagonal with entries on the imaginary axis. In even dimensions $d=2q$, the diagonal entries are given by $\pm i\omega_1,\ldots,\pm i\omega_q$ and typically are nonzero. Using the identification $\mathbb{R}^d\cong\mathbb{C}^q$, we obtain $\dot{p}_j=e^{it\omega_j}v_j(x_0)$ and hence

$$p_j(t)=(1/i\omega_j)e^{it\omega_j}v_j(x_0),\quad j=1,\ldots,q. \quad [7]$$

It follows that $p(t)$ is bounded.

In odd dimensions $d=2q+1$, one of the diagonal entries of $h(x_0)$, without loss of generality the first entry, is forced to vanish and with the identification $\mathbb{R}^d\cong\mathbb{R}\times\mathbb{C}^q$, we have $\dot{p}_1=v_1(x_0)$. Hence,

$$p(t)=ct+O(1),\quad c=(v_1(x_0),0,\ldots,0), \quad [8]$$

and there typically is a linear drift. For periodic solutions, analogous calculations (27) lead similarly to bounded motion for d even and unbounded linear growth for d odd. This constitutes the nonlinear Huygens principle for regular dynamics in isotropic media. This dichotomy may be visualized by looking at the effect of the rotations in even and odd dimensions. In even dimensions, all components of v in Eq. **6** are rotated; in odd dimensions, however, there is an axis of rotation and the corresponding component of v is not subjected to the averaging effect of the rotation. In Fig. 1, we show typical dynamics of the translation variables for $\mathbf{E}(2)$ and $\mathbf{E}(3)$ skew products with underlying regular dynamics. We see clearly bounded motion for $d=2$ and a corkscrew motion along the axis of rotation of $\mathbf{SO}(3)$ for $d=3$.

For chaotic shape dynamics on X, it is convenient to split off the compact group part of the dynamics and rewrite the

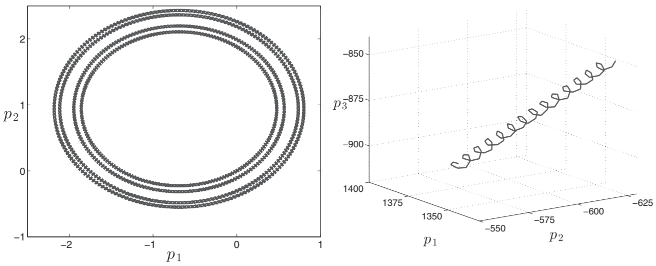

Fig. 1. Typical behavior of the translation variables p for underlying regular behavior for $\mathbf{E}(2)$ (*Left*; bounded) and $\mathbf{E}(3)$ (*Right*; corkscrew).

system Eq. **6** in the same form as in the anisotropic case (Eqs. **2** and **3**), namely

$$\dot{y} = f_h(y), \quad \dot{p} = \phi(y),$$

where $y = (x, A)$, $f_h(x, A) = (f(x), Ah(x))$, and $\phi(x, A) = Av(x)$. Then,

$$p(t) = \int_0^t \phi(y(s))\, ds. \qquad [9]$$

As in the anisotropic case, let $\Lambda \subset X$ be a chaotic attractor with ergodic invariant probability measure μ. The natural invariant probability measure for the y dynamics on $\Lambda \times \mathbf{SO}(d)$ is the product $m = \mu \times \mu'$, where μ' denotes the Haar measure on the rotation group $\mathbf{SO}(d)$. It follows from ref. 28 that m typically is ergodic if the shape dynamics is strongly or weakly chaotic. Because $\int_{\mathbf{SO}(d)} A\, d\mu' = 0$, the time average of $\phi(x, A) = Av(x)$ is zero:

$$\int_{\Lambda \times \mathbf{SO}(d)} \phi\, dm = \int_{\mathbf{SO}(d)} A\, d\mu' \int_{\Lambda} v\, d\mu = 0.$$

Hence, ergodicity of m implies that $\frac{1}{t} p(t) \to 0$, and so there is no linear drift in the isotropic case (11, 12).

Strong convergence to Brownian motion as in Eq. **5** has not been proved in the isotropic case. However, for strongly chaotic dynamics, weak convergence to d-dimensional Brownian motion holds, and the central limit theorem follows. The covariance matrix now is diagonal as a result of the rotation symmetry with $\Sigma = \sigma^2 I_d$ and typically $\sigma^2 > 0$. For rigorous results, we refer the interested reader to refs. 12–15.

The main result of this paper concerns a nonlinear Huygens principle for weakly chaotic systems in isotropic spatially extended media. We establish that weakly chaotic isotropic systems with odd spatial dimensions exhibit superdiffusion, whereas the superdiffusion is suppressed in systems of even dimensions. This dichotomy may be motivated from our results on regular dynamics in isotropic media as stated above. Recall that in the nonisotropic case, the anomalous diffusion is caused by the combination of laminar phases near sticky pockets of regular dynamics interspersed with intermittent chaotic bursts. However, we have seen that in even dimensions these laminar regular phases are averaged out in the isotropic case because of the rotation symmetry. Hence, the mechanism for anomalous diffusion no longer is present. We deduce that for d even, weak chaos leads to Brownian behavior just as for strong chaos; whereas for d odd, the laminar regular phase survives the effect of the rotation symmetry and we expect weak convergence to a Lévy process.

Our predictions are supported both by the above theoretical justification and by numerical investigations described below. (A rigorous mathematical proof is the subject of ongoing work.) The ingredients for the theoretical justification are summarized in a separate paragraph below. The numerical experiments also provide a useful visualization of these phenomena. Fig. 2 (*Lower*) presents results for an isotropic medium with $d = 3$ where our theory predicts anomalous diffusion. The computed solution behaves as a combination of Brownian motion corresponding to the intermittent chaotic bursts (as in the strongly chaotic case) and Lévy flights corresponding to the sticky pockets of regular dynamics. In contrast, in Fig. 3 (*Lower*; isotropic medium $d = 2$), the computed solution behaves like a Brownian motion as before during the chaotic bursts, but the Lévy flights are suppressed during the regular phases. The different behavior of solutions during the regular phases—compare Eqs. **7** and **8** for d even and d odd, respectively, is seen to be the explanation for our nonlinear Huygens principle for anomalous diffusion.

Strong and Weak Chaos

As promised, in this section we provide the definition of strong and weak chaos used throughout this paper. Strongly chaotic systems include Anosov flows [Gallavotti–Cohen chaotic hypothesis (29)] and uniformly hyperbolic (Axiom A) attractors. A more general class of flows comprises those with a Poincaré map modeled by a Young tower with exponential decay of correlations (30). These include Hénon-like attractors, Lorenz-like attractors, and Lorentz gas models. Even more generally, we consider situations in which the Poincaré map is modeled by a Young tower with subexponential decay of correlations (31), distinguishing between the cases in which the decay rate is summable and nonsummable. For us, strongly chaotic flows are precisely those corresponding to the summable case. (This terminology is not completely standard; many authors refer to the entire subexponential case as being weakly chaotic because Lyapunov exponents vanish. However, as evidenced by the results described in this paper, in many respects such systems behave identically to the exponential case provided the decay is summable, and it is the boundary between summable and nonsummable that is significant.)

Roughly speaking, weakly chaotic flows are those corresponding to the nonsummable case, but there is an extra requirement that decay rates be regularly varying functions.* This is not simply a technical hypothesis; regular variation of tails is a necessary condition for convergence to a stable law or Lévy process.

Note that the definition of strong/weak chaos makes assumptions on the decay of correlations for the Poincaré map, but not for the attractor Λ itself. This is important because even Anosov flows are not necessarily mixing and there are mixing uniformly hyperbolic flows with arbitrarily slow decay of correlations (32, 33). In particular, mixing properties for the Poincaré map do not necessarily pass to the flow. In contrast, convergence to a Brownian motion or a Lévy process does pass to the flow (6, 14, 34).

Theoretical Justification

The individual ingredients comprising the theoretical justification for the results in this paper are standard in various subsections of the scientific community but may seem nonstandard when taken collectively. Hence, it seems worth summarizing the ingredients here:

- The underlying pattern-forming system (which, e.g., might be a partial differential equation or the physical system itself) is decomposed in the skew product form Eq. **1** (e.g., ref. 35).
- The assumed form of the shape dynamics $x(t)$ leads to various types of equation $\dot{g}(t) = g\xi(x(t))$ for the group dynamics. There is a well-established theory when the shape dynamics is steady or periodic (27, 36, 37), and there are numerous results in the case in which the shape dynamics is chaotic (12, 13, 15).

*A function $\ell(x)$ is *slowly varying* if $\ell(\lambda x)/\ell(x) \to 1$ as $x \to \infty$ for all $\lambda > 0$; examples are functions that are asymptotically constant and powers and iterates of logarithms. A function of the form $\ell(x)x^q$ is *regularly varying*.

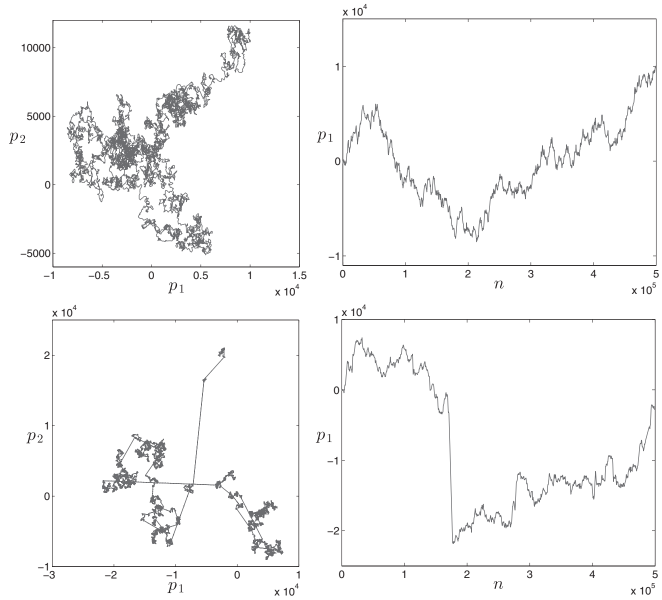

Fig. 2. Isotropic case in odd dimensions: translation variables p_1 and p_2 of an $\mathbf{E}(3)$ skew product driven by the Pomeau–Manneville map Eq. **10**. (*Upper Left* and *Right*) Strongly chaotic case with $\gamma = 0.2$: Brownian motion. (*Lower Left* and *Right*) Weakly chaotic case with $\gamma = 0.7$: Lévy process.

- The "regular dynamics" entries in Table 1 are directly from ref. 27. This already establishes the absence of Lévy flights during regular phases in weakly chaotic systems for even-dimensional isotropic media.
- Passing to a Poincaré cross-section reduces to the conceptually and technically simpler situation of discrete time. Results of ref. 14 guarantee that statistical limit laws for the Poincaré map extend back to the continuous time setting.
- Young towers with exponential decay of correlations (30) have been shown to include most of the classical examples of strongly chaotic dynamical systems (e.g., Hénon, Lorenz) extending far beyond the Anosov/Axiom A setting. A broader class is Young towers with summable decay of correlations (31). The statistical properties of such systems have been the subject of much recent mathematical investigation, and the "strongly chaotic" entries in Table 1 are special cases of these rigorous results (12–15).
- Even more recently, starting with ref. 5, the statistical properties of Young towers with nonsummable tails have come under the spotlight. The prototypical example is provided by Pomeau–Manneville intermittency maps. Here, anomalous diffusion is anticipated, and convergence to a Lévy process is rigorously proved in ref. 6. The bottom left entry in Table 1 is a consequence of this.
- The remaining entries in the bottom row of Table 1 remain conjectural from the point of view of rigorous mathematics, but the theoretical justification is as follows. Intermittent dynamics is a mixture of regular phases and chaotic bursts. Based on refs. 4–6, one is led to anticipate superdiffusive behavior with Lévy flights corresponding to the regular phases. However, we already saw that regular dynamics in isotropic media varies significantly in even and odd dimensions and that the mechanism for Lévy flights exists only in odd dimensions. Consequently, we predict that anomalous diffusion is suppressed in even-dimensional isotropic media and exists only in odd dimensions, and this prediction is supported by the numerical experiments below. [A recent rigorous result of ref. 38 shows that anomalous diffusion indeed is suppressed in the case of two dimensions under the assumption that the rotation component $h \in \mathbf{SO}(2)$ is constant. The assumption on h enables the use of Fourier analysis and simplifies matters significantly. However, the heuristics behind our results do not rely on this assumption.]

In general, there is no convenient way to explicitly determine the skew-product Eq. **1** from the underlying pattern-forming system. To circumvent this, it has become standard in the physics literature to consider lattice models for diffusion in which the underlying system is posed in physical space $\mathbb{R}^d$ [e.g., to model phase dynamics of Josephson junctions and charge-density waves (39, 40) and advection–diffusion of passive tracers in fluid flows (41)]. The Euclidean group is replaced by a discrete group G of rotations and translations, and X is identified with a fundamental domain for the action of G on $\mathbb{R}^d$. In this setting, the transition between the underlying equations and the skew-product system is completely transparent and explicit. Our results hold equally in this situation; see *Supporting Information* for further details and numerical results.

Numerical Results

Intermittent dynamics often is modeled by the prototypical family of Pomeau–Manneville intermittency maps $x_{n+1} = f(x_n)$ with $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & 0 \le x \le \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \le x \le 1 \end{cases}, \qquad [10]$$

where γ is a parameter (3, 42). Pomeau–Manneville intermittency maps are the prototype for the study of intermittency in finite and infinite dimensional systems, in which they have been used to model Poincaré maps (e.g., refs. 43 and 44). If $\gamma \in [0,1)$, there exists a unique, absolutely continuous invariant probability measure (Sinai–Ruelle–Bowen measure) μ. When $\gamma = 0$, this is the doubling map with exponential decay of correlations. For $\gamma \in (0,1)$, it is known (45) that the decay of correlations is polynomial with rate $1/n^{(1/\gamma)-1}$, which is summable for $\gamma < \frac{1}{2}$ and nonsummable for $\gamma \in [\frac{1}{2}, 1)$; according to our definition, the Pomeau–Manneville map Eq. **10** is strongly chaotic for

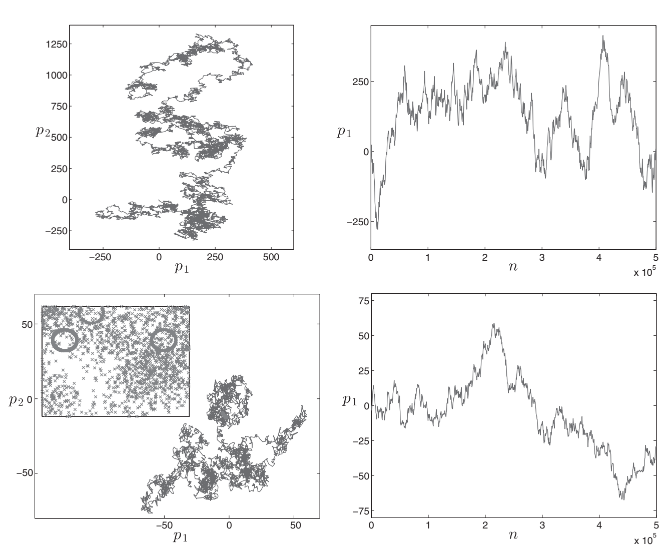

Fig. 3. Isotropic case in even dimensions: translation variables $p = (p_1, p_2)$ of an $\mathbf{E}(2)$ skew product driven by the Pomeau–Manneville map Eq. **10** exhibiting Brownian motion. (*Upper Left* and *Right*) Strongly chaotic case with $\gamma = 0.2$. (*Lower Left* and *Right*) Weakly chaotic case with $\gamma = 0.7$. (*Inset*) Zoom into smaller area illustrating the mechanism of suppression of anomalous diffusion in even space dimensions.

$\gamma \in [0, \frac{1}{2})$ and weakly chaotic for $\gamma \in [\frac{1}{2}, 1)$. Note that for $\gamma > 0$, the fixed point at 0 is indifferent $[f'(0) = 1]$ and plays the role of the sticky regular dynamics. For $\gamma \geq \frac{1}{2}$, the stickiness is strong enough to support superdiffusive phenomena. [In the borderline case $\gamma = \frac{1}{2}$, there still is weak convergence to Brownian motion but with anomalous diffusion rate $\sqrt{t \log t}$ in the anisotropic case (5, 46) and similarly for isotropic systems in odd dimensions.] We now present numerical results for skew products representing the anisotropic and isotropic cases (both even and odd dimensional) in the strongly chaotic and weakly chaotic regimes. We make the obvious modifications to the continuous time case described earlier, so the skew product now is discrete in time.

Anisotropic Case. Here, G is the translation group $\mathbb{R}^d$, and the discrete time skew product for the Pomeau–Manneville map Eq. **10** reads as $(x, p) \mapsto (f(x), p + \phi(x))$ so that

$$p(n) = \sum_{j=0}^{n-1} \phi(x_j). \qquad [11]$$

In Fig. 4, we take $d = 1$ and $\phi(x) = 1 + x$. The translation coordinate $p(n)$ exhibits linear drift cn, where $c = \int_{[0,1]} \phi \, d\mu$, for both strongly and weakly chaotic dynamics. Passing into the comoving frame, the distinction between Gaussian and Lévy-type fluctuations becomes apparent.

Because the fixed point at 0 is indifferent for $\gamma > 0$, an initial condition x_0 that starts close to 0 remains close to 0 for many iterates, say N_0, so that $\phi(x_j)$ is roughly size $\phi(0)$ for $j = 0, 1, \ldots, N_0$. Hence in the comoving frame, $p(n)$ exhibits approximately linear growth, $p(n) \approx (\phi(0) - c)n$, for $n \leq N_0$. In particular, the small jumps $\phi(0) - c$ accumulate into a large jump. This is akin to a particle's ballistic motion with constant velocity $\phi(0) - c$, a common picture of anomalous diffusion (47).

In the strongly chaotic case $\gamma \in [0, \frac{1}{2})$, these large jumps are too rare to cause anomalous diffusion and, as explained above, it follows from ref. 2 that

$$p(n) = cn + W(n) + O\left(n^{\frac{1}{2} - \lambda}\right) \quad \text{a.e.}$$

(compare Eq. **5**) so the dynamics in the comoving frame is Brownian-like. However, in the weakly chaotic case $\gamma \in (\frac{1}{2}, 1)$, Gouëzel (5) demonstrated that the large jumps correspond to Lévy flights. Once the trajectory x_n moves away from 0, the values of ϕ fluctuate erratically, yielding Brownian-like motion for $p(n)$. These two effects combine (5, 6) to produce a Lévy process with diffusion rate t^{γ}.

Isotropic Case. We illustrate the nonlinear Huygens principle by which weakly chaotic dynamics causes anomalous diffusion in isotropic media with odd space dimensions but normal diffusion in even space dimensions. The $\mathbf{E}(2)$ skew product with rotations θ and translations $p = (p_1, p_2)$ may be represented as

$$(x, \theta, p(x)) \mapsto \left(f(x), \theta + h(x), p + e^{i\theta} v(x)\right),$$

where h and v take values in $\mathbb{R}$ and $\mathbb{C}$, respectively. Hence,

$$p(n) = \sum_{j=0}^{n-1} e^{i\theta_j} v(x_j) \quad \theta_j = \sum_{k=0}^{j-1} h(x_k). \qquad [12]$$

For the numerics, we choose $v(x) = 1 + x$ and $h(x) = c_0 \neq 0$. In Fig. 3, we show plots of the translation variables in the (p_1, p_2)-plane. We also show the process $p_1(n)$ as a function of time. The diffusive behavior is seen to be normal in both the strongly and weakly chaotic cases. The mechanism by which anomalous diffusion is suppressed in even dimensions is illustrated nicely: the long laminar phases that in the anisotropic case give rise to large excursions now are bounded in both the strongly and weakly chaotic cases. This is seen in the intermittent circular motion in Fig. 3, *Inset*. When combined with the Brownian-like behavior caused by the chaotic bursts, this leads to an overall Brownian behavior. It is clear from Eq. **12** that if we were to remove the rotation near the fixed point by setting $h(x) \equiv 0$ for x near zero, then the translation variables would exhibit the same behavior as in the anisotropic case (compare Eq. **11**). In particular, this would yield a Lévy process in the weakly chaotic case.

The corresponding plots for the $\mathbf{E}(3)$ skew product are shown in Fig. 2. The anomalous Lévy-type behavior is clearly visible in the weakly chaotic case. See *Supporting Information* for details on the skew-product equations used for the numerics.

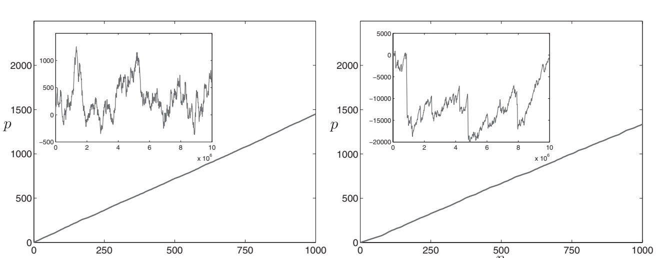

Fig. 4. Anisotropic case: translation variable p as a function of time for an $\mathbb{R}$ skew product driven by the Pomeau–Manneville map Eq. **10**. (*Left*) Strongly chaotic case with $\gamma = 0.2$: linear drift with superimposed Brownian motion. (*Right*) Weakly chaotic case with $\gamma = 0.7$: linear drift with superimposed Lévy process. To highlight the diffusive behavior, we eliminate the linear drift by subtracting the mean from the data. Note that in the weakly chaotic case, the Lévy process is asymmetric; the flights are downward because $\phi(0) - c$ is negative.

Summary and Discussion

We provided a universal view of the type of diffusive behavior that may be expected in spatially extended systems with symmetry. In doing so, we proposed a definition of weak chaos as the boundary between summable and nonsummable correlations. In contrast to the previous view of the onset of weak chaos as the demarcation line of exponential and subexponential decay of correlations, our definition allows a distinction between normal and anomalous behavior. Using this definition, we contributed to the understanding of diffusive behavior in isotropic media, establishing a nonlinear Huygens principle whereby superdiffusion occurs naturally in odd dimensions but not in even dimensions.

The phenomenon of anomalous diffusion has attracted much interest in the past decade, with applications ranging from the motion of metal clusters and large molecules across crystalline surfaces (48), conformational changes in proteins (49), migration of epithelial cells (50), diffusion in plasma membranes of living cells (51), and finance ratios (52) to the foraging strategy of animals (53–55), to name but a few. The mechanisms for anomalous diffusion in these papers are model dependent, specific to the details of the geometry of the various situations. In contrast, ours is a universal perspective (analogous to the classical Huygens principle) driven only by the ambient symmetry and the degree of chaoticity of the underlying dynamics. Our theory sets out the general conditions under which anomalous diffusion may be expected in spatially extended systems with symmetry and may be viewed as a prediction and interpretation of superdiffusive behavior in future experiments.

It is of interest to consider diffusive and superdiffusive behavior in systems with different kinds of spatial symmetry. For systems with rotation $\mathbf{SO}(2)$ symmetry—for example, rotating convection in the plane or on a sphere—trajectories lift naturally to the universal covering group $\mathbb{R}$, and we obtain the same results as for the anisotropic case with $d = 1$.

For systems with $\mathbf{O}(2)$ symmetry (rotations and reflections), it is interesting to consider the effect of the reflection symmetry. There are two entirely different scenarios: (i) There are two disjoint attractors interchanged by reflections, and each behaves as in systems with $\mathbf{SO}(2)$ symmetry. (*ii*) The attractor is invariant (setwise) under reflections and the linear drift vanishes; we predict bounded trajectories for regular dynamics and Brownian motion for strongly chaotic dynamics. However, in the case of weakly chaotic dynamics, we expect diffusion or superdiffusion, depending on whether the sticky regular phase is invariant or not under reflections. Further, in scenario *ii*, the Brownian motions and Lévy processes are symmetric.

Finally, we mention an open problem about systems on a sphere with $\mathbf{SO}(3)$ rotation symmetry (again, these might be reaction–diffusion equations or convection problems). This time, there is no elementary method for passing to a noncompact group in which it makes sense to speak of unbounded growth of trajectories. Our expectation is that locally the results are similar to those in the unbounded plane [$\mathbf{E}(2)$ symmetry], but it is unclear how to make such a statement precise.

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Supporting Information

Gottwald and Melbourne 10.1073/pnas.1217926110

SI Text 1

Lattice Models for Diffusion. As discussed in the main text, it is standard in the physics literature to consider deterministic lattice models for diffusion. We refer to refs. 1–4 and also to the survey article (5). The advantage of this approach is that there is a straightforward correspondence between the equations, and their solutions, for the underlying models and the skew-product systems.

In particular (1–3), consider deterministic models for diffusion and anomalous diffusion on the real line, by considering 1-periodic maps $\tilde{f}:\mathbb{R}\to\mathbb{R}$. The periodicity defines cells of length 1; the map $\tilde{f}$ may map outside the cell, which causes diffusion into other cells. Such systems have discrete translation symmetry $\mathbb{Z}$. We consider extensions of their models to higher dimensions and carry out numerical simulations that confirm the predictions in the main part of our paper. There also is a reflection symmetry in the work of refs. 1–3 that plays no role here and is suppressed throughout (although see the second to last paragraph in *Summary and Discussion* in the main text).

The class of dynamical systems with $\mathbb{Z}$ symmetry on the line is in one-to-one correspondence with the class of skew-product systems on $X\times\mathbb{Z}$ where $X=[0,1)$. The identification $X\times\mathbb{Z}\cong\mathbb{R}$ is given by $(x,k)\mapsto x+k$. Similarly, we can write $\tilde{f}(y)\in\mathbb{R}$ as $\tilde{f}(y)=f(y)+v(y)$, where $f(y)\in X$, $v(y)\in\mathbb{Z}$.

In this way, we obtain the skew product on $X\times\mathbb{Z}$ given by $(x,k)\mapsto(f(x),k+v(x))$, where $f:X\to X$, $v:X\to\mathbb{Z}$ are given by $f(x)\equiv\tilde{f}(x)\bmod 1$ and $v(x)=\tilde{f}(x)-f(x)$.

Note that passing from $\tilde{f}$ to (f,v) introduces discontinuities as does the reverse procedure. Whereas refs. 1–3 initially specify $\tilde{f}$ and then derive (f,v), we take the equivalent approach of specifying (f,v) from the outset (which then implies a choice of $\tilde{f}$). This means we can focus on the fundamental domain X for the action of the symmetry group $\mathbb{Z}$ on $\mathbb{R}$. From this point of view, a convenient choice of map is to take f to be the Pomeau–Manneville intermittency map from the main text (Eq. **10**) and to take v to be any integer-valued map that is continuous (hence constant) and nonzero for x near the neutral fixed point at zero. This corresponds exactly to the approach in ref. 3. The mechanism for superdiffusion in the skew-product formulation is as follows: The dynamics spends a very long time near the neutral fixed point for f, corresponding to ballistic propagation under $\tilde{f}=f+v$ along the axis. This leads to a process on $\mathbb{R}$ that is asymptotically a linear drift (typically nonzero) superimposed with Brownian motion for $\gamma<1/2$ and a stable Lévy process for $\gamma\in(1/2,1)$.

Deterministic Model for Planar Diffusion. Proceeding to two dimensions, we replace the Euclidean group of planar rotations and translations by the discrete group $G=\mathbb{Z}_4\ltimes\mathbb{Z}^2$, where $\mathbb{Z}^2$ consists of translations $(x_1,x_2)\mapsto(x_1+k_1,x_2+k_2)$ for $k_1,k_2\in\mathbb{Z}$, and $\mathbb{Z}_4$ consists of rotations by angle $0,\pi/2,\pi,3\pi/2$ about the origin. The action of $\mathbb{Z}_4$ on $\mathbb{R}^2$ is generated by $(y_1,y_2)\mapsto(-y_2,y_1)$. A fundamental domain for the action of G on $\mathbb{R}^2$ is given by $X=\left[0,\frac{1}{2}\right)\times\left(0,\frac{1}{2}\right]$, and the identification $X\times G\cong\mathbb{R}^2$ is given by $(x,A,k)\mapsto Ax+k$, where $x\in X$, $A\in\mathbb{Z}_4$, and $k\in\mathbb{Z}^2$.

Again, there is a one-to-one correspondence between G-equivariant deterministic diffusion models on $\mathbb{R}^2$ and skew-product maps on $X\times G$ of the form $(x,A,k)\mapsto(f(x),Ah(x),k+Av(x))$, where $f:X\to X$, $h:X\to\mathbb{Z}_4$, $v:X\to\mathbb{Z}^2$. To obtain strongly/weakly chaotic dynamics on X, a simple choice is to take

$$f(x_1,x_2)=\left(f_1(x_1),\tfrac{1}{2}x_2\right), \qquad \textbf{[S1]}$$

with

$$f_1(x_1)=\begin{cases} x_1\left(1+4^\gamma x_1^\gamma\right), & 0\le x_1<\frac{1}{4}\\ 2x_1-\frac{1}{2}, & \frac{1}{4}\le x_1<\frac{1}{2}.\end{cases} \qquad \textbf{[S2]}$$

This map has a neutral fixed point (a nonhyperbolic saddle) at $(0,0)$, and the dynamics is strongly/weakly chaotic for $\gamma\in\left[0,\frac{1}{2}\right)$ and $\gamma\in\left(\frac{1}{2},1\right)$, respectively.

In the strongly chaotic case, we predict normal diffusion. In the anisotropic case (so $h\equiv I_2$), this will be superimposed on a linear drift; in the isotropic case in which rotation symmetry is present, typically the drift term will vanish.

In the weakly chaotic case, we predict superdiffusion superimposed on a linear drift in the anisotropic case. In the isotropic case, again the linear drift vanishes but, moreover, we predict that the anomalous diffusion is suppressed in favor of Brownian motion. These predictions are borne out by the numerical experiments described below.

Deterministic Model for 3D Diffusion. Next, we consider the 3D case, replacing the Euclidean group of rotations and translations by the discrete group $G=\mathbb{O}\ltimes\mathbb{Z}^3$, where $\mathbb{Z}^3$ consists of translations $(x_1,x_2,x_3)\mapsto(x_1+k_1,x_2+k_2,x_3+k_3)$ for $k_1,k_2,k_3\in\mathbb{Z}$, and $\mathbb{O}$ is the 24-element group consisting of rotation symmetries of the cube. The action of $\mathbb{O}$ on $\mathbb{R}^3$ is generated by $(x_1,x_2,x_3)\mapsto(-x_2,x_1,x_3)$ and $(x_1,x_2,x_3)\mapsto(x_1,-x_3,x_2)$. A fundamental domain for the action of G on $\mathbb{R}^3$ is given by $X=\{x\in\left[0,\frac{1}{2}\right]^2: x_2\le x_1, x_3\le x_1\}$ (we choose to be imprecise with regard to the boundaries; this is unimportant because the dynamics sees the boundary only for a set of initial conditions of measure zero), and the identification $X\times G\cong\mathbb{R}^3$ is given by $(x,A,k)\mapsto Ax+k$, where $x\in X$, $A\in\mathbb{O}$, $k\in\mathbb{Z}^3$.

Once again there is a one-to-one correspondence between G-equivariant deterministic diffusion models on $\mathbb{R}^3$ and skew-product maps on $X\times G$ of the form $(x,A,k)\mapsto(f(x),Ah(x),k+Av(x))$, where $f:X\to X$, $h:X\to\mathbb{O}$, $v:X\to\mathbb{Z}^3$. An example of a map that generates strongly/weakly chaotic dynamics is

$$f(x_1,x_2,x_3)=\begin{cases}\begin{pmatrix} x_1\left(1+4^\gamma x_1^\gamma\right)\\ \frac{1}{2}x_2\\ \frac{1}{2}x_3\end{pmatrix} & 0\le x_1<\frac{1}{4}\\[2ex] \begin{pmatrix} 2x_1-\frac{1}{2}\\ \min\left(2x_1-\frac{1}{2},\frac{1}{2}x_2\right)\\ \min\left(2x_1-\frac{1}{2},\frac{1}{2}x_3\right)\end{pmatrix} & \frac{1}{4}\le x_1<\frac{1}{2}.\end{cases}$$

In the strongly chaotic case $\gamma\in\left[0,\frac{1}{2}\right)$ and in the anisotropic case, our predictions are the same as in two dimensions. However, for weak chaos $\gamma\in\left(\frac{1}{2},1\right)$ in the isotropic case, we predict that the anomalous diffusion persists despite the rotation symmetry and that there is a stable Lévy process.

Numerical Experiment. We carry out a numerical verification of our predictions in the case of weakly chaotic dynamics for 2D systems. For the base dynamics $f:X\to X$, we use the map defined in Eqs. **S1** and **S2**. In the numerics, we compute a dynamical orbit (x_n,A_n,k_n) in the skew product and plot the sequence of points $y_n=A_nx_n+k_n$ on the $\mathbb{R}^2$-plane. We do this for both the anisotropic case ($h\equiv I_2$) and the isotropic case (in which we choose h to be rotation by $\pi/2$ independent of x). In both cases, we take $v=(v_1,v_2)$ with

$$v_1(x)=\begin{cases}1 & 0\le x_1\le 0.15\\ -2 & 0.15<x_1\le 0.5\end{cases}$$

$$v_2(x)=\begin{cases}3 & 0\le x_1\le 0.33\\ 1 & 0.33<x_1\le 0.5.\end{cases}$$

The results for the anisotropic and isotropic cases are shown in Figs. S1 and S2, respectively, confirming our theoretical results. In the anisotropic case, the Lévy process is completely antisymmetric for f, an intermittent map with a single neutral fixed point [just as in the one-dimensional case (Fig. 4)]. Hence, the Lévy flights are concentrated along a single direction in the plane.

SI Text 2

Skew-Product System for the E(3) Extension. In the case of $\mathbf{E}(3)=\mathbf{SO}(3)\ltimes\mathbb{R}^3$, it is convenient to make the identification $\mathbf{SO}(3)\cong\mathbf{SU}(2)/\{\pm I_2\}$, where $\mathbf{SU}(2)$ is the special unitary group of 2×2 complex matrices with determinant 1. Such matrices have the form

$$A=\begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix},$$

where $\alpha,\beta\in\mathbb{C}$ and $|\alpha|^2+|\beta|^2=1$. We also identify $\mathbb{R}^3$ with 2×2 complex matrices

$$v=\begin{pmatrix}ia & b\\ -\bar{b} & -ia\end{pmatrix},$$

where $a\in\mathbb{R}$, $b\in\mathbb{C}$. The action of $\mathbf{SO}(3)$ on $\mathbb{R}^3$ is given by $A\cdot v=AvA^{-1}$. The skew product takes the form

$$(x,A,p)\mapsto(f(x),Ah(x),\,p+A\cdot v(x)), \qquad \textbf{[S3]}$$

where $A,h(x)\in\mathbf{SU}(2)$ and $p,v\in\mathbb{R}^3$. Hence,

$$p(n)=\sum_{j=0}^{n-1}A_j\cdot v(x_j),\quad A_j=h(x_0)h(x_1)\cdots h(x_{j-1}). \qquad \textbf{[S4]}$$

Numerical Experiment. For the numerics, we choose

$$a(x)=2+x \quad\text{and}\quad b(x)=(1+i)(2+x),$$

and we represent

$$h=\begin{pmatrix}\cos(c_0)\exp(ic_1) & \sin(c_0)\exp(ic_2)\\ -\sin(c_0)\exp(-ic_2) & \cos(c_0)\exp(-ic_1)\end{pmatrix},$$

where the functions c_i are chosen to be piecewise constant on the subintervals $\left[0,\frac{1}{2}\right]$, $\left[\frac{1}{2},\frac{3}{4}\right]$, $\left[\frac{3}{4},1\right]$ with values chosen from a uniform distribution in the interval $(\pi/5,4\pi/5)$ (so nine different values are chosen at random). In Fig. 2, we show plots of the translation variables in the (p_1,p_2)-plane and the process $p_1(n)$ as a function of time for strongly and weakly underlying dynamics. The anomalous Lévy-type behavior is clearly visible in the weakly chaotic case.

The anomalous diffusion linked to underlying weakly chaotic dynamics of the Pomeau–Manneville map may be motivated by inspecting the skew product Eq. **1** as follows: Let us decompose $v(0)=v^{\parallel}+v^{\perp}$, where $v^{\parallel}$ denotes the component along the axis of rotation of $h(0)$ and $v^{\perp}$ denotes the component perpendicular to the axis of rotation. [In the notation of Eq. **8**, 0 corresponds to x_0 and $v^{\parallel}$ corresponds to $v_1(x_0)$.] Then, within the laminar phase, say with x_j close to the indifferent fixed point for N_0 iterates, the translation variables p are augmented by approximately $v^{\parallel}N_0$ (compare Fig. 1). Noting the expression for A_j in Eq. **S4**, in the weakly chaotic case a requirement for the occurrence of anomalous diffusion is that $v(0)$ must have a nonvanishing component $v^{\parallel}$ along the axis of rotation of $h(0)$.

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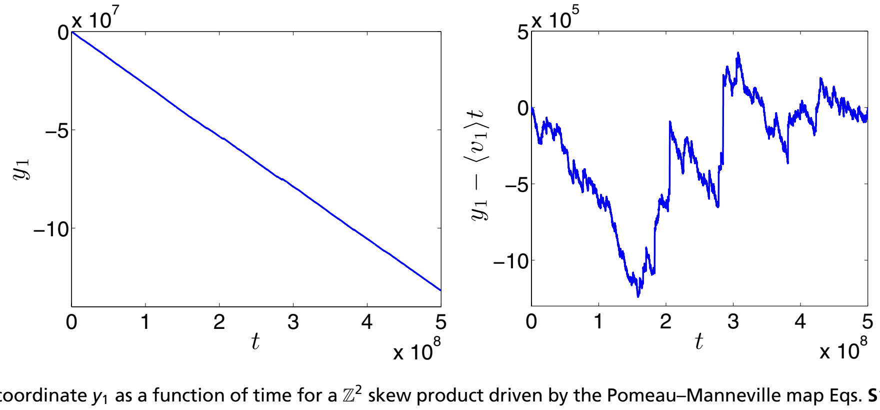

Fig. S1. Anisotropic case: coordinate y_1 as a function of time for a $\mathbb{Z}^2$ skew product driven by the Pomeau–Manneville map Eqs. **S1** and **S2** with $\gamma=0.7$. Shown are the full dynamics including the linear drift (*Left*) and with the linear drift eliminated by subtracting the mean from the data (*Right*).

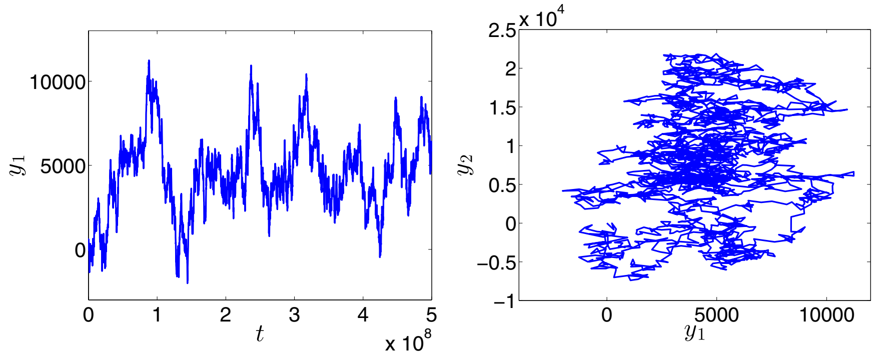

Fig. S2. Isotropic case: coordinates y_1 (*Left*) and (y_1, y_2) (*Right*) as functions of time for a $\mathbb{Z}_4 \ltimes \mathbb{Z}^2$ skew product driven by the Pomeau–Manneville map Eqs. **S1** and **S2** with $\gamma = 0.7$.