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Stable two-dimensional parametric solitons in fluid systems

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Abstract

We derive two-dimensional (2D) envelope equations for models based on linearly coupled Zakharov–Kuznetsov (ZK) and Kadomtsev–Petviashvili (KP) equations which describe the interaction between long nonlinear waves in fluid flows. The asymptotic equations coincide with those describing the second-harmonic generation (SHG) in a 2D optical waveguide, that take into regard both the spatial diffraction and temporal dispersion. The system derived from the ZK and KP equations turn out to be, respectively, fully elliptic and fully hyperbolic with respect to the spatial coordinates. The recently found "light-bullet" solutions to the elliptic SHG equations in nonlinear optics suggest the possible existence of fully localized 2D solitons in the corresponding ZK coupled system. Direct numerical simulations, for which the initial conditions are taken as suggested by an analytical variational approximation (VA), completely corroborate the existence and *stability* of the 2D solitons, with a shape fairly close to that predicted by VA. We also demonstrate that quasi-1D solitons are (numerically) stable against 2D perturbations in *both* the ZK and KP systems. The results suggest that 2D parametric *spatio-temporal* solitons, which are hard to generate experimentally in nonlinear optics, can be generated in certain fluid flows. © 1998 Elsevier Science B.V.

1. INTRODUCTION

Parametric solitons, supported by a competition between the second-harmonic generating (SHG) nonlinearities and temporal dispersion or spatial diffraction, have recently attracted a great deal of attention in nonlinear optics [1]. Experimentally, both one-dimensional (1D) and two-dimensional (2D) *spatial solitons* have been observed, respectively, as self-supporting stripes and cylindrical beams in planar waveguides [2] and in bulk media [3]. A new possibility is to generate fully localized 2D and 3D spatiotemporal solitons (the so-called "light bullets", LB) [4,5], whose existence and stability in media with quadratic nonlinearities was shown, in a nonconstructive way, as early as 1981 [6]. In the same work [6], it was also shown that, unlike LB, spatial solitons, both 1D and 2D, are unstable against modulations or bending (for more details, see Ref. [7]). In real

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experiments, the spatial solitons are nevertheless observed because the samples are too short for development of instability.

A great challenge for the experimentalists is to generate *temporal* parametric solitons, both 1D solitons in fibers and the above-mentioned LBs. Although they are expected to be fully stable, this is a very hard problem because of the lack of sufficiently long SHG fibers and sufficiently large bulk samples that would allow us to observe the (spatio)temporal solitons.

In recent work [8], we have suggested that generation of temporal parametric solitons may be achievable in certain stratified fluid flows, where the roles of the fundamental harmonic (FH) and second harmonic (SH) are played by appropriately selected wave modes, and the SHG nonlinearity is naturally provided by the quadratic terms in the corresponding coupled wave equations, which are typically of the Korteweg–de Vries (KdV) type. Despite the absolutely different physical medium, the mechanism supporting the SHG solitons is essentially the same as in optics. However, the analysis reported in Ref. [8] was confined to 1D solitons. A generalization to the 2D case is both necessary for a realistic description of the corresponding fluid system, and very interesting in itself, as it opens a way to really generate, in a different physical system, the 2D solitons that are currently of great interest in nonlinear optics. This is the subject of the present work.

The paper is organized as follows. In Section 2, we analyze the underlying weakly nonlinear wave equations, demonstrating that, with isotropic or anisotropic background flows, the generic systems are, respectively, two linearly coupled Kadomtsev–Petviashvili (KP) or Zakharov–Kuznetsov (ZK) equations (for a derivation of the KP equations for internal waves, see Ref. [9]). The linear dispersion relation for each system has two branches. We demonstrate that, as well as in the case of two linearly coupled KdV equations considered in Ref. [8], the necessary double resonance between FH and SH (simultaneous coincidence, in the lowest-order approximation, of their phase and group velocities) is possible if the two harmonics belong to *different* dispersion branches. In Section 3, we derive a system of asymptotic equations for slowly varying FH and SH amplitudes, which turn out to have the same general form as the 2D spatiotemporal SHG equations in nonlinear optics [5]. The equations are hyperbolic with respect to the propagation and transverse coordinates if the underlying fluid equations are the coupled KP ones; but in the case when we start from the ZK equations, the asymptotic system turns out to be elliptic, rather than hyperbolic, with respect to the spatial coordinates. Following the line of work in Ref. [5], we expect that only quasi-1D solitons are possible in the former case, and fully localized 2D solitons should be possible in the latter case. Significant issues are the stability of the quasi-1D solitons, and the existence and stability of the 2D ones. In Section 4, these issues are attacked by means of direct numerical simulations of the underlying equations (KP or ZK). As in our previous work [8], the initial configuration is taken as suggested by the variational approximation (VA) developed for the asymptotic SHG equations (VA was elaborated for the 1D case in Ref. [10], and for the 2D case in Ref. [5]). We conclude that, as long as the numerical simulations can be run, the quasi-1D solitons are *stable* in both the (effectively) hyperbolic (KP) and elliptic (ZK) models. This result is quite important for possible experiments in fluids. Another important numerical result is that the 2D solitons in the elliptic system exist and are apparently stable and, moreover, the initial configuration suggested by VA furnishes quite an accurate approximation for them, i.e., the initial configuration undergoes very little change as a result of the evolution in time. The latter result is a considerable contribution to verification of the relevance and accuracy of VA, which is an important issue far beyond the framework of any particular fluid model.

2. Dispersion curves and the double resonance

We consider a model of two interacting weakly nonlinear long waves described by either the coupled ZK equations,

$$u_t + u_{xxx} + u_{yyx} - 6uu_x - v_x = 0\,, \tag{2.1}$$

$$v_t + \delta v_{xxx} + \lambda v_{yyx} + \eta v_x - 6\mu v v_x - \kappa u_x = 0\,, \tag{2.2}$$

or the coupled KP equations,

$$\partial_x (u_t + u_{xxx} - 6uu_x - v_x) + u_{yy} = 0\,, \tag{2.3}$$

$$\partial_x (v_t + \delta v_{xxx} + \eta v_x - 6\mu v v_x - \kappa u_x) + \lambda u_{yy} = 0\,, \tag{2.4}$$

which are written in the most general form. Here, x and y are the propagation and transverse coordinates, η is a group-velocity shift between the coupled modes, δ and λ are the relative longitudinal and transverse dispersion coefficients, and μ and κ are the relative nonlinear and coupling coefficients. In the absence of the transverse variation (i.e., $u_y = v_y = 0$), each system reduces to the same set of coupled KdV equations which are known to describe the interaction of the nonlinear long waves in certain fluid flows (see, for instance, Ref. [9,12]). In each uncoupled KdV subsystem, the linear dispersion relation for waves with the frequency ω and x-wavenumber k has the form

$$\omega = c_0 k + \beta k^3\,, \tag{2.5}$$

where c_0 is the group velocity in the long-wave limit, and β is a dispersion coefficient. In an *isotropic* medium, the two-dimensional counterpart of (2.5) is

$$\omega \approx c_0 \kappa + \beta \kappa^3\,, \tag{2.6}$$

where $\kappa \equiv \sqrt{k^2 + l^2}$, and l is the y-wavenumber. Then, in the nearly 1D case, $l \ll k$, (2.6) yields

$$\omega \approx c_0 k + \frac{c_0}{2}\frac{l^2}{k} + \beta k^3\,, \tag{2.7}$$

which leads to the coupled KP system (2.4). Alternatively, in an *anisotropic* medium, where the anisotropy can, for instance, be provided by an underlying shear flow in the x-direction, or the so-called β-effect in the atmosphere, the 2D counterpart of (2.5) is

$$\omega \approx c_0 k + \beta k \kappa^2\,, \tag{2.8}$$

which leads to the coupled ZK system (2.2).

Two branches of the dispersion relation are, for the coupled ZK model,

$$2\omega = -(\delta + 1)k^3 - (\lambda + 1)l^2 k + \eta k \pm k\sqrt{[(\delta - 1)k^2 + (\lambda - 1)l^2 - \eta]^2 + 4}\,, \tag{2.9}$$

and, for the coupled KP model,

$$2\omega = -(\delta + 1)k^3 + (\lambda + 1)(l^2/k) + \eta k \pm k\sqrt{[(\delta - 1)k^2 - (\lambda - 1)(l/k)^2 - \eta]^2 + 4}\,. \tag{2.10}$$

In this work, we examine the case when the SHG resonance condition, viz., coincidence of the FH and SH phase velocities, can be met at $l = 0$,

$$2\omega(k_r) = \omega(2k_r)\,, \tag{2.11}$$

where k_r is the resonant wavenumber. In order to guarantee a sufficiently strong interaction between the harmonics, the group velocities must also coincide (in the first approximation), so that

$$\omega'(k_r) = \omega'(2k_r)\,. \tag{2.12}$$

The two resonance conditions (2.11) and (2.12) can be met if FH and SH belong, in each model (KP or ZK), to the lower and upper branches, respectively. Hereafter, these branches will be denoted as $\omega_1(k)$ and $\omega_2(k)$, respectively. Thus, the resonance conditions for both models with $l = 0$ reduce to those for the coupled KdV equations, considered earlier in Ref. [8], which determine the resonant x-wavenumber,

$$k_r = \sqrt{5(\eta^2+4)/8\eta(\delta-1)}\,, \tag{2.13}$$

and impose an additional condition on the system's parameters, which is

$$\eta = \sqrt{(16/9\delta)(\delta-1)^2-4}\,, \tag{2.14}$$

implying $\eta > 0$ and $\delta > 4$. In particular, this means that the longitudinal dispersion coefficients in both coupled equations have the same sign.

We will see below that the adopted restriction $l = 0$ does *not* exclude y-dependence in the next-order approximation, when we include a large-scale transverse modulation. For $l \neq 0$, an analytical expression for the resonant wavenumber in the KP model is not available, while in the ZK model it can be obtained by simply replacing the parameter η by $\eta - (\lambda - 1)l^2$.

3. Derivation of the amplitude equations

To investigate the interaction of FH and SH with the wavenumbers k_r and $2k_r$, we perform a multiscale expansion for the resonant waves. Thus, introducing a small parameter ϵ and defining $X = \epsilon x, Y = \epsilon y, T_1 = \epsilon t, T_2 = \epsilon^2 t$, we assume that each mode in the coupled systems (2.1). (2.2) or (2.3), (2.4) is a combination of two harmonics (that will be FH and SH),

$$\begin{aligned} u &\equiv \epsilon^2 A(X,Y,T_1,T_2)\,\mathrm{e}^{\mathrm{i}\theta_1} + \epsilon^2 B(X,Y,T_1,T_2)\,\mathrm{e}^{\mathrm{i}\theta_2} + \epsilon^3 u_3 + \epsilon^4 u_4 + \text{c.c.}\,, \\ v &\equiv \epsilon^2 \xi_1 A(X,Y,T_1,T_2)\,\mathrm{e}^{\mathrm{i}\theta_1} + \epsilon^2 \xi_2 B(X,Y,T_1,T_2)\,\mathrm{e}^{\mathrm{i}\theta_2} + \epsilon^3 v_3 + \epsilon^4 v_4 + \text{c.c.}\,, \end{aligned} \tag{3.1}$$

where $\xi_{1,2}$ are coefficients to be found later, and $\theta_{1,2} \equiv k_{1,2}x - \omega_{1,2}(k_{1,2})t$, with the two carrier wavenumbers $k_{1,2}$ and the corresponding frequencies. Here, $k_1 = k_r + \epsilon\Delta k$ and $k_2 \equiv 2k_1$, Δk accounting for a small deviation from the exact resonance value (2.13). The following analysis stays close to the 1D case considered in Ref. [8] as we will focus on the case of a small but finite y-wavenumber, $l = \mathcal{O}(\Delta k)$. The systems (2.1) and (2.2) and (2.3) and (2.4) can then be written as

$$\hat{\mathcal{L}}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{N}(u,v)\,, \tag{3.2}$$

where the linear operator matrix is

$$\hat{\mathcal{L}}_{uu} = \partial_t + \partial_{xxx} + \partial_{yyx}\,, \qquad \hat{\mathcal{L}}_{uv} = -\partial_x\,, \qquad \hat{\mathcal{L}}_{vu} = -\partial_x\,, \qquad \hat{\mathcal{L}}_{vv} = \partial_t + \eta\partial_x + \delta\partial_{xxx} + \lambda\partial_{yyx}\,,$$

for the ZK model, and

$$\hat{\mathcal{L}}_{uu} = \partial_t + \partial_{xxx} + \partial_x^{-1}\partial_{yy}\,, \qquad \hat{\mathcal{L}}_{uv} = -\partial_x\,, \qquad \hat{\mathcal{L}}_{vu} = -\partial_x\,, \qquad \hat{\mathcal{L}}_{vv} = \partial_t + \eta\partial_x + \delta\partial_{xxx} + \lambda\partial_x^{-1}\partial_{yy}$$

for the KP one. The vector on the right-hand side of (3.2) for both models has the components

$$\mathcal{N}_1(u,v) = 6uu_x\,, \qquad \mathcal{N}_2(u,v) = 6\mu vv_x\,.$$

At the lowest order, $\mathcal{O}(\epsilon^2)$, (3.2) is trivially satisfied by the linear dispersion relation for each mode, yielding the values of the coefficients in (3.1), $\xi_{1,2} = -k_{1,2}^{-1}(\omega_{1,2} + k_{1,2}^3)$. To derive the actual asymptotic equations, it is necessary to continue the analysis at the orders $\mathcal{O}(\epsilon^3)$ and $\mathcal{O}(\epsilon^4)$. Omitting technical details, we will display the final results.

Instead of the rescaled coordinate X defined above, we introduce the traveling coordinate $X' \equiv X - v_G T_1$, where $v_G = \omega_1'(k = k_r, l = 0) \equiv \omega_2'(k = 2k_r, l = 0)$ is the common group velocity of both harmonics, taken in the lowest-order approximation.

In the sequel we will omit the prime on X'. The final equations for the amplitudes A and B can be derived as compatibility conditions for the expansion at the order $\mathcal{O}(\epsilon^4)$,

$$\begin{aligned} &\mathrm{i}A_{T_2} + \Phi_1 A_{XX} + \Psi_1 A_{YY} + 2\mathrm{i}\Phi_1 \Delta k A_X + N_1 A^* B\, \mathrm{e}^{\mathrm{i}\chi} = 0\,, \\ &\mathrm{i}B_{T_2} + \Phi_2 B_{XX} + \Psi_2 B_{YY} + 4\mathrm{i}\Phi_2 \Delta k B_X + N_2 A^2\, \mathrm{e}^{-\mathrm{i}\chi} = 0\,, \end{aligned} \tag{3.3}$$

where the phase mismatch is

$$\chi \equiv \theta_2 - 2\theta_1 \approx -(\Delta k)^2 [2\omega_2''(k = 2k_r, l = 0) - \omega_1''(k = k_r, l = 0)] T_2\,,$$

and the other coefficients are

$$\Phi_1 = \frac{1}{2}\frac{\partial^2 \omega_1}{\partial k^2}(k = k_r, l = 0)\,, \qquad \Phi_2 = \frac{1}{2}\frac{\partial^2 \omega_2}{\partial k^2}(k = 2k_r, l = 0)\,, \tag{3.4}$$

$$\Psi_1 = \frac{1}{2}\frac{\partial^2 \omega_1}{\partial l^2}(k = k_r, l = 0)\,, \qquad \Psi_2 = \frac{1}{2}\frac{\partial^2 \omega_2}{\partial l^2}(k = 2k_r, l = 0)\,, \tag{3.5}$$

$$N_1 = 6k_r \frac{1 + \mu\xi_1^2 \xi_2}{1 + \xi_1^2}\,, \qquad N_2 = 6k_r \frac{1 + \mu\xi_1^2 \xi_2}{1 + \xi_2^2}\,. \tag{3.6}$$

The phase mismatch can be eliminated by the transformation $A = A'\, \mathrm{e}^{-\mathrm{i}\Delta k X - \mathrm{i}\sigma_1 T_2}$, $B = B'\, \mathrm{e}^{-2\mathrm{i}\Delta k X - \mathrm{i}\sigma_2 T_2}$ with

$$2\sigma_1 - \sigma_2 = (4\Phi_2 - 2\Phi_1)(\Delta k)^2\,. \tag{3.7}$$

Finally, again omitting primes, we arrive at the renormalized equations

$$\begin{aligned} &\mathrm{i}A_{T_2} + \Phi_1 A_{XX} + \Psi_1 A_{YY} + S_1 A + N_1 A^* B = 0\,, \\ &\mathrm{i}B_{T_2} + \Phi_2 B_{XX} + \Psi_2 B_{YY} + S_2 B + N_2 A^2 = 0\,, \end{aligned} \tag{3.8}$$

where

$$S_1 \equiv \sigma_1 + \Phi_1 (\Delta k)^2\,, \qquad S_2 \equiv \sigma_2 + 4\Phi_2 (\Delta k)^2 \equiv 2S_1\,. \tag{3.9}$$

We now see that we can set $S_1 = S_2 = 0$ by choosing $\sigma_1 = -\Phi_1 (\Delta k)^2, \sigma_2 = -4\Phi_2 (\Delta k)^2$, which satisfies (3.7). It is easy to see that, in terms of the new variables, the phases are $\theta_1 = \frac{1}{2}\theta_2 = k_r - \omega(k_r)t$ and are evaluated at precisely the resonant wavenumber (2.13). However, it is useful to retain S_1 and $S_2 \equiv 2S_1$ in (3.8), which is equivalent to adding the terms $-S_1 T_2$ and $-S_2 T_2$ to the phases θ_1 and θ_2, or replacing the resonant frequencies $\omega_1(k_r)$ and $\omega_2(2k_r) \equiv 2\omega_1(k_r)$ by $\omega_1(k_r) + \epsilon^2 S_1$ and $\omega_2(2k_r) + \epsilon^2 S_2$. Thus, S_1 may be regarded as an $\mathcal{O}(\epsilon^2)$ frequency detuning from the exact double-resonance point. Note that (3.9) implies that we can either exactly satisfy the first resonance condition (2.11), choosing $\Delta k = 0$, and introduce a frequency detuning via the remaining free parameter σ_1, or exactly satisfy the second resonance condition (2.12), introducing the detuning via the wavenumber mismatch Δk. We choose the former option, setting $\Delta k = 0$. It is important to mention that stationary-wave solutions to (3.8) only exist in the presence of the detuning $S_1 \neq 0$.

For both the ZK and KP models, one can check that $\Phi_{1,2} < 0$ for all values of $\eta = \eta(\delta)$ (see (2.14)) and λ, hence stationary solitary solutions to (3.8) may not exist unless $S_{1,2} > 0$, as one needs $\Phi_{1,2} S_{1,2} < 0$ to provide for the existence of the exponentially decaying soliton's tails far from its center. Then, using the fact that $N_1 N_2 > 0$ according to (3.6), the transformations

$$A = S_1 \sqrt{\Phi_2/2N_1N_2\Phi_1}\, A' , \qquad B = -S_1/N_1 B' ,$$
$$X = \sqrt{|\Phi_1|/S_1}\, X' , \qquad Y = \sqrt{|\Psi_1|/S_1}\, Y' , \qquad T_2 = -1/S_1 T' ,$$

cast the system (3.8) into the eventual form (where the primes are again dropped)

$$\begin{aligned} &\mathrm{i}A_T + A_{XX} + \sigma_A A_{YY} - A + A^* B = 0 , \\ &\tfrac{1}{2}\mathrm{i}\zeta B_T + B_{XX} + \sigma_B \gamma B_{YY} - \zeta B + \tfrac{1}{2}A^2 = 0 , \end{aligned} \tag{3.10}$$

where

$$\zeta = 2\Phi_1/\Phi_2 , \qquad \gamma = (\Phi_1/\Phi_2)|\Psi_2/\Psi_1| ,$$

and $\sigma_A = -\mathrm{sgn}(\Psi_1), \sigma_B = -\mathrm{sgn}(\Psi_2)$, i.e., the sign parameters $\sigma_{A,B}$ determine the ellipticity or hyperbolicity of the dispersion operator in the model. Because $\Phi_{1,2} < 0$, we have $\zeta > 0$. After straightforward algebra, we find that, in the physically meaningful case $\lambda > 0$, the ZK model is fully *elliptic* (i.e., simultaneously elliptic in FH and SH), having $\sigma_A = \sigma_B = +1$, whereas the KP one is fully *hyperbolic*, with $\sigma_A = \sigma_B = -1$.

The system (3.10) coincides with the model of light propagation in a 2D optical waveguide with a quadratic (SHG) nonlinearity [5], if Y and T, X are realized, respectively, as a so-called retarded time and a propagation, a transverse coordinate (in the stationary case, $\partial/\partial T = 0$, ζ plays the role of a *phase mismatch*, while γ is a *relative dispersion* [5]). Note that, in the optical model, a *mixed* case, when the FH equation is elliptic while the SH one is hyperbolic, may also take place (and is, in fact, quite feasible [5]).

Unlike the 1D case, where for $\zeta = 1$ an exact analytical solution to the system (3.10) is known [11], no exact solution is available in the 2D case. Nevertheless, the general stationary soliton solutions can be fairly well approximated by means of the variational approximation (VA). In the case $\sigma_A = +1$, the necessary VA was elaborated in Ref. [5], based on the Gaussian ansatz

$$A = a\,\mathrm{e}^{-\alpha_x X^2 - \alpha_y Y^2} , \qquad B = b\,\mathrm{e}^{-\beta_x X^2 - \beta_y Y^2} , \tag{3.11}$$

where the parameters were found to be

$$\begin{aligned} a^2 &= \frac{(2\alpha_x + \beta_x)(\alpha_x + \alpha_y + 1)(\beta_x + \sigma_B\gamma\beta_y + \zeta)(2\alpha_y + \beta_y)}{2\sqrt{\alpha_x\alpha_y\beta_x\beta_y}} , \\ b &= \tfrac{1}{2}(\alpha_x + \alpha_y + 1)\sqrt{\frac{(2\alpha_x + \beta_x)(2\alpha_y + \beta_y)}{\alpha_x\alpha_y}} , \\ \beta_x &= 4\alpha_x^2(\alpha_y - \alpha_x + 1)^{-1} , \qquad \beta_y = 4\alpha_y^2(\alpha_x - \alpha_y + 1)^{-1} . \end{aligned} \tag{3.12}$$

The remaining parameters $\alpha_{x,y}$ have to be determined as real positive roots of the equations

$$\begin{aligned} 2\beta_x(\beta_x + \sigma_B\gamma\beta_y + \zeta) &= (2\alpha_x + \beta_x)(-\beta_x + \sigma_B\gamma\beta_y + \zeta) , \\ 2\beta_y(\beta_x + \sigma_B\gamma\beta_y + \zeta) &= (2\alpha_y + \beta_y)(\beta_x - \sigma_B\gamma\beta_y + \zeta) . \end{aligned} \tag{3.13}$$

Formally, the solution exists for both signs of σ_B, i.e., for both the fully elliptic and mixed elliptic–hyperbolic systems (3.10). However, comparison with direct numerical simulations has demonstrated that, while VA produces quite reasonable results for the elliptic system, as well as for the marginal case $\sigma_B = 0$, the mixed elliptic–hyperbolic system does *not* support soliton solutions. A cause for this may be because the Gaussian ansatz does not correctly reproduce the exponentially decaying tails of the soliton, which becomes a fatal defect in the mixed case.

VA can also be formally applied to the other mixed hyperbolic-elliptic case, i.e., $\sigma_A = -1$, $\sigma_B = +1$, or the fully hyperbolic case $\sigma_A = -1$, $\sigma_B = -1$, that were not considered in Ref. [5]. However, consideration of the soliton's tails governed by the linearized equations indicates that a genuine 2D soliton is not possible in this case either. One should also bear in mind that, for any sign combination, (3.10) support the quasi-1D soliton corresponding to $\alpha_y = \beta_y = 0$ in (3.11).

Finally, we note that only time-independent real soliton solutions were considered above. A more general class of the solutions includes complex *walking* solitons [13], which, instead of X and Y, are localized with the respect to the "walking" variables $X - v_x T$ and $Y - v_y T$. This generalization is left beyond the framework of the present work.

4. Numerical results

In this section we will test the above approximate analytical results, directly simulating the underlying coupled ZK and KP equations. The main objectives are to test stability of the quasi-1D solitons in both models, and the existence and stability of the 2D soliton in the ZK model, that gives rise to the fully elliptic system (3.10), in which case the 2D soliton has a chance to exist. In all the cases, the simulations will be performed, using the approximate 1D [10] and 2D [5] soliton solutions generated by VA as initial conditions suitably transformed back into the original variables, cf. Ref. [8]. Special care has to be taken when simulating the coupled KP equations with periodic boundary conditions because of a spectral singularity at $k = 0$. To overcome this problem, we modify the initial conditions provided by VA, u_0 and v_0, subtracting from them their mean values, so that the integrals of these functions over the whole x-domain vanish. These corrected initial field configurations fields $u_{\rm corr}$ and $v_{\rm corr}$ are

$$u_{\rm corr}(x,y) \equiv u_0(x,y) - L_x^{-1} \int_0^{L_x} u_0(x,y)\,\mathrm{d}x\,, \tag{4.1}$$

and similarly for v, L_x being the size of the x-domain. For the integration in time we use a pseudo-spectral code, where the linear terms are treated by means of a semi-implicit Crank–Nicholson scheme, and the nonlinear terms are dealt with by means of an explicit leapfrog technique. For the numerical integration, the coupled equations were transformed into a reference frame which moves at the common group velocity v_G of the wave packets.

There are three free parameters, μ, λ and δ, because η is eliminated by means of (2.14). Note that $\mu = 0$ is also a possible case. It is pertinent to mention that when the parameters take values of order unity, the actual length of the wave packet is large, while its amplitude is small to satisfy $\epsilon\Delta k \ll k_r$.

In Fig. 1, we display a typical example of the evolution of a quasi-1D soliton for the linearly coupled KP equations (2.3), (2.4). Quite similar results were also obtained for the coupled ZK equations (2.1), (2.2). Comparing the initial configuration and the one generated by sufficiently long evolution, we conclude that the wave packet slightly readjusts itself to get closer to an exact solution. It has been checked independently that, over the same evolution time, an arbitrary initial wave packet strongly disperses, so that the observed stability of the wave packet predicted by VA is a nontrivial result. A number of simulations have been performed at other values of the parameters, all revealing that the quasi-1D approximate soliton solutions to the system (3.10), predicted by VA, are fairly close to an exact stationary soliton of the original coupled ZK or KP equations. It is pertinent to mention that the evolution of this quasi-1D soliton is exactly the same as in the coupled KdV equations studied in Ref. [8], provided that the initial configuration is strictly y-independent. However, a new result is that, in all the cases simulated, it has been found that the quasi-1D solitons are *stable* against

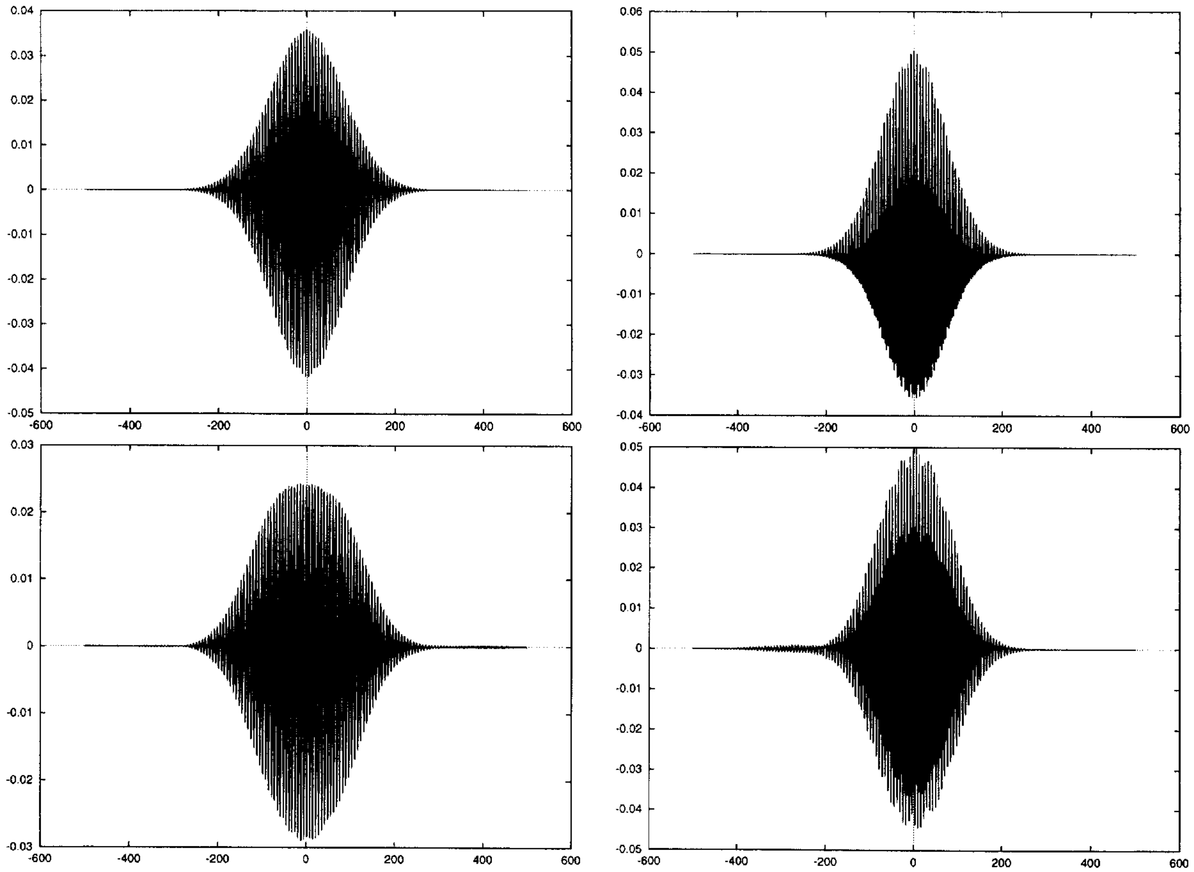

Fig. 1. A typical example of simulations of a quasi-one-dimensional soliton in the system of linearly coupled Kadomtsev–Petviashvili equations (2.3), (2.4). The left and right panels display, respectively, the initial configuration as suggested by the variational approximation, and the result of evolution at $t = 100$. The upper and lower pictures refer to $u(x)$ and $v(x)$. The parameters are $\mu = \eta = 1$, and $\delta = 4.6$. The corresponding resonant wavenumber (2.13) is $k_r = 0.9$. The shaded area of the solitary waves in the right panel stems from the superposition of the two harmonics.

y-dependent perturbations, within the framework of both ZK and KP equations. The stability of the quasi-1D soliton in the former case, when the corresponding asymptotic system (3.10) is purely elliptic, is an *unexpected* result (because in this case a fully localized stable 2D soliton also exists, see below, and hence one could expect that the quasi-1D soliton is apt to be unstable, "losing the competition" to the 2D one).

A typical example of simulations for the 2D soliton governed by the coupled ZK equations (2.1), (2.2) is depicted in Fig. 2. In this case, all the simulations performed at different values of the parameters demonstrate that VA successfully predicts a field configuration very close to a stable 2D soliton. Finally, in Fig. 3 we display an example of the evolution of a 2D KP soliton, which, in full accord with the hyperbolicity of the underlying SHG-equations, is unstable and breaks up.

5. Conclusion

In this work, we have derived asymptotic two-dimensional equations for linearly coupled Zakharov–Kuznetsov and Kadomtsev–Petviashvili equations describing the interaction between weakly nonlinear long waves in certain

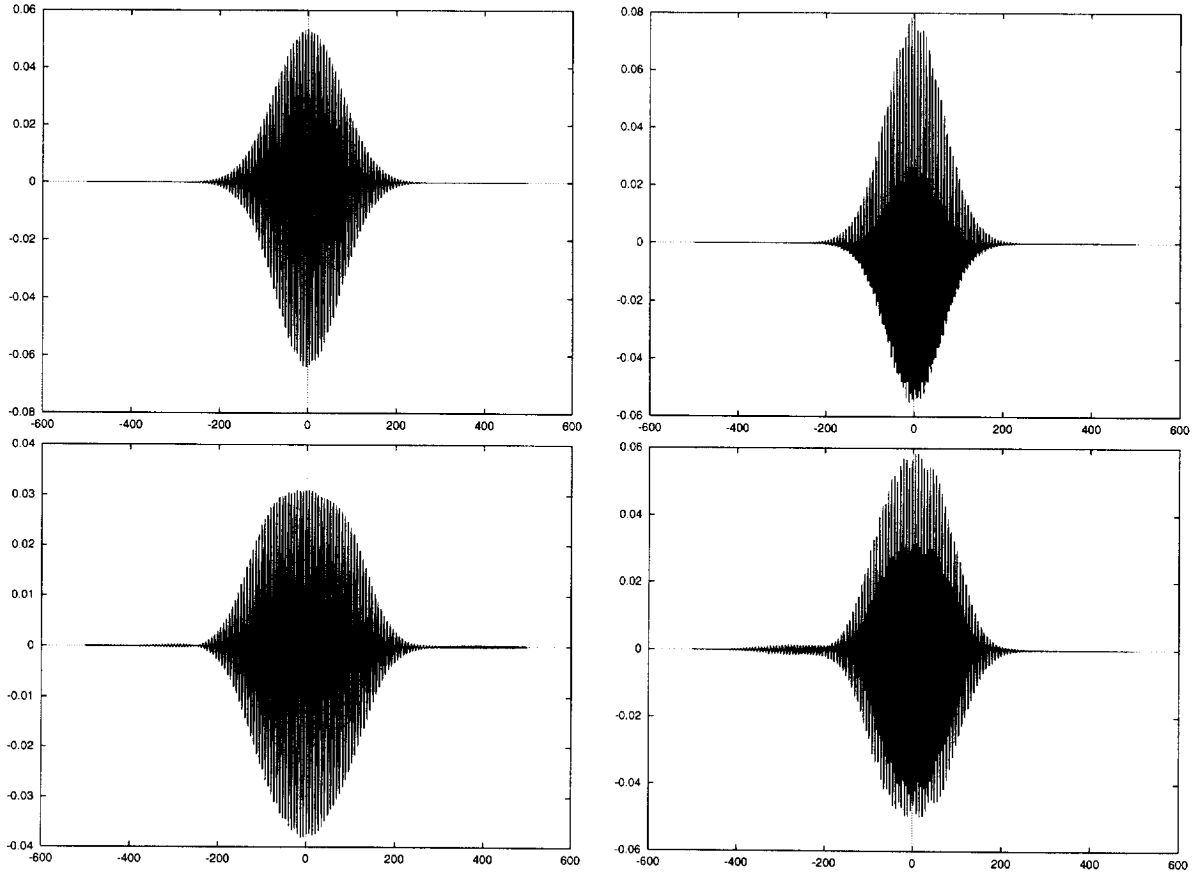

Fig. 2. The cross sections along the x axis (at $y = 0$) of the two-dimensional soliton in the coupled Zakharov-Kuznetsov equations (2.2), shown the same way as the one-dimensional soliton in Fig. 1. The stability of the two-dimensional soliton is clearly seen.

stratified fluid flows. The asymptotic equations coincide with the most general equations describing the second-harmonic generation in a two-dimensional optical waveguide, that take into account both temporal dispersion and spatial diffraction. The system corresponding to the Zakharov–Kuznetsov equations turns out to be fully elliptic, while the one corresponding to the Kadomtsev–Petviashvili equations is fully hyperbolic with respect to the spatial coordinates. The recently found "light-bullet" solutions to the elliptic second-harmonic-generation equations in nonlinear optics suggest existence of fully localized two-dimensional solitons in the corresponding Zakharov–Kuznetsov equations too.

Direct numerical simulations of both the coupled Zakharov–Kuznetsov and Kadomtsev–Petviashvili systems, for which the initial conditions are taken as suggested by the analytical variational approximation, completely corroborate the existence and stability of these solitons, as well as the fact that the variational approximation provides for a fairly high accuracy in predicting their shape. We have also demonstrated that the quasi-one-dimensional solitons are (numerically) stable against two-dimensional perturbations in both the Zakharov–Kuznetsov and Kadomtsev–Petviashvili coupled systems. The results reported in this work suggest that the two-dimensional spatio-temporal solitons, which are very hard to generate experimentally in nonlinear optics, may be generated much easier in certain fluid flows.

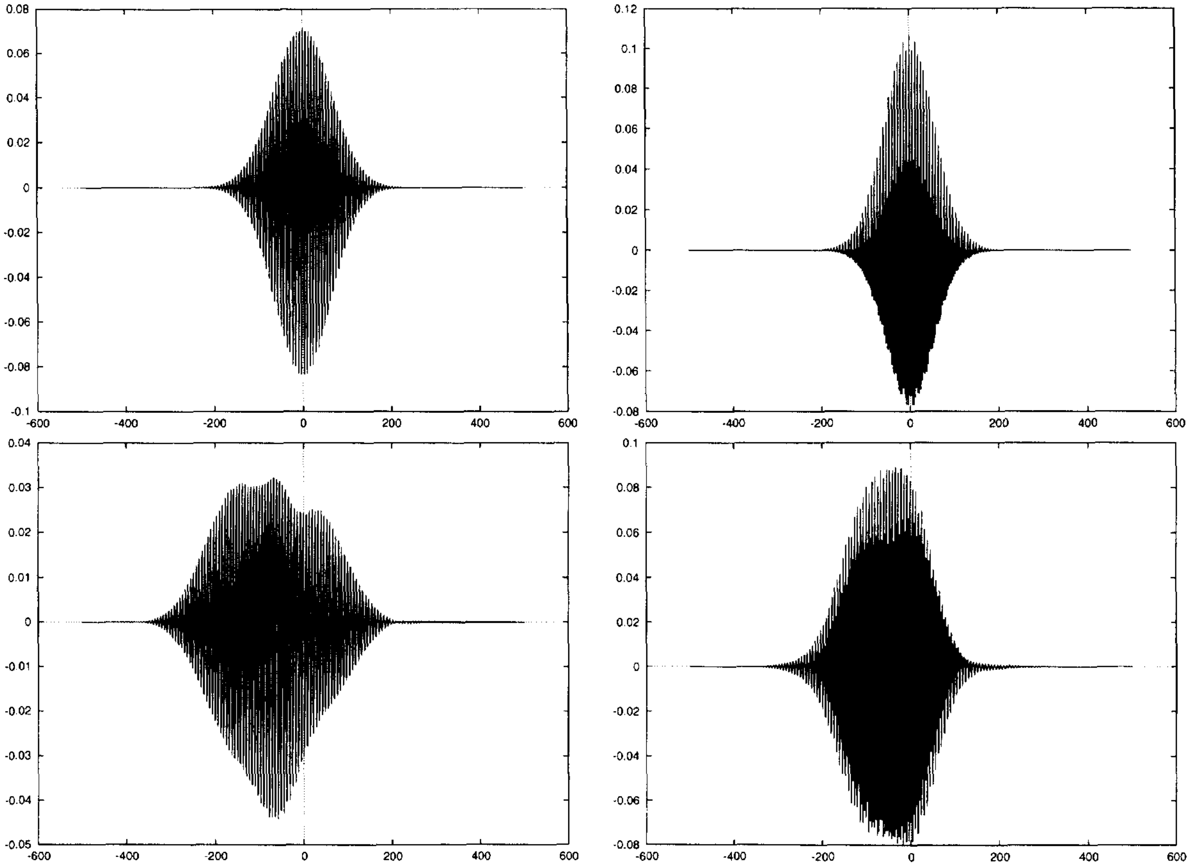

Fig. 3. The cross sections along the x axis (at $y = 0$) of the two-dimensional soliton in the coupled Kadomtsev–Petviashvili equations (2.3), (2.4), shown the same way as the one-dimensional soliton in Fig. 1. The instability of the two-dimensional soliton in this case is obvious.

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References

[1] G.I. Stegeman, D.J. Hagan, L. Torner, Opt. Quant. Electr. 28 (1996) 1691.
[2] R. Schiek, Y. Baek, G.I Stegeman, Phys. Rev. E 53 (1996) 1138.
[3] W.E. Torruellas, Z. Wang, D.J. Hagan, E.W. VanStryland, G.I. Stegeman, L. Torner, C.R. Menyuk, Phys. Rev. Lett. 74 (1995) 5036.
[4] K. Hayata, M. Koshiba, Phys. Rev. Lett. 71 (1993) 3275.
[5] B.A. Malomed, P. Drummond, H. He, D. Anderson, A. Berntson, M. Lisak, Phys. Rev. E 56 (1997) 4725.
[6] A.A. Kanashov, A.M. Rubenchik, Physica D 4 (1981) 122.
[7] A. De Rossi, S. Trillo, A.V. Buryak, Y.S. Kivshar, Opt. Lett. 22 (1997) 868.
[8] G. Gottwald, R. Grimshaw, B.A. Malomed, Phys. Lett. A 227 (1997) 47.
[9] Y. Zhu, R. Grimshaw, Stud. Appl. Math. 92 (1994) 249.
[10] V. Steblina, Y.S. Kivshar, M. Lisak, B.A. Malomed, Opt. Commun. 118 (1995) 345.
[11] Y.N. Karamzin, A.P. Sukhorukov, Sov. Phys. JETP 41 (1976) 414.

[12] G. Gottwald, R. Grimshaw, J. Atmos. Sci., submitted;
J. Gear, R. Grimshaw, Stud. Appl. Math. 70 (1984) 235;
H. Mitsudera, J. Atmos. Sci. 57 (1994) 734.
[13] L. Torner, D. Mazilu, D. Mihalache, Phys. Rev. Lett. 77 (1996) 2455;
C. Etrich, U. Peschel, F. Lederer, B.A. Malomed, Phys. Rev. E 55 (1997) 6155.