Applied Probability Trust (15 June 2009)

LETTER TO THE EDITOR

Dear Editor,

Convex hulls on a hemisphere

We write to describe how recent identities on convex hulls in Euclidean space, proved by this letter's first author (Cowan [2], [3]), can be applied to a study of random points on a hemisphere published by the second author (Miles [4]).

Cowan showed that for points $\mathsf{P}_1, \mathsf{P}_2, ..., \mathsf{P}_n$ distributed exchangeably in R^d, with any probability law μ for the common distribution of each point,

$$\mathbb{E}V_n = \frac{1}{2}\sum_{j=1}^{n-d-1}(-1)^{j-1}\binom{n}{j}\mathbb{E}V_{n-j}, \qquad (n-d) \geq 2 \text{ and even,} \tag{1}$$

where V_j is defined as the volume of H_j, the convex hull of $\mathsf{P}_1, \mathsf{P}_2, ..., \mathsf{P}_j$. We have noticed from Miles [4] that, somewhat surprisingly, the $(d = 2)$-version of this formula also holds on a hemisphere $\mathcal{H}$. So identities like $\mathbb{E}V_4 = 2\mathbb{E}V_3$, $\mathbb{E}V_6 = \frac{1}{2}(6\mathbb{E}V_5 - 15\mathbb{E}V_4 + 20\mathbb{E}V_3)$ and $\mathbb{E}V_8 = \frac{1}{2}(8\mathbb{E}V_7 - 28\mathbb{E}V_6 + 56\mathbb{E}V_5 - 70\mathbb{E}V_4 + 56\mathbb{E}V_3)$ – and so on – hold on the plane *and* on the hemisphere too.

A set X on a topologically-open hemisphere $\mathcal{H}$ (a hemisphere without its great-circle boundary) is convex if all shorter great circle arcs joining pairs of points $\in \mathcal{H}$ lie wholly within the set. With this definition, Miles studied the *polygonal convex hull* of n points uniformly and independently distributed on $\mathcal{H}$ – this being the convex set of minimal area on $\mathcal{H}$ covering all n points. This set's boundary comprises parts of great circles.

His Table 3 gives numerically the expectations of the convex hull's area V_n, perimeter S_n and number of sides N_n. These are given theoretically for $n \geq 1$ (using our notation and replacing equation (8.2) in [4]) by:

$$\mathbb{E}V_n = \pi(2 - n\gamma_{n-1}); \qquad \mathbb{E}S_n = 2\pi(1-\gamma_n); \qquad \mathbb{E}N_n \overset{n\geq 1}{=} \binom{n}{2}\gamma_{n-2}, \tag{2}$$

with $\mathbb{E}N_1 = 1$. Here, for $n \geq 0$,

$$\gamma_n := \int_0^\pi \left(1 - \frac{\theta}{\pi}\right)^n \sin\theta \; d\theta$$

and from this, we get $\gamma_0 = 2, \gamma_1 = 1$ and, for $n \geq 2$, $\gamma_n = 1 - \frac{n(n-1)}{\pi^2}\,\gamma_{n-2}$. Using this γ_n-recurrence it is easily shown that the $(d = 2)$-version of (1) holds. It is also easy to establish that Cowan's identity for $\mathbb{E}N_n$ in R^d, namely

$$\mathbb{E}N_n = \frac{n}{2} + \frac{1}{2}\sum_{j=1}^{n-1}(-1)^{j-1}\binom{n}{j}\mathbb{E}N_{n-j}, \qquad (n-d) \geq 3 \text{ and odd}, \tag{3}$$

augmented by $\mathbb{E}N_n = n$ for $n \leq d+1$, applies also in Miles's hemisphere study; one can readily show that the expression for $\mathbb{E}N_n$ given in (2) satisfies (3). Identity (3) has been proved in [3] under less general conditions than those in identity (1) – *exchangeability* is replaced by *independence* and the probability law μ must give zero measure to any j-dimensional flat $(j < d)$.

Greater geometric insight on why the volume identity (1) applies to a hemisphere can be gained by utilising another identity, proved in [3]:

$$\mathbb{E}\nu(\mathsf{H}_n) = \frac{1}{2}\sum_{j=1}^{n-d-1}(-1)^{j-1}\binom{n}{j}\mathbb{E}\nu(\mathsf{H}_{n-j}), \tag{4}$$

where ν is any measure on the Borel sets of R^d, absolutely continuous with respect to Lebesgue measure. Equation (1) is an example of (4) with ν equal to the volume measure V in R^d.

Let us construct another example by projecting n points, which have been placed randomly on $\mathcal{H}$ using *any* exchangeable probability law, into R^2 via the usual projection $\mathbf{P}$ from an open hemisphere to the plane. This projection is defined as follows. Place the centre of a sphere $\mathcal{S}$ of radius r at the point $C = (0, 0, r) \in \mathsf{R}^3$; so $\mathcal{S}$ is tangential to the xy-plane. The hemisphere of $\mathcal{S}$ lying in the region $0 \leq z < r$ is called $\mathcal{H}$. Then, for $Q \in \mathcal{H}$, $\mathbf{P}(Q) := P$ where P is the unique point in the xy-plane such that C, Q and P are collinear.

Thus we obtain as a result of projection n exchangeable points $\mathsf{P}_1, \mathsf{P}_2, ..., \mathsf{P}_n \in \mathsf{R}^2$ with some common exchangeable law. We now choose the measure ν on R^2

induced by the projection. That is, a Borel set X in R^2 is given the measure $\nu(X)$ equal to the area of the set on the hemisphere which, when projected, gives X. This is obviously absolutely continuous, so (4) holds, initially as a result for the planar convex hulls. But the polygonal convex hull of j points on $\mathcal{H}$ projects into the convex hull of the projections of these j points (and vice versa using the inverse map), so the result holds for the polygonal convex hulls on the hemisphere.

So we have proved that (1) applies to points distributed with any exchangeable probability distribution on $\mathcal{H}$ – a rather more general situation than the independent uniform case studied by Miles. Furthermore, because the hemispherical projection conserves N_n, we have also proved that (3) applies for hemispheres – when points are distributed independently by any probability measure that give zero mass to points and great circles in $\mathcal{H}$.

The findings of this letter also apply to the situation where n points are distributed exchangeably on a sphere, conditional upon all of them being contained in some hemisphere; this conditional construction was also considered in [4] for the independent uniform case.

Remark 1. Miles's formula (6.16) in [4] is incorrect. It should be:

$$\begin{aligned}\gamma_n &= 1 + \sum_{i=1}^{\frac{n}{2}-1} \binom{n}{2i} \frac{(-1)^i (2i)!}{\pi^{2i}} + 2(-1)^{n/2} \frac{n!}{\pi^n}, \qquad n \text{ even} \\ &= 1 + \sum_{i=1}^{\frac{n-1}{2}-1} \binom{n}{2i} \frac{(-1)^i (2i)!}{\pi^{2i}} + (-1)^{(n-1)/2} \frac{n!}{\pi^{n-1}}, \qquad n \text{ odd.}\end{aligned}$$

Remark 2. Formula (1) first appeared in 1990 (see [1]) using the less-general assumption that points are independent and identically distributed. Our argument depends on the more recent result (4). By a similar use of projection arguments, we can prove that (4), with ν replaced by ν^* (an absolutely continuous measure on the Borel sets of $\mathcal{H}$), holds on $\mathcal{H}$.

Remark 3. The findings of this letter apply to hemispheres of higher-dimension, because (1) and (4) are formulae in R^d, and a suitable projection $\mathbf{P}$ mapping $\mathcal{H}$ (which is now part of a d-sphere $\subset \mathsf{R}^{d+1}$) to R^d can be defined.

References

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