Assignment 3

Due Thursday, 1 September, at 16:00 in the assignment box for MATH3405. The box is located on **Level 4** in the Mathematics (Priestley) building (67). It is number 035 in the **brown lot of boxes** (there are two lots, to find ours, turn right as you come from the stairs).

Clearly state your assumptions and conclusions, and justify all steps in your work. Marks will be deducted for sloppy or incomplete working. Please use a cover sheet!

Fix $a, b \in \mathbb{R}$ satisfying $0 < b < a$, and let

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq b\} \subseteq \mathbb{R}^3.$$

Define $f \colon U \to \mathbb{R}^3$ by

$$f(x, y, z) = \Big((a + z\cos x)\cos y, (a + z\cos x)\sin y, z\sin x\Big).$$

E (Exploration)

1. Draw a sketch of $f(U)$ and determine the geometric significance of x, y and z.
2. Find a subset $V \subseteq U$ such that $f(V) = f(U)$ and $f|_V$ is injective.

D (Derivatives)

3. Show that f is differentiable and determine $df_{(x,y,z)} \colon \mathbb{R}^3 \to \mathbb{R}^3$ for each $(x, y, z) \in U$.
4. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. For every $(x, y, z) \in U$, compute the directional derivative of f at (x, y, z) in each of the directions e_i.
5. For each $(x, y, z) \in U$, show that the three directional derivatives from (4) are pairwise orthogonal and determine their lengths.
6. Use your working from (5) to determine all $(x, y, z) \in U$, where $df_{(x,y,z)}$ has full rank.

V (Volumes)

7. Compute the volume of $f(U)$ using a triple integral in the domain.
8. Interpret your answer to (7) geometrically.

C (Curves)

Given $p, q \in \mathbb{Z}$, define the curve $\alpha_{p,q} \colon [0, 2\pi] \to U \subset \mathbb{R}^3$ by $\alpha_{p,q}(t) = (pt, qt, b)$.

9. Compute the length of $\alpha_{p,q}$.
10. Sketch the curves $f \circ \alpha_{1,0}$, $f \circ \alpha_{0,1}$, $f \circ \alpha_{2,0}$, and $f \circ \alpha_{2,3}$. Then describe the curve $f \circ \alpha_{p,q}$ in words, explaining the significance of the parameters p and q.
11. Compute $(f \circ \alpha_{p,q})'(t)$ for all $t \in [0, 2\pi]$ using the chain rule.
12. Determine whether $f \circ \alpha_{p,q}$ is a regular curve.
13. Write down an integral giving the length of $f \circ \alpha_{p,q}$. Simplify the expression as much as possible, but don't evaluate it! Does the integrand (speed) make sense geometrically?
14. Using your integral from the previous part, compute the lengths of $f \circ \alpha_{1,0}$ and $f \circ \alpha_{0,1}$, and compare them with the lengths of $\alpha_{1,0}$ and $\alpha_{0,1}$ respectively.
15. Determine an upper bound for the length of $f \circ \alpha_{p,q}$ in terms of a, b, p, q. Is your bound sharp for $f \circ \alpha_{1,0}$ and $f \circ \alpha_{0,1}$?