Assignment 1

"Unit balls"

Your solutions should be submitted by the beginning of the lecture on Tuesday, 16 August 2011.

Please attach a cover sheet!

Q1 A subset S of a vector space is *convex* if for all $x, y \in S$, we have

$$\{\alpha x + (1-\alpha)y \mid 0 \leq \alpha \leq 1\} \subseteq S.$$

(a) Show that $B(X) = \{x \in X \mid ||x|| \leq 1\}$ is convex, where $X = (V, ||\cdot||)$ is a normed space.

(b) Use (a) to show that the function $||\cdot||: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$||(a,b)|| = (\sqrt{|a|} + \sqrt{|b|})^2$$

does not define a norm on $\mathbb{R}^2$.

Q2 For each $x = (a,b) \in \mathbb{R}^2$, let $||x||_1 = |a| + |b|$ and $||x||_2 = \sqrt{a^2+b^2}$. You may assume that $||\cdot||_1$ and $||\cdot||_2$ are norms on $\mathbb{R}^2$.

(a) For which vectors $x, y \in \mathbb{R}^2$ do we have $||x||_1 + ||y||_1 = ||x+y||_1$?

(b) For which vectors $x, y \in \mathbb{R}^2$ do we have $||x||_2 + ||y||_2 = ||x+y||_2$?

(c) Formulate a general hypothesis about normed spaces X containing linearly independent elements $x, y \in X$ such that $||x|| + ||y|| = ||x+y||$. Can you prove this hypothesis?

Q3 Let V be a real vector space and $\emptyset \neq D \subset V$ a subset satisfying:

(a) If $x, y \in D$ and $\alpha, \beta \in \mathbb{R}$ such that $|\alpha| + |\beta| \leq 1$, then $\alpha x + \beta y \in D$.

(b) If $x \in D$, then there is $\varepsilon > 0$ such that $x + \varepsilon D \subset D$.

(c) For each non-zero $x \in V$, there exist non-zero $\alpha, \beta \in \mathbb{R}$ such that $\alpha x \in D$ and $\beta x \notin D$.

Define

$$||x|| = \inf\{\ t \mid t > 0,\ x \in tD\ \}.$$

Show that this defines a norm on V and, moreover, that D is the open unit ball with respect to this norm. (Hint: First observe that if $x \in D$, then $\alpha x \in D$ for all $\alpha \in [-1, 1]$.)