Problem Set 2

Q11 Show that a linear operator between two normed spaces is continuous if and only if it is continuous at some point.

Q12 Show that $\mathfrak{B}(X,Y)$ is a subspace of $\mathfrak{L}(X,Y)$.

Q13 Verify that the operator norm is indeed a norm and check that the five given descriptions are all equivalent.

Q14 Let $X = (\ C[a,b],\ ||\cdot||_\infty\ )$, where $C[a,b]$ is the real vector space of all continuous functions defined on the interval $[a,b]$, and

$$||\ f\ ||_\infty = \max_{t\in[a,b]} |f(t)|$$

is the uniform norm. Show that

$$I(f) = \int_a^b f(t)dt$$

is a bounded linear functional and determine $||\ I\ ||$.

Q15 Given a linear operator $T\colon X \to Y$, recall that the adjoint $T^*\colon Y^* \to X^*$ satisfies:

$$\langle x, T^*F\rangle = \langle Tx, F\rangle$$

for all $x \in X$ and all $F \in Y^*$.

(a) Show that T^* is the unique linear operator $Y^* \to X^*$ satisfying this equality.
(b) Show that $||T^*|| \leq ||T||$.
(c) Show that $||T^*|| = ||T||$.

Q16 For $T \in \mathfrak{B}(X,Y)$ define

$$\gamma(T) = \inf\{\frac{||Tx||}{||x||} \mid x \neq 0\}.$$

Show that T is invertible with $T^{-1} \in \mathfrak{B}(Y,X)$ if and only if T is surjective and $\gamma(T) > 0$.

Q17 Let $p \geq 1$ and let R and L be the right and left shift operators on l_p:

$$R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots),$$
$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Show that

(a) R and L are linear and bounded and find $||R||$ and $||L||$;
(b) $LR = I$ but $RL \neq I$;
(c) $||L^n x|| \to 0$ for each $x \in l_p$, but $||L^n||$ does not converge to zero.

Q18 Let $X = C^1[0,1]$ be the vector space of all continuous differentiable functions and $Y = C[0,1]$. Let $||\cdot||$ be the supremum norm on both X and Y, and define on X the norm

$$||f||_1 = ||f|| + ||f'||,$$

where $f'(t)$ is the derivative with respect to t. Let D be the differential operator $D(f) = f'$. Show that

(a) $D\colon (X, ||\cdot||_1) \to (Y, ||\cdot||)$ is a bounded linear operator with $||D|| = 1$, and
(b) $D\colon (X, ||\cdot||) \to (Y, ||\cdot||)$ is an unbounded linear operator.