Problem Set 3

Q19 Let X be a normed space and $f \in X'$.
Show that $f \in X^*$ if and only if $\ker f$ is closed (in the norm topology).

Q20 Let X be the normed space with underlying vector space $\mathbb{R}^2$ and norm defined by

$$||(x, y)|| = \max\{|x|, |y|, |x + y|\}.$$

(a) Sketch the unit ball of this norm.

(b) Find the dual norm (i.e. the operator norm) on X^* by determining its unit ball.

(c) Is X isometric with X^*?

Q21 Let X, Y and Z be normed spaces over the same field, and $S \in \mathfrak{B}(X, Y)$, $T \in \mathfrak{B}(Y, Z)$. Show that $(TS)^* = S^*T^*$.

Q22 (a) Show that two equivalent norms on the same vector space induce the same topology.

(b) Show that if $(X, || \cdot ||_0)$ is complete and $|| \cdot ||_1$ equivalent to $|| \cdot ||_0$, then $(X, || \cdot ||_1)$ is complete.

Q23 Show that $C[a, b]$ with the uniform norm is complete.

Q24 Check the details in the proof of the "Completion Theorem 2.28":

(a) $\sim$ is an equivalence relation;

(b) the vector space structure and the norm on $\widehat{X}$ are well-defined;

(c) the candidate sequence $(y_k)_{k\in\mathbb{N}}$ is Cauchy and the limit of $(\hat{x}_n)_{n\in\mathbb{N}}$.

Q25 (a) Let X be a normed space and Z be a subspace that is closed in the norm topology. Define the quotient norm on the quotient space X/Z by:

$$||x + Z|| = \inf\{\ ||y|| \ \mid\ y \in x + Z\} = \inf\{\ ||x + z|| \ \mid\ z \in Z\}.$$

Show that this is a well-defined norm.

(b) Does the subspace Z need to be closed in the above?

(c) Let X and Y be normed spaces and $T \in \mathfrak{B}(X, Y)$. Let $\overline{T}\colon X/\ker(T) \to Y$ be the linear map induced by T. Let $Z = \ker(T)$ and give X/Z the quotient norm (why is this possible?). Show that $\overline{T} \in \mathfrak{B}(X/Z, Y)$ and $||\overline{T}|| = ||T||$.

Q26 Let V be the vector space of all scalar sequences $x = (x_k)_{k=1}^{\infty}$ with at most finitely many non-zero terms. For $1 \le p \le \infty$, let $X_p = (V, || \cdot ||_p)$, where

$$||x||_p = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} \qquad (1 \le p < \infty)$$

and

$$||x||_\infty = \max\{|x_k| \mid 1 \le k \le \infty\}.$$

You may assume that each space X_p is a normed space.
For $1 \le r, s \le \infty$, let $T_{r,s}\colon X_r \to X_s$ be the formal identity map $T_{r,s}x = x$. Then $T_{r,s}$ is a linear operator.

(a) Show that if $r \le s$, then $T_{r,s}$ is bounded and $||T_{r,s}|| = 1$.

(b) Show that if $r > s$, then $T_{r,s}$ is unbounded.