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Algebras of Intermediate Growth*

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We investigate finitely generated associative algebras and Lie algebras in which the dimension of the nth term of the natural filtration grows faster than any polynomial in n but more slowly than any exponential term c^n. As examples, we consider associative and Lie algebras generated by two general vector fields on the real line.

Part I. Preliminary investigations

1.1. Let A be an infinite-dimensional algebra over a field K. We suppose that A is finitely generated and denote by A_n the space of those elements that may be written in the form of a polynomial (of degree at most n) of generators. Henceforth, A will be either an associative algebra with a unit or a Lie algebra. In the former case, we will set $A_0 = K$, and in the latter case, $A_0 = \{0\}$.

The essential feature of A is that the sequence $a_n = \dim A_n$ is an increasing sequence. The numbers a_n grow polynomially for many important examples of algebras, i.e., $a_n \sim cn^d$ as $n \to \infty$.

The number

$$d = \lim_{n \to \infty} \frac{\ln a_n}{\ln n}$$

is sometimes called the *Gelfand–Kirillov dimension* of the algebra A and denoted by Dim A (cf. [9]). Simple verification shows that Dim A is independent of the choice of the generators in A (unlike the coefficient C, which may vary).

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Examples. 1. An algebra of regular functions on an algebraic affine variety X. In this case, $\operatorname{Dim} A = \dim X$ and $c = (1/d!) \deg X$ (cf. [7]).

2. The enveloping algebra $U(\mathfrak{G})$ of a finite-dimensional Lie algebra G. Here $\operatorname{Dim} U(\mathfrak{G}) = \dim \mathfrak{G}$.

3. The subalgebra generated by principal vectors corresponding to simple roots in the contragredient Lie algebra (cf. [5]). In this case, $\operatorname{Dim} A = 0, 1$, or ∞.

In recent years, interest has grown in algebras A for which $\operatorname{Dim} A$ is infinite. The exterior Lie algebra with k generators and its enveloping algebra (which is isomorphic to an exterior associative algebra with k generators or a tensor algebra over a k-dimensional space) is an example of those algebras. In these cases, the numbers a grow exponentially (in the case of a Lie algebra, $a_n \sim n^{-1}k^n$; in the case of an associative algebra, $a_n \sim k^n$).

Algebras for which the numbers a_n grow more slowly than any exponential function $c^n, c > 1$, are of particular interest. We will call them *algebras of intermediate growth*. (The concept of an algebra of subexponential growth was introduced in a somewhat different setting; cf. [1].) The contragredient algebras of infinite growth for which the asymptotic weight multiplicity has been recently calculated [4] would appear to belong to this class (cf. Example 3 above).

1.2. Suppose that $V = \bigoplus_{k=0}^{\infty} V^k$ is a graded vector space over a field K. The series

$$P_V(t) = \sum_{k=0}^{\infty} (\dim V^k) t^k \tag{1}$$

is called the *Poincaré series* of the graded space V. This definition may be generated naturally in two ways. First, we may consider V to be a space with increasing filtration:

$$\{0\} = V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \subset V.$$

With such a space we may associate canonically the graded space $\operatorname{gr} V = \bigoplus_{k=0}^{\infty} \operatorname{gr}^k V$ where $\operatorname{gr}^k V = V_k / V_{k-1}$. By definition, we set $P_V(t) = P_{\operatorname{gr} V}(t)$. Second, we may consider, semigraded spaces. With these spaces, we may associate formal power series of n variables specified by Equation (1), where k denotes the multi-index $(k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, t is the set of variables $(t_1, \ldots, t_n)$, and t^k is the monomial $t_1^{k_1} \ldots t_n^{k_n}$.

We now present certain (well-known) facts about Poincaré series.

The relations

$$P_{V_1 \oplus V_2} = P_{V_1} + P_{V_2}, \qquad P_{V_1 \otimes V_2} = P_{V_1} \cdot P_{V_2} \tag{2}$$

are self-evident: from these, there immediately follow the equalities

$$P_{T^k(V)} = P_V^k, \qquad P_{T(V)} = (1 - P_V)^{-1} \tag{3}$$

where $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$ is a tensor algebra over V.

The relations between P_V and $P_{S(V)}$ and between P_V and $P_{\Lambda(V)}$ is more complicated. Here $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$ is a symmetric algebra over V, and $\Lambda(V) = \bigoplus_{k=n}^{\infty} \Lambda^k(V)$ is an exterior algebra over V. That is, if

$$P_V(t) = \sum a(k) t^k, \qquad \ln P_{S(V)} = \sum b(k) t^k, \qquad \ln P_{\Lambda(V)} = \sum c(k) t^k,$$

then

$$b(k) = \sum_{d|k} d^{-1} a(k/d), \qquad c(k) = \sum_{d|k} (-1)^{d-1} d^{-1} a(k/d), \tag{4}$$

where the summation extends over all natural divisors d of the multi-index k. To prove Equation (4), note that by virtue of the relations

$$S(V_1 \oplus V_2) \cong S(V_1) \otimes S(V_2), \qquad \Lambda(V_1 \oplus V_2) \cong \Lambda(V_1) \otimes \Lambda(V_2)$$

the coefficients $b(k)$ and $c(k)$ depend linearly on the sequence $\{a(k)\}$; therefore it is sufficient to check (4) in the case of a one-dimensional space V.

Equations (4) may be inverted and the coefficients $a(k)$ expressed in terms of $b(k)$ or $c(k)$. In the case of a symmetric algebra, the explicit formula has the form

$$a(k) = \sum_{d|k} \frac{\mu(d)}{d} b(k/d). \tag{5}$$

1.3. The explicit form of the relation between the growth of the sequence $a(n)$ and the behavior of the sum of the series

$$f(t) = \sum_{n \geq 0} a(n) t^n$$

in a neighborhood of $t = 1$ will also be useful to us. If $a(n)$ grows polynomially and if $a(n) \sim cn^d$, the function $f(t)$ will be rational and will have a unique pole of order $d+1$ at the point $t = 1$. At this point, the principal term of the decomposition of f is equal to

$$\frac{c \cdot d!}{(1-t)^{d+1}}.$$

The next case (in terms of the order of growth) was investigated by Ramanujan [8]. Here, if the sequence $a(n)$ grows as $l^{A_n \alpha}$, i.e.,

$$\lim_{n \to \infty} \frac{\ln a(n)}{n^\alpha} = A, \tag{6}$$

where $0 < \alpha < 1$, then at the point $t = 1$, f has an asymptote of the form $l^{\beta}/(1-t)^{\beta}$. More precisely,

$$\lim_{t \to 1} \ln f(t)(1-t)^{\beta} = B, \tag{7}$$

where $\beta = \alpha/(1-\alpha)$ and $B = (1+\alpha)\alpha^{\alpha/(1-\alpha)} A^{1/(1-\alpha)}$.

In particular, if $\ln a(n) \sim A\sqrt{n}$, then $\ln f(t) \sim \frac{1}{4}A^2/(-t)$.

It would be of interest to obtain more exact and more exhaustive information about the correspondence between the growth of $a(n)$ as $n \to \infty$ and the growth of $f(t)$ as $t \to 1$.

That $a(n)$ exhibits intermediate growth is equivalent to the assertion that $f(t)$ possesses a circle of covergence of radius 1 and an essential singularity at the point 1. The following assertion is a consequence of this fact.

Theorem 1. *A Lie algebra $\mathfrak{G}$ of infinite growth has intermediate growth if and only if its enveloping algebra $U(\mathfrak{G})$ possesses this property.*

In fact, a natural filtration such that

$$\operatorname{gr} U(\mathfrak{G}) \cong S(\mathfrak{G})$$

may be defined in $U(\mathfrak{G})$. In the case of infinite growth, the sequences $\{a(n)\}$ and $\{b(n)\}$ connected by Equations (4) and (5) satisfy, as may be easily verified, the bounds:

$$a(n) \le b(n) \le \text{const} \cdot a(n).$$

Therefore, the series $P_G(t)$ and $\ln P_{U(\mathfrak{G})}(t)$ (that is, $P_{U(\mathfrak{G})}(t)$ as well) have the same radius of convergence.

1.4. One tool for the investigation of infinite-dimensional algebras is consideration of the corresponding quotient rings. Let us suppose that an algebra A possesses a filtration $\{A_n\}$ such that the corresponding graded algebra $\operatorname{gr} A$ does not contain any divisors of 0 and has intermediate growth. It turns out that in this case A is an Ore algebra (cf. [3]), i.e., any two nonzero elements x and y in A have a nonzero common right (and left) multiple.

In fact, let us consider the right ideals xA and yA in A. If x and y do not have a common right multiple, these two spaces will intersect only at 0. Therefore, if $x \in A_k$ and $y \in A_l$, then $xA_{n-k} + yA_{n-l} \subset A_n$ and

$$a(n) \ge a(n-k) + a(n-l),$$

where $a(n) = \dim A_n$. Hence, the lower bound

$$a(n) \ge c \cdot 2^{n/\max(k,l)}$$

is easily found, where $c > 0$. This bound contradicts the claim that A is an algebra with intermediate growth (and consequently $\lim \ln a(n)/\ln n = 0$).

Thus, algebras of intermediate growth without zero divisors possess a quotient ring. From Section 1.3, it follows that a Lie algebra of intermediate growth is embedded in a Lie quotient ring (the quotient ring of the enveloping algebra). The study of such quotient rings promises to be of great interest.

Part II. Lie algebra generated by two general vector fields on the real line

2.1. Suppose that Vect R^1 denotes a Lie algebra of smooth vector fields on the real line. We let $L(x, y)$ denote the exterior Lie algebra with generators x and y. Every pair of fields $\xi, \eta \in \text{Vect } R^1$ specifies a homomorphism $\varphi_{\xi,\eta}$ of $L(x, y)$ into Vect R^1 such that $\varphi_{\xi,\eta}(x) + \xi$, and $\varphi_{\xi,\eta}(y) = \eta$. Let $I(\xi, \eta)$ be the kernel of this homomorphism. We set $I = n_{\xi,\eta} I(\xi, \eta)$. We will say that the fields ξ_0 and η_0 are in *general position* if $I(\xi_0, \eta_0) = I$. It may be verified that the fields $\xi = d/dt$ and $\eta = u(t) \cdot d/dt$ are in general position if the functions $u(t), u'(t), u''(t), \ldots$ are algebraically independent.

Lemma 1. *The ideal I coincides with the intersection of those ideals $I(\xi, \eta)$ such that $\xi = d/dt$ and*

$$\eta = \sum_{k=1}^{N} c_k e^{\lambda_k t} d/dt.$$

In fact, any nonzero field ξ may be locally reduced to the form d/dt by an appropriate selection of coordinates. The field η may be approximated locally in the C^{∞}-topology by fields of this form. The lemma is proved.

Our goal in the second part of the article is to investigate the algebra $\mathfrak{A} = L(x, y)/I$.

The algebra $L(x, y)$ is bigraded (by degrees relative to x and y): $L(x, y) = \bigoplus_{k,l} L(x, y)^{k,l}$. It is clear that I is a homogeneous ideal relative to this bigrading and, consequently,

$$I = \bigoplus_{k,l} I^{k,l}, \qquad \mathfrak{A} = \bigoplus_{k,l} \mathfrak{A}^{k,l}.$$

2.2. Let us fix an integer $l \ge 0$. The space $L^{*,l} = \bigoplus_k L(x, y)^{k,l}$ is generated by monomials of the form

$$X_{(k)} = (\operatorname{ad} x)^{k_l} \operatorname{ad} y (\operatorname{ad} x)^{k_{l-1}} \cdots \operatorname{ad} y (\operatorname{ad} x)^{k_1} y, \tag{8}$$

where (k) denotes the set $k_1, \dots, k_l$ of nonnegative integers. We will denote the sum $k_1 + k_2 + \cdots + k_l$ by $|(k)|$.

Lemma 2. *The linear combination $\sum_{|(k)|=k} c_{(k)} X_{(k)}$ belongs to $I^{k,l}$ if and only if it vanishes for any substitution of the form*

$$x \to d/dt, \qquad y \to \sum_{i=1}^{l} a_i e^{\lambda_i t} d/dt,$$

where a_i and λ_i may be thought of as independent parameters.

Proof. The expression is a homogeneous polynomial function of degree l in y. For this function to vanish identically, it is sufficient for it to be equal to 0 on any l-dimensional subspace. Now we need only resort to Lemma 1. Lemma 2 is proved.

By means of the standard transformation of a polynomial function of degree l to a symmetric l-linear form (polarization), it may be proved that the premise of Lemma 2 is equivalent to the assertion that the coefficient of $a_1 a_2 \cdots a_l$, in the expression obtained after the substitution described above, vanishes. This coefficient may be calculated explicitly and has the form

$$\sum_{\sigma \in S(l)} \sum_{(k)} c_{(k)} P_l(\sigma\lambda) \cdot q_{(k)}(\sigma\lambda) \exp\left(t \sum_{i=1}^{l} \lambda_i\right) d/dt,$$

where

$$\sigma\lambda = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(l)}),$$

$$P_l(\lambda) = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \lambda_3) \cdots (\lambda_1 + \cdots + \lambda_{l-1} - \lambda_l), \tag{9}$$

$$q_{(k)}(\lambda) = \lambda_1^{k_1}(\lambda_1 + \lambda_2)^{k_2} \cdots (\lambda_1 + \cdots + \lambda_l)^{k_l}. \tag{10}$$

Let Sym denote the natural projector in the space $\mathbf{C}[\lambda_1, \dots, \lambda_l]$ onto a subspace of symmetric polynomials:

$$\operatorname{Sym} p(\lambda) = \frac{1}{l!} \sum_{\sigma \in S(l)} p(\sigma\lambda).$$

Now we define the mapping of the space $L^{\varkappa,l}$ into $\operatorname{Sym} \mathbf{C}[\lambda_1, \dots, \lambda_l]$ by setting

$$\varkappa(X_{(k)}) = \operatorname{Sym} p_l q(k). \tag{11}$$

The preceding line of reasoning may be summarized as follows.

Theorem 2. *The kernel of the mapping $\varkappa$ coincides with the space*

$$I^{*,l} = \bigoplus_{k=0}^{\infty} I^{k,l}.$$

Thus, the mapping $\varkappa$ specifies an isomorphism of the space $\mathfrak{A}^{*,l} = \bigoplus_{k=0}^{\infty} \mathfrak{A}^{k,l}$ onto some subspace of $\operatorname{Sym} \mathbf{C}[\lambda_1, \dots, \lambda_l]$.

We denote this space by K_l. It is graded in a natural way by degree relative to the set of variables λ_i. Obviously, the resulting isomorphism between $\mathfrak{A}^{*,l}$ and K_l is a homogeneous mapping of degree $\deg P = l - 1$.

2.3. Let us now study the structure of the space K_l. We denote by J_l the ideal in $\mathbf{C}[\lambda_1, \dots, \lambda_l]$ generated by the $l!$ polynomials

$$p_{\sigma,l}(\lambda) = p_l(\sigma\lambda), \qquad \sigma \in S(l).$$

Theorem 3. *The following equality is satisfied:*

$$K_l = \operatorname{Symp} J_l. \tag{12}$$

Proof. By definition, K_l is the linear hull of polynomials of the form $\operatorname{Sym} p_l q(k)$ and, consequently, is embedded in $\operatorname{Sym} J_l$. To prove the converse embedding, note that every element $\operatorname{Sym} J_l$ has the form

$$\operatorname{Sym} \sum_\sigma p_{\sigma,l} U_\sigma = \sum_\sigma \operatorname{Sym} p_{\sigma,l} U_\sigma = \sum_\sigma \operatorname{Sym} p_l \sigma^{-1} U_\sigma \in \operatorname{Sym} p_l \mathbf{C}[\lambda, \dots, \lambda_l].$$

It remains for us to prove that the linear hull of the polynomials $q_{(k)}$ coincides with $\mathbf{C}[\lambda_1, \dots, \lambda_l]$. This is immediately evident if we introduce the change of variables $\mu_1 = \lambda_1, \mu_l = \lambda_1 + \lambda_2, \dots, \mu_l = \lambda_1 + \cdots + \lambda_l$, since

$$q_{(k)}(\lambda) = \mu_1^{k_1} \cdots \mu_l^{k_l}.$$

Corollary. *Suppose that $q_1, q_2, \dots, q_l!$ is a basis in $\mathbf{C}[\lambda_1, \dots, \lambda_l]$ as a module over $\operatorname{Sym} \mathbf{C}[\lambda_1, \dots, \lambda_l]$ (see [2]). The space K_l is an ideal in $\operatorname{Sym} \mathbf{C}[\lambda_1 \dots \lambda_l]$ generated by the elements of $\operatorname{Sym} p_l q_i$, $1 \le i \le l!$.*

2.4. We let R_l denote the radical of the ideal J_l. From general theorem of algebraic geometry (see [7]), it follows that R_l is a homogeneous ideal, i.e., $R_l = \bigoplus_{x=0}^{\infty} R_l^k$, and that for large enough k, we have $R_l^k = J_l^k$

Our hypothesis is that

$$R_l = J_l, \tag{13}$$

i.e., that J_l is a radical ideal.

Let us investigate the ideal R_l. Suppose that $X_l \subset p^l(\mathbf{C})$ is the common zero set of the ideals R_l and J_l. This set may be described explicitly. That is, we denote by E_l the set of those points belonging to Z^l which possess the following properties.

(a) All coordinates take values from the set

$$\{1, 0, -1, -2, -3, \ldots\},$$

(b) the sum of the coordinates is less than 2.

The points E_l will be called *admissible l-sets*. Every such set $(\lambda_1, \ldots, \lambda_l)$ determines a point of the projective space $(\lambda_1 : \lambda_2 : \ldots : \lambda_l) \in p^l(\mathbf{C})$, which we will also call admissible.

Theorem 4. *The set X_l with $1 \geq 2$ consists of all the admissible points; moreover, all these points have multiplicity* 1.

Proof. First note certain extremely simple properties of admissible sets.

1. Every admissible set contains at least two units.
2. If two admissible sets are proportional, they are equal.
3. The number of admissible sets is equal to the number of monomials of degree at most $(l-2)$ in l variables, i.e.,

$$\binom{2l-2}{l-2}.$$

4. As regards the action of the group $S(l)$ of permutations of coordinates, the set E_l may be decomposed into orbits, of which there are $\sum_{j=0}^{l-2} p(j)$, where $p(n)$ is the number of partitions of n into a sum of unordered nonnegative integral terms.

The first two properties are self-evident. The last two follow from the one-to-one correspondence between E_l and the set of monomials of degree at most $l-2$ in l variables; i.e., with the set $(\mathscr{E}_1, \ldots, \mathscr{E}_l)$ we may associate the monomial

$$\lambda_1^{1-\mathscr{E}_1} \lambda_2^{1-\mathscr{E}_2} \cdots \lambda_l^{1-\mathscr{E}_l}.$$

Let us now prove the theorem. We first analyze the case $l = 2$. The ideal J_2 is generated by the single generator $\lambda_1 - \lambda_2 = p_2(\lambda)$. The set E_2 consists of the single point $(1, 1)$. In this case, the assertion of the theorem is true and, moreover, $J_2 = R_2$.

Suppose that $l > 2$ and that $\mathscr{E} = (\mathscr{E}_1, \ldots, \mathscr{E}_l)$ is an admissible set. We must verify that $p_{\sigma,l}(\mathscr{E}) = 0$ for all $\sigma \in S(l)$ and that the differentials $dp_{\sigma,l}(\mathscr{E})$ generate the orthogonal complement of the vector $\mathscr{E}$.

If $\mathscr{E} = (\mathscr{E}_{\sigma(1)}, \ldots, \mathscr{E}_{\sigma(l-1)})$ is an admissible $(l-1)$-set, then, by the inductive hypothesis $p_{\sigma,l}(\mathscr{E}) = 0$, since the first $(l-2)$ factors in $p_{\sigma,l}$ (cf. Equation (9)) when multiplied together yield $p_{\sigma',l-1}$. But if this set is not admissible, the only way this can be so is if condition (b) is not satisfied. However, this is possible only if $\mathscr{E}_{\sigma(l)} = 1$ and $\sum_{i=1}^{l} \mathscr{E}_i = 2$. In this case the last factor in the product (9) vanishes. That is, $p_{\sigma,l}(\mathscr{E}) = 0$.

Now suppose that the vector $\gamma = (\gamma_1, \ldots, \gamma_l)$ is orthogonal to all $dp_{\sigma,l}(\mathscr{E})$. Let us prove that it is proportional to $\mathscr{E}$. We first assume that $\mathscr{E}' = (\mathscr{E}_{\sigma(1)}, \ldots, \mathscr{E}_{\sigma(l-1)})$ is an admissible $(l-1)$-set for all $\sigma \in S(l)$. Then

$$dp_{\sigma,l}(\mathscr{E}) = dp_{\sigma',l-1}(\mathscr{E}') \cdot (\mathscr{E}_{\delta(1)} + \cdots + \mathscr{E}_{\sigma(l-1)} - \mathscr{E}_{\sigma(l)}),$$

while the last factor does not vanish, since $\mathscr{E}_{\sigma(1)} + \cdots + \mathscr{E}_{\sigma(l-1)} \geq 2$ and $\mathscr{E}_{\sigma(l)} \leq 1$. By the inductive hypothesis, the vector γ' is proportional to $\mathscr{E}'$. Thus, the vectors γ and $\mathscr{E}$ become proportional when any coordinate is discarded. If $l > 2$, this is possible only if they are themselves proportional. In essence, this line of reasoning proves the assertion required in the case in which the vector $\mathscr{E}$ has at least two coordinates not equal to 1. (Discarding them in this case results in admissible sets $\mathscr{E}'$.) It remains for us to analyze the case in which all the coordinates of $\mathscr{E}$ other than one coordinate are units, and the sum of the coordinates is equal to 2. Suppose that $\mathscr{E}_{\sigma(l)} = 1$. Then the set $\mathscr{E}'$ is not admissible. In this case,

$$dp_{\sigma,l}(\mathscr{E}) = p_{\sigma',l-1}(\mathscr{E}')(d\lambda_{\sigma(1)} + \cdots + d\lambda_{\sigma(l-1)} - d\lambda_{\sigma(l)}),$$

where the first cofactor is nonzero, as we will prove below. That is, the vector γ possesses the property that $\gamma_{\sigma(l)} = \sum_{k=1}^{l-1} \gamma_{\sigma(k)}$. Hence, it follows that γ and $\mathscr{E}$ are proportional. It remains for us to verify that x_l contains only admissible points. Suppose that $\mathscr{E} \in x_l$. Representing $p_{\sigma,l}$ in the form of the product of $p_{\sigma',l-1}$ and $(\mathscr{E}_{\sigma(1)} + \cdots + \mathscr{E}_{\sigma(l-1)} - \mathscr{E}_{\sigma(l)})$, it is clear that either $\mathscr{E}'$ is proportional to an admissible set or the equality $\mathscr{E}_{\sigma(l)} = \sum_{k=1}^{l-1} \mathscr{E}_{\sigma(k)}$ is satisfied. If $l > 2$, the latter equality cannot hold true for all $\sigma \in S(l)$ if $\mathscr{E} \neq 0$. Therefore, by replacing $\mathscr{E}$ by a proportional vector we may assume that $\mathscr{E}' \in E_{l-1}$. Let us consider the set δ' that is obtained by replacing one of the unit coordinates in $\mathscr{E}'$ by the coordinate $\mathscr{E}_{\sigma(l)}$. If the first case of the above alternative is satisfied, then δ' will be proportional to an admissible set and will differ from $\mathscr{E}'$ in only a single coordinate. Hence, it follows that δ' is an admissible set, and therefore is admissible in $\mathscr{E}$. In the second case, the equality

$$1 = \sum_{k=1}^{l-1} \mathscr{E}_{\sigma(k)} - 1 + \mathscr{E}_{\sigma(l)}$$

holds, whence $\mathscr{E}$ is likewise admissible. The theorem is proved.

2.5. Let us introduce the bilinear form $\langle , \rangle$ in the space $A_l = \mathbb{C}[\lambda_1, \dots \lambda_l]$, setting

$$\langle P, Q\rangle = P(\partial_1, \dots \partial_l)Q|_{\lambda_1 = \dots = \lambda_l = 0}. \tag{14}$$

Clearly, the subspaces A_l^k of homogeneous polynomials of degree k are pairwise orthogonal relative to the form (14). The monomials $\lambda^k/\sqrt{k!}$ form an orthonormalized basis in A_l^k, where the multi-index k runs through all nonnegative integer-valued vectors such that $|k| = k$, and $k!$ denotes $k_1!\,k_2! \cdots k_l!$. This form may also be specified by the equality

$$\langle P, Q\rangle = \int_{\mathbf{R}^l} PQ \exp(-|\lambda|^2/2)\, d^l\lambda. \tag{15}$$

An important property of this form is described by the following simple assertion.

Lemma 3. *The operator for multiplication by λ_i and the operator for differentiation with respect to λ_i are adjoint.*

Theorem 5. (a) *The space $J_l^\perp$ orthogonal to J_l consists of all polynomial solutions q of the system of equations*

$$(c_l) \quad p_{\sigma,l}(\partial_1, \dots \partial_l)q = 0, \quad \sigma \in S(l);$$

(b) *the space $R_l^\perp$ orthogonal to R_l is generated by the monomials*

$$q_{\mathscr{E},k}(\lambda) = (\lambda, \mathscr{E})^k, \qquad k = 0, 1, 2, \dots, \mathscr{E} \in E_l. \tag{16}$$

Proof. Since the spaces $J_l^\perp$ and $R_l^\perp$ are invariant under dilatations, it is sufficient to verify the assertion of the theorem for homogeneous components of A_l^k. Assertion (a) follows at once from the definition of the ideal J_l and Equation (14). To verify assertion (b), note that the value of a homogeneous polynomial q of degree k at the point $\mathscr{E}$ may be written in the following form, using Taylor's formula:

$$q(\mathscr{E}) = \sum_{|k|=k} \frac{1}{k!}(\partial^k q)(0)\mathscr{E}^k = \frac{1}{|k|!}\langle q, q_{\mathscr{E},k}\rangle.$$

The hypothesis (13) is consequently equivalent to the following assertion:

Every polynomial solution of the system (C_l) is a linear combination of the monomials (16).

If $l = 2$, this assertion assumes the following form:

Every polynomial solution of the equation $(\partial x - \partial y)q = 0$ is a polynomial in $(x + y)$. This is obviously true.

If $l = 3$, it is necessary to investigate the solutions of the system

$$(c_3) \quad \begin{cases} (\partial x - \partial y)(\partial x + \partial y - \partial z)q = 0, \\ (\partial x - \partial z)(\partial x + \partial z - \partial y)q = 0. \end{cases}$$

In this case, it may also be simply verified that all the solutions of degree k are generated by the monomials

$$(x + y + z)^k, (x + y)^k, (y + z)^k, (k + z)^k.$$

Theorem 6. *The family of monomials* (16) *is linearly independent if $k \ge 2l - 4$.*

An equivalent formulation is as follows:

If $k \ge 2l - 4$, any function on E_l may be obtained by a restriction of a homogeneous polynomial of degree k.

Proof. Since the function $\lambda_1 + \cdots + \lambda_l$ is everwhere nonzero on E_l, it is sufficient to analyze the case $k = 2l - 4$. We will construct explicitly a polynomial of degree $2l - 4$ that is nonzero only at a single (moreover, arbitrarily specified) point $\mathscr{E}$ of the set E_l. Without loss of generality, it may be assumed that the coordinates of $\mathscr{E}$ are not increasing: $\mathscr{E}_1 \ge \mathscr{E}_2 \ge \cdots \ge \mathscr{E}_l$. In particular, $\mathscr{E}_1 = \mathscr{E}_2 = 1$ (cf. Property 1, Section 2.4). Let us consider the following polynomial of degree $(l - 2)$:

$$\tilde{p}(\lambda) = (\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4) \cdots (\lambda_1 + \cdots + \lambda_{l-1} - \lambda_l).$$

Lemma 4. *The polynomial $\tilde{p}$ is nonzero only at the points of the set E_l such that $\lambda_1 = \lambda_2 = 1$.*

Proof. In the course of proving Theorem 4, we established that the last factor of the polynomial $p_{\sigma,l}$ is equal to 0 only on those admissible sets that become admissible when the coordinates $\lambda_{\sigma(l)}$ are discarded. That is, $\tilde{p}$ is nonzero only on those sets that remain admissible when $\lambda_l, \lambda_{l-1}, \dots \lambda_3$ are discarded. The only admissible 2-set is $(1, 1)$. The lemma is proved.

To complete the proof of the theorem, it remains for us to construct a polynomial $\tilde{q}$ of degree $(l - 2)$ that is equal to 0 at all points of E_l of the form $(1, 1, \delta_3, \dots, \delta_l)$, other than the point $\mathscr{E}$, and is not equal to 0 at this point. We will find it in the form

$$\tilde{q}(\lambda_1, \dots, \lambda_l) = r\left(\frac{\lambda_1 - \lambda_3}{\lambda_1}, \dots, \frac{\lambda_1 - \lambda_l}{\lambda_1}\right)\lambda_1^{l-2},$$

where r is a polynomial of degree at most $l - 2$ in $(l - 2)$ variables. The conditions imposed on r may be stated in the following way. Suppose that $\tilde{E}_l$ is a collection of nonnegative integer-valued l-sets such that $\sum_{i=1}^{l} \lambda_i \le l$. Then the restriction of r to $\tilde{E}_l$ may be nonzero at only a single fixed point. That this condition is satisfied follows from the next assertion.

Lemma 5. *The restriction of the space of homogeneous polynomials of degree l in l variables to the set $\tilde{E}_l$ is an isomorphism of linear spaces.*

To prove the assertion, note that all the derivatives of a polynomial of degree at most l may be calculated successively in terms of its difference derivatives at the point 0, while the latter may be expressed in terms of the values of the polynomial on $\tilde{E}_l$.

2.6. Now, using the results of Section 2.5 and hypothesis (13) from Section 2.4, we may obtain an explicit formula for the dimension of the space $\mathfrak{A}^{k,l}$.

Let $p^k(n)$ denote the number of partitions of n into an (unordered) sum of k positive integral terms. We set

$$p_k(n) = \sum_{j=0}^{k} p^j(n) \quad \text{and} \quad p(n) = \sum_{k=0}^{\infty} p^k(n).$$

Then

$$\dim \mathfrak{A}^{k,l} = p_k(k+l-1) + p_l(k+l-1) - p(k+l-1). \tag{17}$$

From Theorems 2 and 3, it follows that the desired dimension is equal to $\dim \operatorname{Sym} J_l^{k+l-1}$. If hypothesis (13) is valid, it will be equal to

$$\dim \operatorname{Sym} R_l^{k+l-1}.$$

Note that the space $\operatorname{Sym}^k \mathbb{C}[\lambda_1, \ldots, \lambda_e]$ of homogeneous symmetric polynomials of degree k in l variables has dimension $p_l(k)$. By Theorem 6, in this space the codimension R_l^k, with $k \ge 2l - 4$, is equal to the number of orbits of the group $S(l)$ in E_l, i.e.,

$$\sum_{m=0}^{l-2} p(m).$$

Thus, the quantity we wish to find assumes the form

$$p_e(k+l-1) - \sum_{m=0}^{l-2} p(m).$$

Now we use the simple combinatorial identity

$$p^k(n) = p_k(n-k). \tag{18}$$

whence it follows that when $k \ge l$,

$$p(k+l-1) = p_k(k+l-1) = \sum_{j \ge k+1} p^j(k+l-1)$$

$$= \sum_{j \ge kw} p_j(l+k-j-1) = \sum_{m=0}^{l-2} p(m),$$

since $p_j(n) = p(n)$ when $j \ge n$. Moreover, Equation (17) is proved when $k \ge l$. Since neither side of the equality is altered if k and l are interchanged, the formula has been proved completely.

Let us calculate the generating function $f_G(x, y)$ of the binary sequence $a_{k,l} = \dim \mathfrak{A}^{k,l}$.

From the identity (18), it may easily be proved that

$$\sum_{\substack{k \ge 0 \\ n \ge 0}} p^k(n) x^k y^n = \prod_{l=1}^{\infty} (1 - xy^l)^{-1}, \tag{19}$$

The right-hand side of (17) may be represented in the form

$$\sum_{j=0}^{k} p^j(k+l-1) + \sum_{j=0}^{l} p^j(k+l-1) - \sum_{j=0}^{k+l-1} p^j(k+l-1).$$

Multiplying this expression by $x^k y^l$ and summing over k and l, we arrive at the series

$$\sum_{d,n} p^j(n) \cdot p_{j,n}(x, y),$$

where

$$p_{j,n}(x, y) = \sum_{\substack{k \ge j, \mathscr{S} \ge 0 \\ k+l=n+1}} x^k y^l + \sum_{\substack{k \ge 0, l \ge j \\ k+l=n+1}} x^k y^l - \sum_{\substack{k \ge 0, l \ge 0 \\ k+l=m+1}} x^k y^l$$

$$= \sum_{d \le x \le n+1} x^k y^{n+1-k} \sum_{0 \le l \le j} x^{n+1-l} y^l = \frac{x^{n+2-j} y^j - x^j y^{n+2-j}}{x - y}.$$

Hence

$$f_{\mathfrak{A}}(x, y) = \frac{1}{x-y} \left(x^2 \prod_{l \ge 0} (1 - k^l y)^{-1} - y^2 \prod_{l \ge 0} (1 - xy^l)^{-1} \right). \tag{20}$$

Suppose that $a_n = \sum_{k+l=n} a_{k,l}$. The function $\varphi_{\mathfrak{A}}(t) = \sum a_n t^n$ may be obtained from $f_{\mathfrak{A}}(x, y)$ if we set $x = y = t$.

Finding the values of the indeterminate form in Equation (20) by means of L'Hôpital's rule, we obtain

$$\varphi_{\mathfrak{A}}(t) = \partial/\partial x\left[\frac{x^2}{\prod_{l\geq 0}(1+x^l y)} - \frac{y^2}{\prod_{l\geq 0}(1-xy^l)}\right]\Bigg|_{x=y=t}$$

$$= \frac{2x + x^2 \sum_{l\geq 0}\frac{lx^{l-1}y}{1-x^l y}}{\prod_{l\geq 0}(1-x^l y)} - \frac{y^2\sum_{l\geq 0}\frac{y^l}{1-xy^l}}{\prod_{l\geq 0}(1-xy^l)}\Bigg|_{x=y=t}$$

$$= \frac{t}{\prod_{n\geq 1}(1-t^n)}\left(2 + \sum_{\substack{l\geq 0\\ j\geq 0}}(l-2)t^{lj}\right).$$

We set

$$\mathscr{P}(t) = \sum_{n\geq 0} p^{(n)t^n} = \prod_{b\geq 1}(1-t^b)^{-1}.$$

Then

$$\mathscr{P}'(t) = \mathscr{P}(t)\sum_{l\geq 1}\frac{lt^{l-1}}{1-t^l} = \frac{\mathscr{P}(t)}{t}\sum_{\substack{l\geq 1\\ j\geq 1}} lt^{lj}.$$

Substituting this in the expression obtained above, we arrive at the equality

$$\varphi_{\mathfrak{A}}(t) = t^2\mathscr{P}'(t) + 2t\mathscr{P}(t) - 2t\mathscr{P}(t)\mathscr{D}(t) \tag{21}$$

where

$$\mathscr{D}(t) = \sum_{\substack{l\geq 1\\ j\geq 1}} t^{lj} = \sum_{n\geq 1} d(n)t^n,$$

and $d(n)$ denotes the number of divisors of n. We finally have

$$a_n = (n+1)p(n-1) - 2\sum_{k=1}^{n-1} d(k)p(n-k-1), \tag{22}$$

or

$$a_n = 2p(n-1) + \sum_{k=1}^{n-1}[\sigma(k) - 2d(k)]p(n-k-1), \tag{23}$$

that is, $a_n = 2p(n-1) - p(n-2) - p(n-3) + p(n-5) + 2p(n-6) + 4p(n-7) + 4p(n-8)\ldots$, where $\sigma(k)$ is the sum of the divisors of k, and $d(k)$ is the number of divisors of k.

Using Ramanujan's formula **[8]**, we obtain the asymptotic expression

$$a_n \sim \frac{1}{4\sqrt{3}}\exp(\pi\sqrt{2n/3}). \tag{24}$$

Without using the hypothesis (13), only the following weaker assertion may be obtained:

$$\ln a_n \sim \pi\sqrt{\frac{2}{3}n}.$$

Part III. Associative algebra generated by two general vector fields on the real line

3.1. Suppose that ξ and η are two vector fields from the space Vect R^1 that are in general position (cf. Section 2.1).

We are interested in the dimension of the homogeneous components of the space $\mu = A(x, y)/I(\xi, \eta)$, where $A(x, y)$ is a free associative algebra with generators x and y and $I(\xi, \eta)$ is the kernel of a homomorphism of $A(x, y)$ into the algebra of differential operators on the real line: $x \to \xi$, $y \to \eta$.

The space μ is bigraded by the degrees of x and y, and $\mu = \bigoplus_{k,l}\mu^{k,l}$. Our problem is to find the dimension of the space $\mu^{k,l}$.

We introduce the notation $\mathbf{C}_k[x_1, \ldots, x_l]$ to denote the space of homogeneous polynomials in the variables $x_1, \ldots, x_l$ of degree k, where $\mathbf{C}_{\leq k}[x_1, \ldots, x_l]$ is the space of polynomials of degree at most k.

In the case $l = 1$, the dimension of the space $\mu^{\mu,l}$ may be easily calculated directly. We therefore set $l \geq 2$.

In the same way as Section 2.2, it may be proved that the dimension of $\mu^{k,l}$ coincides with the dimension of the space $\mathrm{Sym}(p(x, d)\mathbf{C}_k[x, d])$, where $x = (x_1, \ldots, x_l)$ and $d = d/dt$,

$$p(x, d) = (x_1 + d)(x_1 + x_2 + d)\cdots(x_1 + \cdots + x_{l-1} + d),$$

it may be assumed that d is an independent variable) and the projector Sym acts only on the variables $x_1, \ldots, x_l$.

For the sake of convenience in further calculations, we may set $d = 1$, and let

$$I^{k,l} = \mathrm{Sym}(p(x)\mathbf{C}_{\leq k}[x]), \qquad J^{k,l} = \mathrm{Sym}(p(x)\mathbf{C}_k[x]), \qquad p(x) = p(x, 1).$$

Using the methods developed in Theorems 4, 5, and 6 of Part 2, and applying them to the study of the ideal $I^{k,l}$, we may obtain the following bound on the dimension of the space $I^{k,l}$:

$$\dim I^{k,l} \leq \sum_{s=0}^{r+l-1}(p_l(s) + p_k(s) - p(s)). \tag{25}$$

Our problem is to prove the opposite equality.

Let us decompose the polynomial

$$p(x) = (x_1 + 1)(x_1 + x_2 + 1)(x_1 + x_2 + x_3 + 1) \cdots (x_1 + \cdots + x_{l-1} + 1)$$

into homogeneous terms:

$$p(x) = p_0(x) + p_1(x) + \cdots + p_{l-1}(x).$$

Lemma 6. *If $k \geq l - 2$,*

$$\mathrm{Sym}(p_{l-1}\mathbf{C}_k[x]) = \mathrm{Sym}\,\mathbf{C}_{k+l-1}[x].$$

For the proof, see Lemma 2 in [**6**].

Since the dimension of the space of all homogeneous symmetric polynomials of degree $(k + l)$ of l variables is equal to $p_l(k + l)$, then, using the self-evident representation

$$I^{k+1,l} = I^{k,l} + J^{k+1,l}$$

from Lemma 6, the following inequality may be obtained:

$$\dim I^{k+1,l} \geq \dim I^{k,l} + p_l(k + l), \qquad k \geq l - 3. \tag{26}$$

Considering $I^{k,2}$ separately, we find that

$$\dim I^{k,2} = \dim I^{2,k} = \sum_{s=1}^{k+1} p_2(s).$$

Using Equation (26) and the fact that $\dim I^{k,l}$ is symmetric with respect to the indices k and l, we obtain the inequality

$$\dim I^{k,l} \geq \sum_{s=0}^{k+l-1} (p_l(s) + p_k(s) - p(s)).$$

Comparing this with (25) leads us to the following conclusion:

Theorem 7. *The dimension of the space $\mu^{k,l}$ of homogeneous components of the algebra $A(x, y)/I$, where $I = I(\xi, \eta)$ and ξ and η are vector fields of general position, may be calculated from the formula*

$$\dim U^{k,l} = \sum_{s=0}^{k+l-1} (p_l(s) + p_k(s) - p(s)).$$

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