Generators of classical $\mathcal{W}$-algebras

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equipped with the derivation ∂,

$$\partial(X_i^{(r)}) = X_i^{(r+1)}$$

for all $i = 1, \dots, d$ and $r \geqslant 0$.

Introduce the λ-bracket on $\mathcal{V}$ as a linear map

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and the Leibniz rule $(a, b, c \in \mathcal{V})$:

$$\{a_\lambda bc\} = \{a_\lambda b\}c + \{a_\lambda c\}b.$$

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$$\rho : \mathcal{V} \to \mathcal{V}(\mathfrak{p}), \qquad \rho(X) = \pi_{\mathfrak{p}}(X) + (f|X), \qquad X \in \mathfrak{g}.$$

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The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is defined by

$$\mathcal{W}(\mathfrak{g}) = \{P \in \mathcal{V}(\mathfrak{p}) \mid \rho\{X_\lambda P\} = 0 \quad \text{for all} \quad X \in \mathfrak{n}_+\}.$$

The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is a Poisson vertex algebra equipped with the λ-bracket

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Motivation: Hamiltonian equations

$$\frac{\partial u}{\partial t} = \{H\,{}_\lambda\,u\}\big|_{\lambda=0}$$

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De Sole, Kac and Valeri, 2013-14; Drinfeld and Sokolov, 1985.

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The Hamiltonian equation with $H = \dfrac{u^2}{2}$ is equivalent to

the KdV equation

$$\frac{\partial u}{\partial t} = 3\,u\,u' - \frac{1}{2}\,u'''.$$

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We will work with the algebra $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$,

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The invariant symmetric bilinear form on $\mathfrak{gl}_n$ is defined by

$$(X|Y) = \operatorname{tr} XY, \qquad X, Y \in \mathfrak{gl}_n.$$

Expand the determinant with entries in $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$,

$$\det \begin{bmatrix} \partial + E_{11} & 1 & 0 & 0 & \dots & 0 \\ E_{21} & \partial + E_{22} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ E_{n-1\,1} & E_{n-1\,2} & E_{n-1\,3} & \dots & \dots & 1 \\ E_{n\,1} & E_{n\,2} & E_{n\,3} & \dots & \dots & \partial + E_{n\,n} \end{bmatrix}$$

$$= \partial^{\,n} + w_1\,\partial^{\,n-1} + \dots + w_n.$$

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$$= \partial^{\,n} + w_1\, \partial^{\,n-1} + \dots + w_n.$$

Theorem. All elements $w_1, \dots, w_n$ belong to $\mathcal{W}(\mathfrak{gl}_n)$. Moreover,

$$\mathcal{W}(\mathfrak{gl}_n) = \mathbb{C}[w_1^{(r)}, \dots, w_n^{(r)} \mid r \geqslant 0].$$

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Taking $w_1 = 0$, for $\mathfrak{sl}_2$ we find

$$w_2 = -u = -\frac{h^2}{4} - \frac{h'}{2} - f.$$

MacMahon Master Theorem

Let

$$A = \begin{bmatrix} a_{11} & 1 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1\,1} & a_{n-1\,2} & a_{n-1\,3} & \dots & \dots & 1 \\ a_{n\,1} & a_{n\,2} & a_{n\,3} & \dots & \dots & a_{n\,n} \end{bmatrix}.$$

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Define elements e_m by the expansion

$$\det(1 + tA) = \sum_{m=0}^{n} e_m t^m.$$

The elements e_m are then found by

$$e_m = \sum_{s=1}^{m} \sum_{i_k < j_{k+1}} (-1)^{m-s} a_{i_1 j_1} \dots a_{i_s j_s},$$

summed over $i_1, \dots, i_s$ and $j_1, \dots, j_s$ such that

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Example.

$$e_1 = a_{11} + \cdots + a_{nn},$$

$$e_2 = \sum_{i<j} a_{ii} a_{jj} - \sum_{i} a_{i+1\,i},$$

$$e_3 = \sum_{i<j<k} a_{ii} a_{jj} a_{kk} - \sum_{i+1<j} a_{i+1\,i} a_{jj} - \sum_{i<j} a_{ii} a_{j+1\,j} + \sum_{i} a_{i+2\,i}.$$

Set

$$h_m = \sum_{s=1}^{m} \sum_{i_k \geqslant j_{k+1}} a_{i_1 j_1} \dots a_{i_s j_s},$$

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Equivalently, for $m \geqslant 1$

$$h_m - h_{m-1} e_1 + h_{m-2} e_2 - \cdots + (-1)^n h_{m-n} e_n = 0,$$

assuming $h_m = e_m = 0$ for $m < 0$.

Now specialize the matrix A,

$$a_{ij} = \delta_{ij}\partial + E_{ij}, \qquad i \geqslant j,$$

and write

$$e_m = e_{m\,0} + e_{m\,1}\,\partial + \cdots + e_{m\,m}\,\partial^m,$$

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In particular,

$$h_{m\,0} = \sum_{s=1}^{m} \sum_{i_k \geqslant j_{k+1}} \big(\delta_{i_1 j_1}\,\partial + E_{i_1 j_1}\big) \ldots \big(\delta_{i_s j_s}\,\partial + E_{i_s j_s}\big)\, 1.$$

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Corollary. All elements $h_{m\,i}$ belong to $\mathcal{W}(\mathfrak{gl}_n)$. Moreover,

$$\mathcal{W}(\mathfrak{gl}_n) = \mathbb{C}\,[h^{(r)}_{1\,0}, \dots, h^{(r)}_{n\,0} \mid r \geqslant 0].$$

Generators of $\mathcal{W}(\mathfrak{o}_{2n+1})$

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The Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements

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is the orthogonal Lie algebra $\mathfrak{o}_N$.

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$$f = F_{21} + F_{32} + \cdots + F_{n+1\,n} \in \mathfrak{o}_{2n+1}.$$

Expand the determinant of the matrix

$$\begin{bmatrix} \partial + F_{1\,1} & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ F_{2\,1} & \partial + F_{2\,2} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots \\ F_{n\,1} & F_{n\,2} & \dots & \partial + F_{n\,n} & 1 & 0 & \dots & 0 & 0 \\ F_{n+1\,1} & F_{n+1\,2} & \dots & F_{n+1\,n} & \partial & -1 & \dots & 0 & 0 \\ F_{n'\,1} & F_{n'\,2} & \dots & 0 & F_{n'\,n+1} & \partial + F_{n'\,n'} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ F_{2'\,1} & 0 & \dots & \dots & F_{2'\,n+1} & \dots & \dots & \partial + F_{2'\,2'} & -1 \\ 0 & F_{1'\,2} & \dots & \dots & F_{1'\,n+1} & \dots & \dots & F_{1'\,2'} & \partial + F_{1'\,1'} \end{bmatrix}$$

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F_{n1} & F_{n2} & \dots & \partial + F_{nn} & 1 & 0 & \dots & 0 & 0 \\
F_{n+1\,1} & F_{n+1\,2} & \dots & F_{n+1\,n} & \partial & -1 & \dots & 0 & 0 \\
F_{n'\,1} & F_{n'\,2} & \dots & 0 & F_{n'\,n+1} & \partial + F_{n'\,n'} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\
F_{2'\,1} & 0 & \dots & \dots & F_{2'\,n+1} & \dots & \dots & \partial + F_{2'\,2'} & -1 \\
0 & F_{1'\,2} & \dots & \dots & F_{1'\,n+1} & \dots & \dots & F_{1'\,2'} & \partial + F_{1'\,1'}
\end{bmatrix}$$

as a differential operator

$$\partial^{2n+1} + w_2\,\partial^{2n-1} + w_3\,\partial^{2n-2} + \cdots + w_{2n+1}, \qquad w_i \in \mathcal{V}(\mathfrak{p}).$$

Theorem. All elements $w_2, \ldots, w_{2n+1}$ belong to $\mathcal{W}(\mathfrak{o}_{2n+1})$.

Moreover,

$$\mathcal{W}(\mathfrak{o}_{2n+1}) = \mathbb{C}\left[w_2^{(r)}, w_4^{(r)}, \ldots, w_{2n}^{(r)} \mid r \geqslant 0\right].$$

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One proof is based on the folding procedure. The subalgebra $\mathfrak{o}_{2n+1} \subset \mathfrak{gl}_{2n+1}$ is considered as the fixed point subalgebra for an involutive automorphism of $\mathfrak{gl}_{2n+1}$.

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One proof is based on the folding procedure. The subalgebra $\mathfrak{o}_{2n+1} \subset \mathfrak{gl}_{2n+1}$ is considered as the fixed point subalgebra for an involutive automorphism of $\mathfrak{gl}_{2n+1}$. For the principal nilpotent we have

$$f \mapsto \widetilde{f} = E_{2\,1} + E_{3\,2} + \dots + E_{n+1\,n} - E_{n+2\,n+1} - \dots - E_{2n+1\,2n}.$$

Generators of $\mathcal{W}(\mathfrak{sp}_{2n})$

The Lie subalgebra of $\mathfrak{gl}_{2n}$ spanned by the elements

$$F_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{j'i'}, \qquad i,j = 1, \ldots, 2n,$$

is the symplectic Lie algebra $\mathfrak{sp}_{2n}$, where

$\varepsilon_i = 1$ for $i = 1, \ldots, n$ and $\varepsilon_i = -1$ for $i = n+1, \ldots, 2n$.

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The Lie subalgebra of $\mathfrak{gl}_{2n}$ spanned by the elements

$$F_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{j'i'}, \qquad i,j = 1,\dots,2n,$$

is the symplectic Lie algebra $\mathfrak{sp}_{2n}$, where

$\varepsilon_i = 1$ for $i = 1,\dots,n$ and $\varepsilon_i = -1$ for $i = n+1,\dots,2n$.

Cartan subalgebra $\mathfrak{h} =$ span of $\{F_{11},\dots,F_{nn}\}$.

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$$f = F_{21} + F_{32} + \dots + F_{n\,n-1} + \frac{1}{2} F_{n'\,n} \in \mathfrak{sp}_{2n}.$$

Expand the determinant of the matrix

$$\begin{bmatrix}
\partial + F_{1\,1} & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
F_{2\,1} & \partial + F_{2\,2} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots \\
F_{n\,1} & F_{n\,2} & \dots & \partial + F_{n\,n} & 1 & 0 & \dots & 0 & 0 \\
F_{n'\,1} & F_{n'\,2} & \dots & F_{n'\,n} & \partial + F_{n'\,n'} & -1 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
F_{2'\,1} & F_{2'\,2} & \dots & F_{2'\,n} & F_{2'\,n'} & \dots & \dots & \partial + F_{2'\,2'} & -1 \\
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as a differential operator

$$\partial^{2n} + w_2\,\partial^{2n-2} + w_3\,\partial^{2n-3} + \dots + w_{2n}, \qquad w_i \in \mathcal{V}(\mathfrak{p}).$$

Theorem. All elements $w_2, \dots, w_{2n}$ belong to $\mathcal{W}(\mathfrak{sp}_{2n})$.

Moreover,

$$\mathcal{W}(\mathfrak{sp}_{2n}) = \mathbb{C}\left[w_2^{(r)}, w_4^{(r)}, \dots, w_{2n}^{(r)} \mid r \geqslant 0\right].$$

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This can be proved by using the folding procedure for the subalgebra $\mathfrak{sp}_{2n} \subset \mathfrak{gl}_{2n}$. For the principal nilpotent we have

$$f \mapsto \widetilde{f} = E_{2\,1} + E_{3\,2} + \dots + E_{n+1\,n} - E_{n+2\,n+1} - \dots - E_{2n\,2n-1}.$$

Generators of $\mathcal{W}(\mathfrak{o}_{2n})$

Introduce the algebra of pseudo-differential operators

$\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}((\partial^{-1}))$,

$$\partial^{-1} F_{ij}^{(r)} = \sum_{s=0}^{\infty} (-1)^s F_{ij}^{(r+s)} \partial^{-s-1}.$$

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Take the principal nilpotent element $f \in \mathfrak{o}_{2n}$ in the form

$$f = F_{21} + F_{32} + \dots + F_{n\,n-1} + F_{n'\,n-1}.$$

Remark. Under the embedding $\mathfrak{o}_{2n} \subset \mathfrak{gl}_{2n}$, $f \mapsto \widetilde{f}$,

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$$\widetilde{f} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathbf{1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathbf{1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \mathbf{1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \mathbf{-1} & \mathbf{-1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \mathbf{-1} & 0 \end{bmatrix}$$

Expand the determinant of the $(2n+1)\times(2n+1)$ matrix

$$\begin{bmatrix}
\partial + F_{1\,1} & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
F_{2\,1} & \partial + F_{2\,2} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots \\
F_{n\,1} - F_{n'\,1} & F_{n\,2} - F_{n'\,2} & \dots & \partial + F_{n\,n} & 0 & -2\,\partial & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \partial^{-1} & 0 & \dots & 0 & 0 \\
F_{n'\,1} & F_{n'\,2} & \dots & 0 & 0 & \partial + F_{n'\,n'} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\
F_{2'\,1} & 0 & \dots & \dots & 0 & F_{2'\,n'} - F_{2'\,n} & \dots & \partial + F_{2'\,2'} & -1 \\
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F_{n'\,1} & F_{n'\,2} & \ldots & 0 & 0 & \partial + F_{n'\,n'} & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\
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$$

as a pseudo-differential operator

$$\partial^{2n-1} + w_2\,\partial^{2n-3} + w_3\,\partial^{2n-4} + \cdots + w_{2n-1} + (-1)^n\, y_n\,\partial^{-1}\, y_n.$$

Theorem. All elements $w_2, w_3, \dots, w_{2n-1}$ and y_n belong to $\mathcal{W}(\mathfrak{o}_{2n})$. Moreover,

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We have

$$y_n = \det \begin{bmatrix} \partial + F_{11} & 1 & 0 & 0 & \dots & 0 \\ F_{21} & \partial + F_{22} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ F_{n-1\,1} & F_{n-1\,2} & F_{n-1\,3} & \dots & \dots & 1 \\ F_{n\,1} - F_{n'\,1} & F_{n\,2} - F_{n'\,2} & F_{n\,3} - F_{n'\,3} & \dots & \dots & \partial + F_{n\,n} \end{bmatrix} 1.$$

Generators of $\mathcal{W}(\mathfrak{g}_2)$

$\mathfrak{g}_2$ is the simple Lie algebra of type G_2 with the Cartan matrix

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

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We let α and β be the simple roots, and the positive roots are

$$\alpha, \quad \beta, \quad \alpha+\beta, \quad \alpha+2\beta, \quad \alpha+3\beta, \quad 2\alpha+3\beta.$$

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The Cartan subalgebra $\mathfrak{h}$ is spanned by H_α and H_β.

Consider the 7×7 matrix

$$\begin{bmatrix} \partial + \widetilde{F}_{11} & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\,Y_{\beta} & \partial + \widetilde{F}_{22} & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\,Y_{\alpha+\beta} & Y_{\alpha} & \partial + \widetilde{F}_{33} & 1 & 0 & 0 & 0 \\ \frac{4}{9}\,Y_{\alpha+2\,\beta} & -\frac{2}{3}\,Y_{\alpha+\beta} & \frac{2}{3}\,Y_{\beta} & \partial & -1 & 0 & 0 \\ -\frac{4}{9}\,Y_{\alpha+3\,\beta} & \frac{4}{9}\,Y_{\alpha+2\,\beta} & 0 & -\frac{2}{3}\,Y_{\beta} & \partial - \widetilde{F}_{33} & -1 & 0 \\ \frac{4}{9}\,Y_{2\,\alpha+3\,\beta} & 0 & -\frac{4}{9}\,Y_{\alpha+2\,\beta} & \frac{2}{3}\,Y_{\alpha+\beta} & -Y_{\alpha} & \partial - \widetilde{F}_{22} & -1 \\ 0 & -\frac{4}{9}\,Y_{2\,\alpha+3\,\beta} & \frac{4}{9}\,Y_{\alpha+3\,\beta} & -\frac{4}{9}\,Y_{\alpha+2\,\beta} & -\frac{1}{3}\,Y_{\alpha+\beta} & -\frac{1}{3}\,Y_{\beta} & \partial - \widetilde{F}_{11} \end{bmatrix},$$

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where

$$\widetilde{F}_{11} = -H_{\alpha} - \frac{2}{3}\,H_{\beta}, \qquad \widetilde{F}_{22} = -H_{\alpha} - \frac{1}{3}\,H_{\beta}, \qquad \widetilde{F}_{33} = -\frac{1}{3}\,H_{\beta}.$$

Write its determinant as a differential operator

$$\partial^7 + w_2\,\partial^5 + w_3\,\partial^4 + w_4\,\partial^3 + w_5\,\partial^2 + w_6\,\partial + w_7, \qquad w_i \in \mathcal{V}(\mathfrak{p}).$$

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Theorem. The elements $w_2, \dots, w_7$ belong to $\mathcal{W}(\mathfrak{g}_2)$. Moreover,

$$\mathcal{W}(\mathfrak{g}_2) = \mathbb{C}\,[w_2^{(r)}, w_6^{(r)} \mid r \geqslant 0].$$

Chevalley-type theorem

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Let

$$\phi : \mathcal{V}(\mathfrak{p}) \to \mathcal{V}(\mathfrak{h})$$

denote the homomorphism of differential algebras defined on the generators as the projection $\mathfrak{p} \to \mathfrak{h}$ with the kernel $\mathfrak{n}_-$.

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denote the homomorphism of differential algebras defined on the generators as the projection $\mathfrak{p} \to \mathfrak{h}$ with the kernel $\mathfrak{n}_-$.

The restriction of ϕ to $\mathcal{W}(\mathfrak{g})$ is injective. The embedding

$$\phi : \mathcal{W}(\mathfrak{g}) \hookrightarrow \mathcal{V}(\mathfrak{h})$$

is known as the Miura transformation.

To describe the image $\widetilde{\mathcal{W}}(\mathfrak{g}) = \phi\big(\mathcal{W}(\mathfrak{g})\big)$, introduce the

screening operators

$$V_i : \mathcal{V}(\mathfrak{h}) \to \mathcal{V}(\mathfrak{h}), \qquad i = 1, \dots, n.$$

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Let $h_1, \dots, h_n$ be basis elements of $\mathfrak{h}$. Then

$$V_i = \sum_{r=0}^{\infty} V_{i\,r} \sum_{j=1}^{n} a_{ji} \frac{\partial}{\partial h_j^{(r)}},$$

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$$V_i = \sum_{r=0}^{\infty} V_{i\,r} \sum_{j=1}^{n} a_{ji} \, \frac{\partial}{\partial h_j^{(r)}},$$

where $A = [a_{ij}]$ is the Cartan matrix,

$$\sum_{r=0}^{\infty} \frac{V_{i\,r}\, z^r}{r!} = \exp\Big(-\sum_{m=1}^{\infty} \frac{h_i^{(m-1)}\, z^m}{\epsilon_i\, m!}\Big),$$

and $B = D^{-1}A$ is symmetric for $D = \operatorname{diag}[\epsilon_1, \ldots, \epsilon_n]$.

Proposition.

The restriction of the homomorphism ϕ to the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ yields an isomorphism

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where $\widetilde{\mathcal{W}}(\mathfrak{g})$ is the subalgebra of $\mathcal{V}(\mathfrak{h})$ which consists of the elements annihilated by all screening operators V_i,

$$\widetilde{\mathcal{W}}(\mathfrak{g}) = \bigcap_{i=1}^{n} \ker V_i.$$

For $\mathfrak{g} = \mathfrak{gl}_n$, the image of the determinant

$$\det \begin{bmatrix} \partial + E_{11} & 1 & 0 & 0 & \dots & 0 \\ E_{21} & \partial + E_{22} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ E_{n-1\,1} & E_{n-1\,2} & E_{n-1\,3} & \dots & \dots & 1 \\ E_{n\,1} & E_{n\,2} & E_{n\,3} & \dots & \dots & \partial + E_{n\,n} \end{bmatrix}$$

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equals

$$(\partial + E_{11}) \dots (\partial + E_{n\,n}) = \partial^{\,n} + \widetilde{w}_1\, \partial^{\,n-1} + \dots + \widetilde{w}_n.$$

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equals

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Therefore, we recover the Adler–Gelfand–Dickey generators:

$$\widetilde{\mathcal{W}}(\mathfrak{gl}_n) = \phi\big(\mathcal{W}(\mathfrak{gl}_n)\big) = \mathbb{C}\,\big[\widetilde{w}_1^{(r)}, \dots, \widetilde{w}_n^{(r)} \mid r \geqslant 0\big].$$

Explicitly,

$$\widetilde{w}_m = \sum_{i_1 < \cdots < i_m} (\partial + E_{i_1\, i_1}) \ldots (\partial + E_{i_m\, i_m})\, 1.$$

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Define elements $\widetilde{u}_m$ by

$$\widetilde{u}_m = \sum_{i_1 \geqslant \cdots \geqslant i_m} (\partial + E_{i_1 i_1}) \dots (\partial + E_{i_m i_m})\, 1.$$

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$$\widetilde{w}_m = \sum_{i_1 < \dots < i_m} (\partial + E_{i_1\, i_1}) \dots (\partial + E_{i_m\, i_m})\, 1.$$

Define elements $\widetilde{u}_m$ by

$$\widetilde{u}_m = \sum_{i_1 \geqslant \dots \geqslant i_m} (\partial + E_{i_1\, i_1}) \dots (\partial + E_{i_m\, i_m})\, 1.$$

Then

$$\widetilde{\mathcal{W}}(\mathfrak{gl}_n) = \mathbb{C}\left[\widetilde{u}_1^{(r)}, \dots, \widetilde{u}_n^{(r)} \mid r \geqslant 0\right].$$

Drinfeld–Sokolov generators for $\mathfrak{o}_{2n+1}$:

$$(\partial + F_{11}) \dots (\partial + F_{nn})\, \partial\, (\partial - F_{nn}) \dots (\partial - F_{11})$$

$$= \partial^{\,2n+1} + \widetilde{w}_2\, \partial^{\,2n-1} + \widetilde{w}_3\, \partial^{\,2n-2} + \dots + \widetilde{w}_{2n+1},$$

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$$\widetilde{\mathcal{W}}(\mathfrak{o}_{2n+1}) = \mathbb{C}\left[\widetilde{w}_2^{(r)}, \widetilde{w}_4^{(r)}, \dots, \widetilde{w}_{2n}^{(r)} \mid r \geqslant 0\right].$$

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Drinfeld–Sokolov generators for $\mathfrak{o}_{2n}$:

$$(\partial + F_{11}) \dots (\partial + F_{nn})\, \partial^{-1} (\partial - F_{nn}) \dots (\partial - F_{11})$$

$$= \partial^{2n-1} + \widetilde{w}_2\, \partial^{2n-3} + \widetilde{w}_3\, \partial^{2n-4} + \dots + \widetilde{w}_{2n-1} + (-1)^n \widetilde{y}_n\, \partial^{-1} \widetilde{y}_n.$$

Drinfeld–Sokolov generators for $\mathfrak{o}_{2n}$:

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Then

$$\widetilde{\mathcal{W}}(\mathfrak{o}_{2n}) = \mathbb{C}\,\big[\widetilde{w}_2^{(r)}, \widetilde{w}_4^{(r)}, \dots, \widetilde{w}_{2n-2}^{(r)}, \widetilde{y}_n^{(r)} \,|\; r \geqslant 0\big].$$

Miura transformation for $\mathfrak{g}_2$. Recall

$$\widetilde{F}_{11} = -H_\alpha - \frac{2}{3}\,H_\beta, \qquad \widetilde{F}_{22} = -H_\alpha - \frac{1}{3}\,H_\beta, \qquad \widetilde{F}_{33} = -\frac{1}{3}\,H_\beta.$$

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The image of the determinant under the Miura transformation is

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$$= \partial^7 + \widetilde{w}_2\,\partial^5 + \widetilde{w}_3\,\partial^4 + \widetilde{w}_4\,\partial^3 + \widetilde{w}_5\,\partial^2 + \widetilde{w}_6\,\partial + \widetilde{w}_7.$$

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Corollary.

$$\widetilde{\mathcal{W}}(\mathfrak{g}_2) = \mathbb{C}\, [\widetilde{w}_2^{(r)}, \widetilde{w}_6^{(r)} \mid r \geqslant 0].$$

Center at the critical level

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The Feigin–Frenkel center consists of its $\mathfrak{g}[t]$-invariants,

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\,v = 0\}.$$

Consider the (affine) Harish-Chandra homomorphism

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where the classical $\mathcal{W}$-algebra $\widetilde{\mathcal{W}}({}^L\mathfrak{g})$ is associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$ [Feigin and Frenkel, 1992].