Higher Sugawara operators and the classical W-algebra for $\mathfrak{gl}_n$

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joint work with

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$\mathbf{1}$ is a vacuum vector $\mathbf{1} \in V$,

and the infinitesimal translation T is an operator

$$T : V \to V.$$

These data must satisfy certain axioms. For $a \in V$ we write

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Each formal series $Y(a, z) \in \operatorname{End} V[[z, z^{-1}]]$ must be a field:

for any $b \in V$ we must have $a_{(n)} b = 0$ for $n \gg 0$.

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Remarks. T is determined by Y: $\qquad Ta = a_{(-2)}\mathbf{1}$.

Moreover, $\qquad Y(Ta,z) = \partial_z Y(a,z)$.

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A general vertex algebra can be viewed as a vector space with the multiplication depending on z:

$$a_z b = Y(a,z)b.$$

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The commutator of Fourier coefficients is given by

the Borcherds identity:

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- Hence, if $a_{(n)}\, b = 0$ for all $a \in V$ and $n \geqslant 0$, then all Fourier coefficients $b_{(n)}$ belong to the center of this Lie subalgebra.

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  $a\,b := a_{(-1)} b, \qquad a, b \in \mathcal{Z}(V).$
- The vacuum vector $\mathbf{1}$ is a unit, T is a derivation.

Vertex algebra associated with $\widehat{\mathfrak{gl}}_n$

The affine Kac–Moody algebra $\widehat{\mathfrak{gl}}_n = \mathfrak{gl}_n[t, t^{-1}] \oplus \mathbb{C}K$ has the commutation relations

$$\big[e_{ij}[r], e_{kl}[s]\big] = \delta_{kj}\, e_{i\,l}[r+s] - \delta_{i\,l}\, e_{kj}[r+s] + K\Big(\delta_{kj}\delta_{i\,l} - \frac{\delta_{ij}\delta_{kl}}{n}\Big)\, r\,\delta_{r,-s},$$

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In particular, for any r the element $e_{11}[r] + \dots + e_{nn}[r]$ belongs to the center of $\widehat{\mathfrak{gl}}_n$.

Let $\kappa \in \mathbb{C}$.

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Introduce the vector space $V_\kappa(\mathfrak{gl}_n)$ as the quotient of

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We view $V_\kappa(\mathfrak{gl}_n)$ as a $\widehat{\mathfrak{gl}}_n$-module. It is called

the vacuum representation of level κ.

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The state-field correspondence Y is defined as follows. First,

$$Y(e_{ij}[-1], z) = \sum_{m\in\mathbb{Z}} e_{ij}[m]\, z^{-m-1} =: e_{ij}(z).$$

Furthermore, for any $r \geqslant 0$ we get

$$Y(e_{ij}[-r-1], z) = \frac{1}{r!} Y(T^r e_{ij}[-1], z) = \frac{1}{r!} \partial_z^r e_{ij}(z).$$

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In order to define $Y(e_{i_1 j_1}[-r_1 - 1] \dots e_{i_m j_m}[-r_m - 1], z)$,

we need to use normal ordering.

Let

$$a(z) = \sum_{r\in\mathbb{Z}} a_{(r)} z^{-r-1} \qquad \text{and} \qquad b(w) = \sum_{r\in\mathbb{Z}} b_{(r)} w^{-r-1}.$$

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Now, for any $r_i \geqslant 0$ we have

$$Y(e_{i_1 j_1}[-r_1 - 1] \dots e_{i_m j_m}[-r_m - 1], z) = \frac{1}{r_1! \dots r_m!} :\partial_z^{r_1} e_{i_1 j_1}(z) \dots \partial_z^{r_m} e_{i_m j_m}(z):,$$

with the convention that the ordered product is read

from right to left.

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Hence, for the Fourier coefficients we have

$$\big(e_{ij}[-1]\,e_{kl}[-1]\big)_{(m)} = \sum_{r<0} e_{ij}[r]\,e_{kl}[m-r-1] + \sum_{r\geqslant 0} e_{kl}[m-r-1]\,e_{ij}[r].$$

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The local completion of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{gl}}_n)$ at the level κ is the Lie algebra $\mathrm{U}_\kappa(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ spanned by the Fourier coefficients of the fields $Y(a, z)$ with $a \in V_\kappa(\mathfrak{gl}_n)$.

The center of $V_\kappa(\mathfrak{gl}_n)$

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By a Segal–Sugawara vector S we will mean any element of the center of the vertex algebra $V_\kappa(\mathfrak{gl}_n)$, that is, any element $S \in V_\kappa(\mathfrak{gl}_n)$ satisfying $\mathfrak{gl}_n[t]\, S = 0$.

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If $\kappa \neq -n$, then the center of $V_\kappa(\mathfrak{gl}_n)$ is trivial, i.e., coincides with the algebra of polynomials in

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Remark. $n = h^\vee$ is the dual Coxeter number for $\mathfrak{sl}_n$.

From now on suppose $\kappa = -n$, the critical level.

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The quadratic element

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Remark. If $\kappa \neq -n$ then the Fourier coefficients of the field

$$\frac{1}{2(\kappa + n)} Y(S, z)$$

generate an action of the Virasoro algebra on $V_{\kappa}(\mathfrak{sl}_n)$

(the Sugawara construction).

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For any element $S \in \mathrm{U}(t^{-1}\mathfrak{gl}_n[t^{-1}])$ denote by $\overline{S}$ its highest degree component with respect to the natural filtration in the universal enveloping algebra.

Segal–Sugawara vectors

$$S_1, \ldots, S_n \in \mathrm{U}(t^{-1}\mathfrak{gl}_n[t^{-1}])$$

form a complete set of Segal–Sugawara vectors, if the highest degree components $\overline{S}_1, \ldots, \overline{S}_n$ coincide with the images of certain algebraically independent generators of the algebra of invariants $\mathrm{S}(\mathfrak{gl}_n)^{\mathfrak{gl}_n}$ under the embedding $\mathrm{S}(\mathfrak{gl}_n) \hookrightarrow \mathrm{S}(t^{-1}\mathfrak{gl}_n[t^{-1}])$ defined by $e_{ij} \mapsto e_{ij}[-1]$.

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There exists a complete set $S_1, \dots, S_n$ of Segal–Sugawara vectors and

$$\mathfrak{z}(\widehat{\mathfrak{gl}}_n) = \mathbb{C}[T^r S_l \mid l = 1, \dots, n, \;\; r \geqslant 0].$$

Explicit formulas for Segal–Sugawara vectors

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We will need the extended Lie algebra $\widehat{\mathfrak{gl}}_n \oplus \mathbb{C}\tau$, where for the element τ we have the relations

$$\big[\tau, e_{ij}[r]\big] = -r\, e_{ij}[r-1], \qquad \big[\tau, K\big] = 0.$$

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Note that $\quad T\, a = [\tau, a] \quad$ for any $\quad a \in V_{-n}(\mathfrak{gl}_n)$.

For an arbitrary $n \times n$ matrix $A = [a_{ij}]$ with entries in a ring we define its column-determinant $\operatorname{cdet} A$ by the formula

$$\operatorname{cdet} A = \sum_{\sigma} \operatorname{sgn} \sigma \cdot a_{\sigma(1)1} \dots a_{\sigma(n)n},$$

summed over all permutations σ of the set $\{1, \dots, n\}$.

Consider the $n \times n$ matrix $\tau + E[-1]$ given by

$$\tau + E[-1] = \begin{bmatrix} \tau + e_{11}[-1] & e_{12}[-1] & \dots & e_{1n}[-1] \\ e_{21}[-1] & \tau + e_{22}[-1] & \dots & e_{2n}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1}[-1] & e_{n2}[-1] & \dots & \tau + e_{nn}[-1] \end{bmatrix}.$$

Theorem (A. Chervov & A. M. ’09).

The coefficients $S_1, \dots, S_n$ of the polynomial

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^n + S_1\,\tau^{n-1} + \dots + S_{n-1}\,\tau + S_n$$

form a complete set of Segal–Sugawara vectors in $V_{-n}(\mathfrak{gl}_n)$.

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Hence, $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ is the algebra of polynomials,

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with

$$S_1 = e_{11}[-1] + e_{22}[-1],$$

$$S_2 = e_{11}[-1]\,e_{22}[-1] - e_{21}[-1]\,e_{12}[-1] + e_{22}[-2].$$

Regarding the Lie algebra $\mathfrak{sl}_n$ as the quotient of $\mathfrak{gl}_n$ by the relation $e_{11} + \cdots + e_{nn} = 0$, we obtain the respective complete set of Segal–Sugawara vectors in $V_{-n}(\mathfrak{sl}_n)$. In particular, the vector S_1 vanishes, while S_2 coincides with the canonical quadratic element, up to a constant factor.

Proof. A matrix $A = [a_{ij}]$ over a ring is a Manin matrix if

$$a_{ij}\, a_{kl} - a_{kl}\, a_{ij} = a_{kj}\, a_{il} - a_{il}\, a_{kj} \qquad \text{for all possible} \quad i, j, k, l.$$

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Lemma. The matrix $\tau + E[-1]$ with entries in the algebra $\mathrm{U}(t^{-1}\mathfrak{gl}_n[t^{-1}] \oplus \mathbb{C}\tau)$ is a Manin matrix.

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Check that for all i, j

$$e_{ij}[0]\, \mathrm{cdet}\big(\tau + E[-1]\big) = 0 \qquad \text{and}$$

$$e_{nn}[1]\, \mathrm{cdet}\big(\tau + E[-1]\big) = 0$$

in the $\widehat{\mathfrak{gl}}_n$-module $V_{-n}(\mathfrak{gl}_n) \otimes \mathbb{C}[\tau]$.

Corollary. For any $k \geqslant 0$ all coefficients P_{kl} in the expansion

$$\mathrm{tr}(\tau + E[-1])^k = P_{k\,0}\,\tau^k + P_{k1}\,\tau^{k-1} + \cdots + P_{kk}$$

are Segal–Sugawara vectors in $V_{-n}(\mathfrak{gl}_n)$.

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$$\mathrm{tr}(\tau + E[-1])^k = P_{k\,0}\,\tau^k + P_{k1}\,\tau^{k-1} + \cdots + P_{kk}$$

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Hence, $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ is the algebra of polynomials,

$$\mathfrak{z}(\widehat{\mathfrak{gl}}_n) = \mathbb{C}[T^{\,r} P_{l\,l} \mid l = 1, \ldots, n;\; r \geqslant 0].$$

Proof is based on the Newton formula

(A. Chervov & G. Falqui, '08):

$$\mathrm{cdet}(u + \tau + E[-1])^{-1} \cdot \partial_u \, \mathrm{cdet}(u + \tau + E[-1])$$

$$= \sum_{k=0}^{\infty} (-1)^k \, u^{-k-1} \, \mathrm{tr}(\tau + E[-1])^k.$$

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Examples. We have

$$P_{10}=n,\qquad P_{11}=\mathrm{tr}\,E[-1]$$

$$P_{20}=n,\qquad P_{21}=2\,\mathrm{tr}\,E[-1],\qquad P_{22}=\mathrm{tr}\,E[-1]^2+\mathrm{tr}\,E[-2],$$

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Center of the local completion

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Recall that in the vertex algebra $V_{-n}(\mathfrak{gl}_n)$ we have

$e_{ij}(z) = Y(e_{ij}[-1], z)$ with

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Recall also that the local completion of $\mathrm{U}(\widehat{\mathfrak{gl}}_n)$ at the critical level $\kappa = -n$ is the Lie algebra $\mathrm{U}_{-n}(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ spanned by the Fourier coefficients of the fields $Y(a, z)$ with $a \in V_{-n}(\mathfrak{gl}_n)$.

Introduce the $n \times n$ matrix $\partial_z + E(z)$ by

$$\partial_z + E(z) = \begin{bmatrix} \partial_z + e_{11}(z) & e_{12}(z) & \dots & e_{1n}(z) \\ e_{21}(z) & \partial_z + e_{22}(z) & \dots & e_{2n}(z) \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1}(z) & e_{n2}(z) & \dots & \partial_z + e_{nn}(z) \end{bmatrix}.$$

Expand the normally ordered column-determinant

$$: \mathrm{cdet}(\partial_z + E(z)) : \; = \partial_z^n + S_1(z)\,\partial_z^{n-1} + \cdots + S_{n-1}(z)\,\partial_z + S_n(z).$$

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The fields $P_{kl}(z) = Y(P_{kl}, z)$ corresponding to the

Segal–Sugawara vectors P_{kl} are given by

$$:\mathrm{tr}\big(\partial_z + E(z)\big)^k:\; = P_{k\,0}(z)\,\partial_z^k + P_{k1}(z)\,\partial_z^{k-1} + \dots + P_{kk}(z).$$

The center of the local completion $\mathrm{U}_{-n}(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ at the critical level is the vector subspace $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ which consists of the elements commuting with $\widehat{\mathfrak{gl}}_n$.

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Corollary. The center $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ of the local completion $\mathrm{U}_{-n}(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ consists of the Fourier coefficients of all differential polynomials in either family of the fields $S_1(z), \dots, S_n(z)$ or $P_{11}(z), \dots, P_{nn}(z)$.

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K\,\xi &= -n\,\xi.
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Write

$$S_l(z) = \sum_{r \in \mathbb{Z}} S_{l,(r)} z^{-r-1}.$$

If η is a singular vector, then so is $S_{l,(r)}\,\eta$ for any $l = 1, \dots, n$ and $r \leqslant l - 2$.

Corollary. If $\lambda_i - \lambda_j + j - i \notin \{0, 1, \dots\}$ for all $i < j$, then the space of singular vectors of $M(\lambda)$ is

$$\mathbb{C}\,[S_{l,(l-2)}, S_{l,(l-3)}, \dots \mid l = 1, \dots, n]\,\xi.$$

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Example. For $n = 2$ we have $S_{1,(r)} = e_{11}[r] + e_{22}[r]$ and

$$S_{2,(r)} = \sum_{s<0}^{\infty} \Big(e_{11}[s]\, e_{22}[r-s-1] - e_{21}[s]\, e_{12}[r-s-1] \Big)$$
$$+ \sum_{s\geqslant 0}^{\infty} \Big(e_{22}[r-s-1]\, e_{11}[s] - e_{12}[r-s-1]\, e_{21}[s] \Big)$$
$$- r\, e_{22}[r-1].$$

Commutative subalgebras in $\mathrm{U}\big(t^{-1}\mathfrak{gl}_n[t^{-1}]\big)$

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By the vacuum axiom of a vertex algebra, the application of the fields $S_l(z)$ and $P_{kl}(z)$ to the vacuum vector 1 of $V_{-n}(\mathfrak{gl}_n)$ yields power series in z which we denote respectively by

$$S_l(z)_+ = \sum_{r<0} S^+_{l,(r)} z^{-r-1} \qquad \text{and} \qquad P_{kl}(z)_+ = \sum_{r<0} P^+_{kl,(r)} z^{-r-1}.$$

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More explicitly, set

$$e_{ij}(z)_+ = \sum_{r<0} e_{ij}[r]\, z^{-r-1}, \qquad i,j = 1,\dots,n.$$

Consider the matrix

$$\partial_z + E(z)_+ = \begin{bmatrix} \partial_z + e_{11}(z)_+ & e_{12}(z)_+ & \dots & e_{1n}(z)_+ \\ e_{21}(z)_+ & \partial_z + e_{22}(z)_+ & \dots & e_{2n}(z)_+ \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1}(z)_+ & e_{n2}(z)_+ & \dots & \partial_z + e_{nn}(z)_+ \end{bmatrix}.$$

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Then

$$\mathrm{cdet}(\partial_z + E(z)_+) = \partial_z^n + S_1(z)_+ \, \partial_z^{n-1} + \cdots + S_{n-1}(z)_+ \, \partial_z + S_n(z)_+$$

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$$\mathrm{tr}\big(\partial_z + E(z)_+\big)^k = P_{k\,0}(z)_+ \, \partial_z^k + P_{k1}(z)_+ \, \partial_z^{k-1} + \dots + P_{kk}(z)_+.$$

Corollary. The elements of each of the families

$$S^+_{l,(r)} \quad \text{with} \quad l = 1, \dots, n \qquad \text{and} \qquad r < 0,$$

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Moreover, $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ is the algebra of polynomials

$$\begin{aligned}\mathfrak{z}(\widehat{\mathfrak{gl}}_n) &= \mathbb{C}[S^{+}_{l,(r)} \mid l = 1, \dots, n, \;\; r < 0] \\ &= \mathbb{C}[P^{+}_{l\,l,(r)} \mid l = 1, \dots, n; \;\; r < 0].\end{aligned}$$

Remarks. The first family of commuting elements in $\mathrm{U}(t^{-1}\mathfrak{gl}_n[t^{-1}])$ was originally discovered by D. Talalaev ’06 in a slightly different form.

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The second subalgebra was constructed earlier by B. Feigin, E. Frenkel and N. Reshetikhin, '94.

Classical $\mathcal{W}$-algebra for $\mathfrak{gl}_n$

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Let π_0 denote the algebra of polynomials

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in the variables $b_i[r]$, which we consider as a (commutative) vertex algebra.

The translation operator on π_0 is defined by

$$T\,1 = 0, \qquad \big[T, b_i[r]\big] = -r\, b_i[r-1].$$

Introduce the operators

$$Q_i : \pi_0 \to \pi_0, \qquad i = 1, \ldots, n-1,$$

by

$$Q_i = \sum_{r=0}^{\infty} \sum_{\lambda \vdash r} \frac{\mathbf{b}_i(\lambda)}{z_\lambda} \Big(\frac{\partial}{\partial b_i[-r-1]} - \frac{\partial}{\partial b_{i+1}[-r-1]} \Big).$$

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Here,

$$\mathbf{b}_i(\lambda) = \big(b_i[-\lambda_1] - b_{i+1}[-\lambda_1]\big) \dots \big(b_i[-\lambda_p] - b_{i+1}[-\lambda_p]\big),$$

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \dots r^{m_r} m_r!,$$

where m_k is the multiplicity of k in λ.

The first few terms:

$$Q_i = \frac{\partial}{\partial b_i[-1]} - \frac{\partial}{\partial b_{i+1}[-1]}$$

$$+ \Big(b_i[-1] - b_{i+1}[-1]\Big)\Big(\frac{\partial}{\partial b_i[-2]} - \frac{\partial}{\partial b_{i+1}[-2]}\Big)$$

$$+ \frac{b_i[-2] - b_{i+1}[-2] + \big(b_i[-1] - b_{i+1}[-1]\big)^2}{2} \times \Big(\frac{\partial}{\partial b_i[-3]} - \frac{\partial}{\partial b_{i+1}[-3]}\Big) + \dots$$

The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{gl}_n)$ consists of the elements of π_0, annihilated by all operators Q_i,

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The Weyl algebra $\mathcal{A}(\mathfrak{gl}_n)$ is generated by the elements $a_{ij}[r]$ with $r \in \mathbb{Z}$, $i, j = 1, \dots, n$ and $i \neq j$ and the defining relations

$$\big[a_{ij}[r], a_{kl}[s]\big] = \delta_{kj}\delta_{il}\delta_{r,-s} \qquad \text{for} \quad i < j;$$

all other pairs of the generators commute.

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The Fock representation $M(\mathfrak{gl}_n)$ of $\mathcal{A}(\mathfrak{gl}_n)$ is generated by a vector $|0\rangle$ such that for $i < j$ we have

$$a_{ij}[r]|0\rangle = 0, \quad r \geqslant 0 \qquad \text{and} \qquad a_{ji}[r]|0\rangle = 0, \quad r > 0.$$

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The Fock representation $M(\mathfrak{gl}_2)$ is generated by a vector $|0\rangle$ such that

$$a_{12}[r]|0\rangle = 0, \quad r \geqslant 0 \qquad \text{and} \qquad a_{21}[r]|0\rangle = 0, \quad r > 0.$$

The elements of $M(\mathfrak{gl}_2)$ are polynomials in the $a_{12}[r]$ with $r < 0$ and $a_{21}[r]$ with $r \leqslant 0$ applied to $|0\rangle$.

The vector space $M(\mathfrak{gl}_n)$ carries a vertex algebra structure. In particular, $|0\rangle$ is the vacuum vector, and for $i < j$ we have

$$Y(a_{ij}[-1]\,|0\rangle, z) = \sum_{r\in\mathbb{Z}} a_{ij}[r]\,z^{-r-1} =: a_{ij}(z)$$

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Key fact (M. Wakimoto '86, B. Feigin & E. Frenkel '88).

There exists a vertex algebra homomorphism

$$\rho : V_{-n}(\mathfrak{gl}_n) \to M(\mathfrak{gl}_n) \otimes \pi_0.$$

Example. For $n = 2$ the explicit formulas are

$$e_{12}(z) \mapsto a_{12}(z)$$

$$e_{11}(z) \mapsto \; - \; : a_{21}(z)\, a_{12}(z) : \; + b_1(z)$$

$$e_{22}(z) \mapsto \; : a_{21}(z)\, a_{12}(z) : \; + b_2(z)$$

$$e_{21}(z) \mapsto \; - \; : a_{21}(z)^2\, a_{12}(z) : \; - 2\,\partial_z\, a_{21}(z) + a_{21}(z)\,\big(b_1(z) - b_2(z)\big),$$

where

$$b_i(z) = \sum_{r<0} b_i[r]\, z^{-r-1}.$$

The image of the center $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ of the vertex algebra $V_{-n}(\mathfrak{gl}_n)$ under the homomorphism ρ is contained in $\pi_0 \cong 1 \otimes \pi_0$.

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This image coincides with the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{gl}_n)$.

Corollary.

$$\rho : \mathrm{cdet}(\tau + E[-1]) \mapsto \big(\tau + b_n[-1]\big) \cdots \big(\tau + b_1[-1]\big),$$

where $\quad \big[\tau, b_i[r]\big] = -r\, b_i[r-1]$.

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Hence, $\mathcal{W}(\mathfrak{gl}_n) = \mathbb{C}[T^r B_i \mid i = 1, \dots, n, \quad r \geqslant 0]$, where

$$\big(\tau + b_n[-1]\big) \cdots \big(\tau + b_1[-1]\big) = \tau^n + B_1\, \tau^{n-1} + \cdots + B_n.$$

Corollary.

$$
\begin{aligned}
\rho : \sum_{k=0}^{\infty} t^k \operatorname{tr}(\tau + E[-1])^k
&\mapsto \sum_{i=1}^{n} \Big(1 - t\left(\tau + b_1[-1]\right)\Big)^{-1} \cdots \Big(1 - t\left(\tau + b_i[-1]\right)\Big)^{-1} \\
&\qquad \times \Big(1 - t\left(\tau + b_{i-1}[-1]\right)\Big) \cdots \Big(1 - t\left(\tau + b_1[-1]\right)\Big),
\end{aligned}
$$

where t is a complex variable.

Eigenvalues in the Wakimoto modules

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Take an n-tuple

$$\chi(t) = \big(\chi_1(t), \dots, \chi_n(t)\big), \qquad \chi_i(t) = \sum_{r\in\mathbb{Z}} \chi_i[r]\, t^{-r-1} \in \mathbb{C}((t)).$$

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We obtain the Wakimoto modules of critical level $W_{\chi(t)}$.

Example. For $n = 2$ the explicit formulas are

$$e_{12}(z) \mapsto a_{12}(z)$$

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The elements of the center $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ of $\mathrm{U}_{-n}(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ act on the Wakimoto modules $W_{\chi(t)}$ as multiplications by scalars.

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$$: \mathrm{cdet}(\partial_z + E(z)) : \ \mapsto \big(\partial_z + \chi_n(z)\big) \dots \big(\partial_z + \chi_1(z)\big)$$

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Corollary.

$$: \mathrm{cdet}(\partial_z + E(z)) : \; \mapsto \big(\partial_z + \chi_n(z)\big) \dots \big(\partial_z + \chi_1(z)\big)$$

and

$$\sum_{k=0}^{\infty} t^k \; : \mathrm{tr}\big(\partial_z + E(z)\big)^k :$$

$$\mapsto \sum_{i=1}^{n} \Big(1 - t\left(\partial_z + \chi_1(z)\right)\Big)^{-1} \cdots \Big(1 - t\left(\partial_z + \chi_i(z)\right)\Big)^{-1}$$

$$\times \Big(1 - t\left(\partial_z + \chi_{i-1}(z)\right)\Big) \cdots \Big(1 - t\left(\partial_z + \chi_1(z)\right)\Big).$$

Example. If $n = 3$, then

$$:\mathrm{cdet}(\partial_z + E(z)) := \partial_z^3 + S_1(z)\,\partial_z^2 + S_2(z)\,\partial_z + S_3(z)$$

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$$: \mathrm{cdet}(\partial_z + E(z)) := \partial_z^3 + S_1(z)\,\partial_z^2 + S_2(z)\,\partial_z + S_3(z)$$

and

$$S_1(z) \mapsto \chi_1(z) + \chi_2(z) + \chi_3(z),$$

$$S_2(z) \mapsto \chi_1(z)\,\chi_2(z) + \chi_1(z)\,\chi_3(z) + \chi_2(z)\,\chi_3(z) + 2\chi_1'(z) + \chi_2'(z),$$

$$S_3(z) \mapsto \chi_1(z)\,\chi_2(z)\,\chi_3(z) + \chi_1'(z)\,\chi_2(z) + \chi_1'(z)\,\chi_3(z)$$

$$+ \chi_1(z)\,\chi_2'(z) + \chi_1''(z).$$