Classical Lie algebras and Yangians

Alexander Molev

University of Sydney

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Lecture 1. Casimir elements for classical Lie algebras

Lecture 2. Yangians: algebraic structure

Lecture 3. Yangians: representations

Classical Lie algebras over $\mathbb{C}$

A type: special linear Lie algebra $\mathfrak{sl}_N$

general linear Lie algebra $\mathfrak{gl}_N$

B type: orthogonal Lie algebra $\mathfrak{o}_{2n+1}$

C type: symplectic Lie algebra $\mathfrak{sp}_{2n}$

D type: orthogonal Lie algebra $\mathfrak{o}_{2n}$

General linear Lie algebra

The Lie algebra $\mathfrak{gl}_N$ has the basis of the standard matrix units E_{ij} with $1 \leqslant i, j \leqslant N$ so that $\dim \mathfrak{gl}_N = N^2$. The commutation relations are

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}.$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$ is the associative algebra with generators E_{ij} and the defining relations

$$E_{ij}\, E_{kl} - E_{kl}\, E_{ij} = \delta_{kj} E_{il} - \delta_{il} E_{kj}.$$

By the Poincaré–Birkhoff–Witt theorem, given any ordering on the set of generators $\{E_{ij}\}$, any element of $\mathrm{U}(\mathfrak{gl}_N)$ can be uniquely written as a linear combination of the ordered monomials in the E_{ij}.

The center $\mathrm{Z}(\mathfrak{gl}_N)$ of $\mathrm{U}(\mathfrak{gl}_N)$ is

$$\mathrm{Z}(\mathfrak{gl}_N) = \big\{ z \in \mathrm{U}(\mathfrak{gl}_N) \mid z\,x = x\,z \quad \text{for all} \quad x \in \mathrm{U}(\mathfrak{gl}_N) \big\}.$$

The Casimir elements for $\mathfrak{gl}_N$ are elements of $\mathrm{Z}(\mathfrak{gl}_N)$.

Given an N-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_N)$ the Verma module $M(\lambda)$ for $\mathfrak{gl}_N$ is the quotient of $\mathrm{U}(\mathfrak{gl}_N)$ by the left ideal generated by the elements

$$E_{ij}, \quad i < j, \qquad \text{and} \qquad E_{ii} - \lambda_i, \quad i = 1, \dots, N.$$

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The Verma module has a unique maximal submodule K.

Set

$$L(\lambda) = M(\lambda)/K,$$

the unique irreducible quotient of $M(\lambda)$.

Equivalently, $L(\lambda)$ is an irreducible module generated by a nonzero vector ζ such that

$$E_{ij}\,\zeta = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$E_{ii}\,\zeta = \lambda_i\,\zeta \qquad \text{for} \quad 1 \leqslant i \leqslant N.$$

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The representation $L(\lambda)$ is finite-dimensional if and only if

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \text{for all} \quad i = 1, \dots, N-1.$$

Any Casimir element $z \in \mathrm{Z}(\mathfrak{gl}_N)$ acts as a multiplication by a scalar $\chi(z)$ in $L(\lambda)$. This scalar is a polynomial in $\lambda_1, \dots, \lambda_N$; this polynomial is symmetric in the shifted variables

$$l_1 = \lambda_1, \quad l_2 = \lambda_2 - 1, \quad \dots, \quad l_N = \lambda_N - N + 1.$$

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Example. $\chi : E_{11} + \dots + E_{NN} \mapsto \lambda_1 + \dots + \lambda_N$

$$= l_1 + \dots + l_N - N(N-1)/2.$$

Orthogonal and symplectic Lie algebras

For $N = 2n$ or $N = 2n + 1$, respectively, set

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We will number the rows and columns of $N \times N$ matrices by the indices $\{-n, \ldots, -1, 0, 1, \ldots, n\}$ if $N = 2n + 1$, and by $\{-n, \ldots, -1, 1, \ldots, n\}$ if $N = 2n$.

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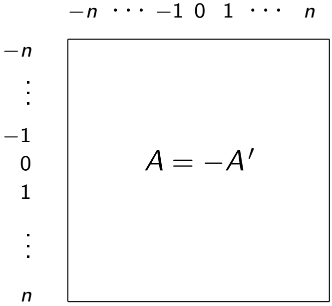
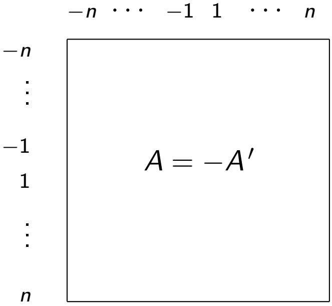
The Lie algebra $\mathfrak{g}_N = \mathfrak{o}_N$ is spanned by the elements

$$F_{ij} = E_{ij} - E_{-j,-i}, \qquad -n \leqslant i, j \leqslant n.$$

$\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ $\qquad\qquad$ $\mathfrak{g}_N = \mathfrak{o}_{2n}$





Skew-symmetric matrices with respect to the second diagonal.

The Lie algebra $\mathfrak{g}_N = \mathfrak{sp}_N$ with $N = 2n$ is spanned by the elements

$$F_{ij} = E_{ij} - \operatorname{sgn} i \cdot \operatorname{sgn} j \cdot E_{-j,-i}, \qquad -n \leqslant i, j \leqslant n.$$

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$$F_{ij} = E_{ij} - \operatorname{sgn} i \cdot \operatorname{sgn} j \cdot E_{-j,-i}, \qquad -n \leqslant i, j \leqslant n.$$

$$\begin{array}{c|c|c|}
 & -n \ \cdots \ -1 & 1 \ \cdots \ n \\
\hline
-n, \dots, -1 & A & B = B' \\
\hline
1, \dots, n & C = C' & -A' \\
\hline
\end{array}$$

Commutation relations in $\mathfrak{g}_N$:

$$F_{-j,-i} = -\theta_{ij}\, F_{ij} \qquad \text{and}$$

$$[F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj} - \theta_{k,-j}\, \delta_{i,-k} F_{-j,l} + \theta_{i,-l}\, \delta_{-l,j} F_{k,-i},$$

where

$$\theta_{ij} = \begin{cases} 1 & \text{in the orthogonal case,} \\ \operatorname{sgn} i \cdot \operatorname{sgn} j & \text{in the symplectic case.} \end{cases}$$

For any n-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n)$ the corresponding irreducible highest weight representation $V(\lambda)$ of $\mathfrak{g}_N$ is generated by a nonzero vector ξ such that

$$F_{ij}\,\xi = 0 \qquad \text{for} \quad -n \leqslant i < j \leqslant n, \qquad \text{and}$$

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$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1, \dots, n-1$$

and

$$-\lambda_1 - \lambda_2 \in \mathbb{Z}_+ \qquad \text{if} \quad \mathfrak{g}_N = \mathfrak{o}_{2n},$$

$$-\lambda_1 \in \mathbb{Z}_+ \qquad \text{if} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n},$$

$$-2\,\lambda_1 \in \mathbb{Z}_+ \qquad \text{if} \quad \mathfrak{g}_N = \mathfrak{o}_{2n+1}.$$

Any element $z \in \mathrm{Z}(\mathfrak{g}_N)$ of the center of $\mathrm{U}(\mathfrak{g}_N)$ acts as a multiplication by a scalar $\chi(z)$ in $V(\lambda)$. This scalar is a polynomial in $\lambda_1, \ldots, \lambda_n$. In the B and C cases, this polynomial is symmetric in the variables $l_1^2, \ldots, l_n^2$, where $l_i = \lambda_i + \rho_i$ and

$$\rho_i = -\rho_{-i} = \begin{cases} -i+1 & \text{for} \quad \mathfrak{g}_N = \mathfrak{o}_{2n}, \\ -i+\frac{1}{2} & \text{for} \quad \mathfrak{g}_N = \mathfrak{o}_{2n+1}, \\ -i & \text{for} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}, \end{cases}$$

for $i = 1, \ldots, n$. Also, $\rho_0 = 1/2$ in the case $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$.

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In the D case $\chi(z)$ is the sum of a symmetric polynomial in $l_1^2, \dots, l_n^2$ and $l_1 \dots l_n$ times a symmetric polynomial in $l_1^2, \dots, l_n^2$.

The map

$$\chi : \mathrm{Z}(\mathfrak{g}_N) \to \quad \text{algebra of polynomials}$$

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Example. For $\mathfrak{g}_N = \mathfrak{o}_N$

$$\sum_{m=1}^{n} \Big((F_{mm} + \rho_m)^2 + 2 \sum_{-m<i<m} F_{mi} F_{im} \Big)$$

is the second degree Casimir element. Its Harish-Chandra image is

$$l_1^2 + \cdots + l_n^2.$$

Newton's formulas

Denote by E the $N \times N$ matrix whose ij-th entry if E_{ij}. Denote by $\mathcal{C}(u)$ the Capelli determinant

$$\mathcal{C}(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot (u + E)_{p(1),1} \dots (u + E - N + 1)_{p(N),N}.$$

This is a polynomial in u with coefficients in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$,

$$\mathcal{C}(u) = u^N + \mathcal{C}_1\, u^{N-1} + \dots + \mathcal{C}_N, \qquad \mathcal{C}_i \in \mathrm{U}(\mathfrak{gl}_N).$$

Example. For $N = 2$ we have

$$\begin{aligned}\mathcal{C}(u) &= (u + E_{11})\,(u + E_{22} - 1) - E_{21}\,E_{12} \\ &= u^2 + (E_{11} + E_{22} - 1)\,u + E_{11}\,(E_{22} - 1) - E_{21}\,E_{12}.\end{aligned}$$

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Note that

$$\mathcal{C}_1 = E_{11} + E_{22} - 1, \qquad \mathcal{C}_2 = E_{11}\,(E_{22} - 1) - E_{21}\,E_{12}$$

are Casimir elements for $\mathfrak{gl}_2$ and

$$\chi(\mathcal{C}_1) = l_1 + l_2,$$
$$\chi(\mathcal{C}_2) = l_1\,l_2.$$

Theorem

The coefficients $\mathcal{C}_1, \dots, \mathcal{C}_N$ *belong to* $\mathrm{Z}(\mathfrak{gl}_N)$. *The image of* $\mathcal{C}(u)$ *under the Harish-Chandra isomorphism is given by*

$$\chi : \mathcal{C}(u) \mapsto (u + l_1) \dots (u + l_N),$$

so that $\chi(\mathcal{C}_k)$ *is the elementary symmetric polynomial of degree* k *in* $l_1, \dots, l_N$,

$$\chi(\mathcal{C}_k) = \sum_{i_1 < \dots < i_k} l_{i_1} \dots l_{i_k}.$$

Moreover, the algebra $\mathrm{Z}(\mathfrak{gl}_N)$ *is generated by* $\mathcal{C}_1, \dots, \mathcal{C}_N$.

Gelfand invariants

are the elements of $\mathrm{U}(\mathfrak{gl}_N)$ defined by

$$\operatorname{tr} E^k = \sum_{i_1, i_2, \dots, i_k = 1}^{N} E_{i_1 i_2} E_{i_2 i_3} \dots E_{i_k i_1}, \qquad k = 0, 1, \dots.$$

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$$\mathrm{tr}\, E = E_{11} + E_{22},$$

$$\mathrm{tr}\, E^2 = E_{11}^2 + E_{12}\, E_{21} + E_{21}\, E_{12} + E_{22}^2.$$

These are Casimir elements and

$$\chi(\mathrm{tr}\, E) = l_1 + l_2 - 1,$$

$$\chi(\mathrm{tr}\, E^2) = l_1^2 + l_2^2 + l_1 + l_2.$$

A noncommutative analogue of the classical Newton formula:

Theorem

We have the equality of power series in u^{-1}

$$1+\sum_{k=0}^{\infty}\frac{(-1)^k\,\mathrm{tr}\,E^k}{(u-N+1)^{k+1}}=\frac{\mathcal{C}(u+1)}{\mathcal{C}(u)}.$$

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Proof.

This is equivalent to the Perelomov–Popov formulas

$$1+\sum_{k=0}^{\infty}\frac{(-1)^k\,\chi(\mathrm{tr}\,E^k)}{(u-N+1)^{k+1}}=\prod_{i=1}^{N}\frac{u+l_i+1}{u+l_i}.$$

□

Capelli-type determinant for $\mathfrak{g}_N$

Introduce a special map

$$\varphi_N : \mathfrak{S}_N \to \mathfrak{S}_N, \qquad p \mapsto p'$$

from the symmetric group $\mathfrak{S}_N$ into itself.

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If $N = 2$ we define φ_2 as the map $\mathfrak{S}_2 \to \mathfrak{S}_2$ whose image is the identity permutation.

Given a set of positive integers $a_1 < \cdots < a_N$ we regard $\mathfrak{S}_N$ as the group of their permutations.

For $N > 2$ define a map from the set of ordered pairs (a_k, a_l) with $k \neq l$ into itself by the rule

$$(a_k, a_l) \mapsto (a_l, a_k), \qquad\qquad k, l < N,$$

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\begin{aligned}
(a_k, a_l) &\mapsto (a_l, a_k), && k, l < N, \\
(a_k, a_N) &\mapsto (a_{N-1}, a_k), && k < N-1, \\
(a_N, a_k) &\mapsto (a_k, a_{N-1}), && k < N-1,
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\end{aligned}$$

Let $p = (p_1, \ldots, p_N)$ be a permutation of the indices $a_1, \ldots, a_N$. Its image under the map φ_N is the permutation of the form $p' = (p'_1, \ldots, p'_{N-1}, a_N)$.

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Then the pair (p'_2, p'_{N-2}) is found as the image of (p_2, p_{N-1}) under the above map, etc.

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$$(N, \color{red}{N-1}, \ldots, \color{red}{2}, 1) \mapsto (1, 2, \ldots, N-2, N-1, N) = \mathrm{id}.$$

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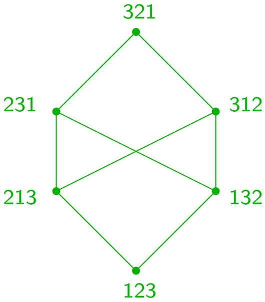

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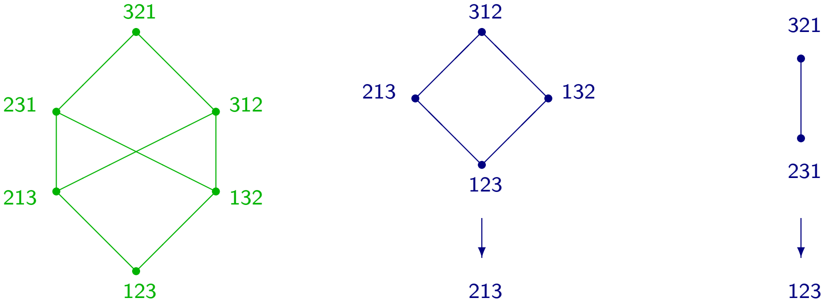

Denote by F the $N \times N$ matrix whose ij-th entry is F_{ij}. Introduce the Capelli-type determinant

$$\mathcal{C}(u) = (-1)^n \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p\, p' \cdot (u + \rho_{-n} + F)_{-b_{p(1)},\, b_{p'(1)}} \times \cdots \times (u + \rho_n + F)_{-b_{p(N)},\, b_{p'(N)}},$$

where $(b_1, \ldots, b_N)$ is a fixed permutation of the indices $(-n, \ldots, n)$ and p' is the image of p under the map φ_N.

Theorem

The polynomial $\mathcal{C}(u)$ does not depend on the choice of the permutation $(b_1, \dots, b_N)$. All coefficients of $\mathcal{C}(u)$ belong to $\mathrm{Z}(\mathfrak{g}_N)$. Moreover, the image of $\mathcal{C}(u)$ under the Harish-Chandra isomorphism is given by

$$\chi : \mathcal{C}(u) \mapsto \prod_{i=1}^{n} (u^2 - l_i^2), \qquad \textit{if} \quad N = 2n,$$

and

$$\chi : \mathcal{C}(u) \mapsto \left(u + \frac{1}{2}\right) \prod_{i=1}^{n} (u^2 - l_i^2), \qquad \textit{if} \quad N = 2n + 1.$$

Examples. For $\mathfrak{g}_N = \mathfrak{sp}_2$ take $(b_1, b_2) = (-1, 1)$.

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$$\mathcal{C}(u) = (u + F_{-1,-1} + 1)\,(u + F_{11} - 1) - F_{1,-1}\,F_{-1,1}$$

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$$\chi : \mathcal{C}(u) \mapsto u^2 - l_1^2.$$

For $\mathfrak{g}_N = \mathfrak{o}_3$ take $(b_1, b_2, b_3) = (-1, 0, 1)$.

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Hence

$$\mathcal{C}(u) = (u + 1/2)\,(u^2 - (F_{11} - 1/2)^2 - 2\,F_{10}\,F_{01})$$

and

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Gelfand invariants

are the elements of $\mathrm{U}(\mathfrak{g}_N)$ defined by

$$\operatorname{tr} F^k = \sum_{i_1, i_2, \dots, i_k = -n}^{n} F_{i_1 i_2} F_{i_2 i_3} \dots F_{i_k i_1}, \qquad k = 0, 1, \dots .$$

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We have $\operatorname{tr} F = 0$,

$$\operatorname{tr} F^2 = \sum_{i,j=-n}^{n} F_{ij} F_{ji}$$

and

$$\chi(\operatorname{tr} F^2) = 2 \sum_{i=1}^{n} (l_i^2 - \rho_i^2).$$

Theorem

If $N = 2n$ *then*

$$1 + \frac{2u+1}{2u+1\mp 1}\sum_{k=0}^{\infty}\frac{(-1)^k\,\mathrm{tr}\,F^k}{(u+\rho_n)^{k+1}} = \frac{\mathcal{C}(u+1)}{\mathcal{C}(u)},$$

where the upper sign is taken in the orthogonal case and the lower sign in the symplectic case. If $N = 2n+1$ *then*

$$1 + \frac{2u+1}{2u}\sum_{k=0}^{\infty}\frac{(-1)^k\,\mathrm{tr}\,F^k}{(u+\rho_n)^{k+1}} = \frac{\overline{\mathcal{C}}(u+1)}{\overline{\mathcal{C}}(u)},$$

where

$$\overline{\mathcal{C}}(u) = \frac{2u}{2u+1}\,\mathcal{C}(u).$$

All Gelfand invariants $\mathrm{tr}\, F^k$ belong to $\mathrm{Z}(\mathfrak{g}_N)$.

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Their images under the Harish-Chandra isomorphism are found by the Perelomov–Popov formulas

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where the zero index is skipped in the product if $N=2n$, while for $N=2n+1$ one should set $l_0=0$.

Noncommutative Cayley–Hamilton theorem

$\mathcal{C}(u)$ denotes the Capelli determinant for $\mathfrak{gl}_N$

or the Capelli-type determinant for $\mathfrak{g}_N = \mathfrak{o}_{2n}, \ \ \mathfrak{sp}_{2n}, \ \ \mathfrak{o}_{2n+1}$.

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Theorem

(i) *For* $\mathfrak{gl}_N$ *we have*

$$\mathcal{C}(-E + N - 1) = 0 \qquad \textit{and} \qquad \mathcal{C}(-E^{\,t}) = 0.$$

(ii) *For* $\mathfrak{g}_N$ *we have*

$$\mathcal{C}(-F - \rho_n) = 0.$$

Corollary (Characteristic identities of Bracken and Green)

(i) The image of the matrix E in the representation $L(\lambda)$ of $\mathfrak{gl}_N$ satisfies

$$\prod_{i=1}^{N}(E - l_i - N + 1) = 0 \qquad \text{and} \qquad \prod_{i=1}^{N}(E^{\,t} - l_i) = 0.$$

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(ii) The image of the matrix F in the representation $V(\lambda)$ of $\mathfrak{g}_N$ satisfies

$$\prod_{i=-n}^{n}(F - l_i + \rho_n) = 0,$$

The zero index is skipped in the product if $N = 2n$, while for $N = 2n + 1$ one should set $l_0 = 1/2$.

Noncommutative power sums Casimir elements

For $1 \leqslant m \leqslant N$ and any positive integer k set

$$\Phi_k^{(m)} = \sum \frac{k}{\alpha(I)+1}\, \mathcal{E}_{m\, i_1} \mathcal{E}_{i_1 i_2} \dots \mathcal{E}_{i_{k-1} m},$$

summed over $i_1, \dots, i_{k-1} \in \{1, \dots, m\}$,

where $\mathcal{E}_{ij} = E_{ij} - \delta_{ij}(m-1)$ and

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Example.

$$\Phi_1^{(m)} = E_{mm} - m + 1$$

$$\Phi_2^{(m)} = (E_{mm} - m + 1)^2 + 2 \sum_{i=1}^{m-1} E_{m\, i} E_{i\, m}.$$

Theorem. For any $k \geqslant 1$ the element

$$\Phi_k = \Phi_k^{(1)} + \cdots + \Phi_k^{(N)}$$

belongs to $\mathrm{Z}(\mathfrak{gl}_N)$. Moreover,

$$\chi(\Phi_k) = l_1^k + \cdots + l_N^k.$$

(Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon, '95).

$$\Phi_1 = \sum_{m=1}^{N} (E_{mm} - m + 1),$$

$$\Phi_2 = \sum_{m=1}^{N} (E_{mm} - m + 1)^2 + 2 \sum_{1 \leqslant l < m \leqslant N} E_{ml} \, E_{lm}.$$

Orthogonal and symplectic case

For $1 \leqslant m \leqslant n$ and any positive integer k set

$$\Phi_{2k}^{(m)} = \sum \frac{2k}{\alpha(I)+1}\, \mathcal{F}_{m\, i_1} \mathcal{F}_{i_1 i_2} \dots \mathcal{F}_{i_{2k-1} m},$$

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summed over $i_1, \dots, i_{2k-1} \in \{-m+1, \dots, m\}$,

where $\mathcal{F}_{ij} = F_{ij} + \delta_{ij}\, \rho_m$ and

$\alpha(I)$ is the multiplicity of m in the multiset $I = \{i_1, \dots, i_{2k-1}\}$.

Example. We have

$$\Phi_2^{(m)} = (F_{mm} + \rho_m)^2 + 2 \sum_{-m \leqslant i < m} F_{m\,i} F_{i\,m}, \qquad \text{and}$$

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Theorem. For any $k \geqslant 1$ the element

$$\Phi_{2k} = \Phi_{2k}^{(1)} + \widehat{\Phi}_{2k}^{(1)} + \dots + \Phi_{2k}^{(n)} + \widehat{\Phi}_{2k}^{(n)}$$

belongs to $\mathrm{Z}(\mathfrak{g}_N)$. Moreover,

$$\chi(\Phi_{2k}) = 2\left(l_1^{2k} + \dots + l_n^{2k}\right).$$

If $\mathfrak{g}_N = \mathfrak{o}_N$ then the second order Casimir element is

$$\Phi_2 = 2 \sum_{m=1}^{n} \Big((F_{mm} + \rho_m)^2 + 2 \sum_{-m<i<m} F_{m\,i} F_{i\,m} \Big).$$

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$$\Phi_2 = 2\sum_{m=1}^{n}\Big((F_{mm} - m)^2 + F_{m,-m}F_{-m,m} + 2\sum_{-m<i<m} F_{m\,i} F_{i\,m}\Big).$$

Quantum immanants for $\mathfrak{gl}_N$

A diagram (or partition) is a sequence $\mu = (\mu_1, \dots, \mu_N)$ of integers μ_i such that $\mu_1 \geqslant \dots \geqslant \mu_N \geqslant 0$, depicted as an array of unit cells (or boxes).

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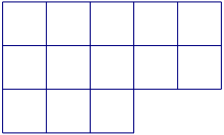
Example. The diagram $\mu = (5, 5, 3)$ is



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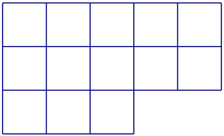

$|\mu| = 13$ $\qquad$ $\ell(\mu) = 3$

The number of cells is the weight of the diagram, denoted $|\mu|$.

The number of nonzero rows is its length, denoted $\ell(\mu)$.

For a diagram μ with $\ell(\mu) \leqslant N$ and $|\mu| = k$ consider the row tableau T_0 obtained by filling in the cells by the numbers $1, \ldots, k$ from left to right in successive rows:

1	2	3	4	5
6	7	8		
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Let R_μ and C_μ denote the row symmetrizer and column antisymmetrizer of T_0 respectively:

$$R_\mu = \sum_\sigma \sigma, \qquad C_\mu = \sum_\tau \operatorname{sgn} \tau \cdot \tau.$$

Set $c_\mu(r) = j - i$ if the cell (i, j) of T_0 is occupied by r.

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Consider the matrix E as the element

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \mathrm{End}\,(\mathbb{C}^N) \otimes \mathrm{U}(\mathfrak{gl}_N)$$

and define the quantum immanant $\mathbb{S}_\mu$ by

$$\mathbb{S}_\mu = \frac{1}{h(\mu)} \mathrm{tr}\,(E - c_\mu(1)) \otimes \cdots \otimes (E - c_\mu(k)) \cdot R_\mu C_\mu,$$

where $h(\mu)$ is the product of the hooks of μ (Okounkov, 96).

The symmetric group $\mathfrak{S}_k$ acts in a natural way in the tensor space $(\mathbb{C}^N)^{\otimes k}$. We identify elements of $\mathfrak{S}_k$ and hence R_μ and C_μ with the corresponding operators.

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$$T_0 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

Hence

$$\mathbb{S}_{(2,1)} = \frac{1}{3}\,\mathrm{tr}\, E \otimes (E-1) \otimes (E+1) \cdot (1+P_{12})(1-P_{13}).$$

Explicitly,

$$
\begin{aligned}
&E \otimes (E-1) \otimes (E+1) \\
&\quad = \sum e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes e_{i_3 j_3} \otimes E_{i_1 j_1} \, (E_{i_2 j_2} - \delta_{i_2 j_2}) \, (E_{i_3 j_3} + \delta_{i_3 j_3}).
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$$
\begin{aligned}
\mathbb{S}_{(2,1)} = \frac{1}{3} \sum_{i_1, i_2, i_3} \Big(& E_{i_1 i_1} \, (E_{i_2 i_2} - 1) \, (E_{i_3 i_3} + 1) \\
& + E_{i_1 i_2} \, (E_{i_2 i_1} - \delta_{i_2 i_1}) \, (E_{i_3 i_3} + 1) \\
& - E_{i_1 i_3} \, (E_{i_2 i_2} - 1) \, (E_{i_3 i_1} + \delta_{i_3 i_1}) \\
& - E_{i_1 i_2} \, (E_{i_2 i_3} - \delta_{i_2 i_3}) \, (E_{i_3 i_1} + \delta_{i_3 i_1}) \Big),
\end{aligned}
$$

summed over the indices $i_1, i_2, i_3 \in \{1, \ldots, N\}$.

Examples. Capelli elements (quantum minors)

$$\mathbb{S}_{(1^k)} = \sum_{i_1 > \cdots > i_k} \sum_{p \in \mathfrak{S}_k} \operatorname{sgn} p \cdot E_{i_1, i_{p(1)}} \dots (E + k - 1)_{i_k, i_{p(k)}}.$$

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Quantum permanents

$$\mathbb{S}_{(k)} = \sum_{i_1 \geqslant \dots \geqslant i_k} \frac{1}{\alpha_1! \dots \alpha_n!} \sum_{p \in \mathfrak{S}_k} E_{i_1, i_{p(1)}} \dots (E - k + 1)_{i_k, i_{p(k)}},$$

where α_i is the multiplicity of i in $i_1, \dots, i_k$, each $i_r \in \{1, \dots, N\}$.

Theorem

The quantum immanants $\mathbb{S}_\mu$ *with* $\ell(\mu) \leqslant N$ *form a basis of the center of the universal enveloping algebra* $\mathrm{U}(\mathfrak{gl}_N)$.

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Moreover,

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where $s^*_\mu = s^*_\mu(\lambda)$ *is the shifted symmetric polynomial.*

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Moreover,

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where $s^*_\mu = s^*_\mu(\lambda)$ *is the shifted symmetric polynomial.*

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The $s^*_\mu(\lambda)$ are certain symmetric polynomials in $l_1, \dots, l_N$.

Explicit formula:

$$s_\mu^*(\lambda) = \sum_{\mathrm{sh}(T)=\mu} \prod_{\alpha\in\mu} \big(\lambda_{T(\alpha)} - c(\alpha)\big),$$

summed over all reverse μ-tableaux T with entries in $\{1, \dots, N\}$ such that the entries of T weakly decrease along the rows and strictly decrease down the columns.

Here $c(\alpha) = j - i$ for $\alpha = (i, j)$ and $T(\alpha)$ is the entry of T in the cell α.

Example. For $\mu = (2,1)$ the reverse tableaux are

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \text{with} \quad i \geqslant j \quad \text{and} \quad i > k$$

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Hence

$$s^*_{(2,1)}(\lambda) = \sum_{i \geqslant j,\ i > k} \lambda_i \, (\lambda_j - 1) \, (\lambda_k + 1).$$

Examples. Harish-Chandra images of the Capelli elements

$$\chi(\mathbb{S}_{(1^k)}) = \sum_{i_1 > \dots > i_k} \lambda_{i_1} (\lambda_{i_2} + 1) \dots (\lambda_{i_k} + k - 1).$$

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These are symmetric polynomials in $\quad l_1, \dots, l_N,$

$l_1 = \lambda_1, \quad \dots, \quad l_N = \lambda_N - N + 1.$

Noncommutative Pfaffians and Hafnians

The Pfaffian $\operatorname{Pf} A$ of a $2k \times 2k$ matrix $A = [A_{ij}]$ is defined by

$$\operatorname{Pf} A = \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} \operatorname{sgn} \sigma \cdot A_{\sigma(1),\sigma(2)} \ldots A_{\sigma(2k-1),\sigma(2k)}.$$

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Let $\mathfrak{g}_N = \mathfrak{o}_N$. For any subset I of $\{-n, \dots, n\}$ containing $2k$ elements $i_1 < \dots < i_{2k}$, the $2k \times 2k$ matrix $[F_{i_p,-i_q}]$ is skew-symmetric. We denote its Pfaffian by

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Set

$$C_k = (-1)^k \cdot \sum_I \Phi_I \, \Phi_{I^*}, \qquad I^* = \{-i_{2k}, \dots, -i_1\}.$$

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Theorem

For all $k = 1, \dots, n$ the C_k are Casimir elements for $\mathfrak{o}_N$. Moreover, the image of C_k under the Harish-Chandra isomorphism is given by

$$\chi : C_k \mapsto (-1)^k \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} (l_{i_1}^2 - \rho_{i_1}^2) \dots (l_{i_k}^2 - \rho_{i_k - k + 1}^2).$$

If $N = 2n$ then $C_n = (-1)^n (\mathrm{Pf}\, F)^2$.

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Corollary.

$$\frac{\mathcal{C}(u)}{(u + \rho_{-n}) \dots (u + \rho_n)} = 1 + \sum_{k=1}^{n} \frac{C_k}{(u^2 - \rho_{n-k+1}^2) \dots (u^2 - \rho_n^2)}.$$

For any $k \geqslant 1$ let $I = \{i_1, \ldots, i_{2k}\}$ be a multiset whose elements belong to $\{-n, \ldots, n\}$.

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The Hafnian $\mathrm{Hf}\, A_I$ of the matrix A_I is defined by

$$\mathrm{Hf}\, A_I = \frac{1}{2^k k!} \sum_{\sigma \in \mathfrak{S}_{2k}} A_{i_{\sigma(1)}, i_{\sigma(2)}} \ldots A_{i_{\sigma(2k-1)}, i_{\sigma(2k)}}.$$

(Caianiello, '56).

Let $\mathfrak{g}_N = \mathfrak{sp}_{2n}$. Set $\widetilde{F}_{ij} = \operatorname{sgn} i \cdot F_{ij}$. Then we have $\widetilde{F}_{i,-j} = \widetilde{F}_{j,-i}$. Let I be any sequence of length $2k$ of elements from the set $\{-n, \dots, n\}$. Denote the multiplicity of an element i in I by α_i. Denote the Hafnian of the symmetric matrix $[\widetilde{F}_{i_p,-i_q}]$ by

$$\Psi_I = \operatorname{Hf}[\widetilde{F}_{i_p,-i_q}], \qquad p, q = 1, \dots, 2k.$$

Set

$$D_k = \sum_I \frac{\operatorname{sgn}(i_1 \dots i_{2k})}{\alpha_{-n}! \dots \alpha_n!} \cdot \Psi_I \Psi_{I^*}, \qquad I^* = \{-i_{2k}, \dots, -i_1\}.$$

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Corollary.

$$\Big(\frac{\mathcal{C}(u)}{(u + \rho_{-n}) \ldots (u + \rho_n)} \Big)^{-1} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k D_k}{(u^2 - (n+1)^2) \ldots (u^2 - (n+k)^2)}.$$