Affine center at the critical level and quantum Mishchenko–Fomenko subalgebras

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  - Generators of quantum Mishchenko–Fomenko subalgebras.

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The subalgebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \subset \mathrm{S}(\mathfrak{g})$ coincides with

the Poisson center of $\mathrm{S}(\mathfrak{g})$.

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Denote by $\overline{\mathcal{A}}_\mu$ the subalgebra of $\mathrm{S}(\mathfrak{g})$ generated by all the μ-shifts $P_{(i)}$ associated with all invariants $P \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$.

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- Moreover, $\overline{\mathcal{A}}_\mu$ is a maximal Poisson commutative subalgebra of $\mathrm{S}(\mathfrak{g})$ [D. Panyushev and O. Yakimova 2008].

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Is it possible to quantize the subalgebra $\overline{\mathcal{A}}_\mu$ of $\mathrm{S}(\mathfrak{g})$?

We would like to find a commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{g})$ (together with its free generators) such that $\mathrm{gr}\,\mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$.

A solution via Yangian approach: classical types with regular semisimple μ [M. Nazarov and G. Olshanski 1996].

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The uniqueness of the solution in this case is established [A. Tarasov 2003].

Affine center at the critical level

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Note that $T = -\dfrac{d}{dt}$ is a derivation of $\widehat{\mathfrak{g}}$.

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^{\vee}$.

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$\mathfrak{z}(\widehat{\mathfrak{g}})$ is a T-invariant commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

As a differential algebra, $\mathfrak{z}(\widehat{\mathfrak{g}})$ possesses free generators

$S_1, \dots, S_n$ (a complete set of Segal–Sugawara vectors),

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Equivalently, $\mathfrak{z}(\widehat{\mathfrak{g}})$ can be defined as the centralizer in

$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ of the canonical Segal–Sugawara vector

$$S = \sum_{i=1}^{l} X_i[-1]^2,$$

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[L. Rybnikov 2008; also O. Yakimova 2019].

Connection with Casimir elements

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For any nonzero $z \in \mathbb{C}$, the images of $S_1, \dots, S_n$ under the evaluation homomorphism

$$\varrho_z : \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g}), \qquad X[r] \mapsto X z^r,$$

are free generators of the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$.

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Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of m of length $\ell = \ell(\lambda)$, so that $\lambda_1 \geqslant \dots \geqslant \lambda_\ell > 0$ and $\lambda_1 + \dots + \lambda_\ell = m$.

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$$c_\lambda = \frac{m!}{1^{\alpha_1}\alpha_1!\, 2^{\alpha_2}\alpha_2! \dots m^{\alpha_m}\alpha_m!}$$

for $\lambda = (1^{\alpha_1} 2^{\alpha_2} \dots m^{\alpha_m})$.

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$$D(\lambda) = \frac{1}{\ell!} \sum_{i_1,\dots,i_\ell=1}^{N} \sum_{\sigma\in\mathfrak{S}_\ell} \operatorname{sgn}\sigma \cdot E_{i_{\sigma(1)}\, i_1}[-\lambda_1] \dots E_{i_{\sigma(\ell)}\, i_\ell}[-\lambda_\ell]$$

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$$\phi_m = \sum_{\lambda \vdash m} \binom{N}{\ell}^{-1} c_\lambda \, D(\lambda)$$

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Moreover, each family $\phi_1, \dots, \phi_N$ and $\psi_1, \dots, \psi_N$ is a complete set of Segal–Sugawara vectors for $\mathfrak{gl}_N$.

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form a complete set of Segal–Sugawara vectors for $\mathfrak{gl}_N$ [A. Chervov and D. Talalaev 2006; A. Chervov and M. 2009].

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Verify that $\phi_m^\circ = \binom{N}{m}\phi_m$ for all m.

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Define the orthogonal Lie algebra $\mathfrak{o}_N$ with $N = 2n$ and $N = 2n + 1$ and symplectic Lie algebra $\mathfrak{sp}_N$ with $N = 2n$ as subalgebras of $\mathfrak{gl}_N$ spanned by the elements F_{ij},

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We use the involution $i \mapsto i' = N - i + 1$ on the set $\{1, \dots, N\}$, and in the symplectic case we set

$$\varepsilon_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n \\ -1 & \text{for} \quad i = n + 1, \dots, 2n. \end{cases}$$

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For $\mathfrak{g} = \mathfrak{o}_N$ the symmetrized λ-permanent is defined by

$$P(\lambda) = \frac{1}{\ell!} \sum_{i_1,\dots,i_\ell=1}^{N} \sum_{\sigma\in\mathfrak{S}_\ell} F_{i_{\sigma(1)}\, i_1}[-\lambda_1] \dots F_{i_{\sigma(\ell)}\, i_\ell}[-\lambda_\ell].$$

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Both $D(\lambda)$ and $P(\lambda)$ are zero unless $\ell(\lambda)$ is even.

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$$\phi_m = \sum_{\lambda \vdash m} \binom{2n+1}{\ell}^{-1} c_\lambda \, D(\lambda) \qquad \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n},$$

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belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$.

In the case $\mathfrak{g} = \mathfrak{o}_{2n}$, the Pfaffian

$$\operatorname{Pf} F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1)\,\sigma(2)'}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1]$$

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Theorem (2021). The family $\phi_2, \phi_4, \dots, \phi_{2n}$ is a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathfrak{g} = \mathfrak{o}_{2n+1}$,

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Theorem (2021). The family $\phi_2, \phi_4, \dots, \phi_{2n}$ is a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathfrak{g} = \mathfrak{o}_{2n+1}$, and $\phi_2, \phi_4, \dots, \phi_{2n-2}, \mathrm{Pf}\, F[-1]$ is a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{o}_{2n}$.

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The linear map $\varpi : \mathrm{S}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ defined by

$$\varpi : X_1 \dots X_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} X_{\sigma(1)} \dots X_{\sigma(n)}, \qquad X_i \in \mathfrak{g}.$$

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Hence we have a vector space isomorphism

$$\varpi : \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{Z}(\mathfrak{g}).$$

Casimir elements in type A

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Consider the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & \dots & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}$$

with entries in the symmetric algebra $\mathrm{S}(\mathfrak{gl}_N)$.

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We have

$$\mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[\Phi_1, \dots, \Phi_N] = \mathbb{C}[\Psi_1, \dots, \Psi_N].$$

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$$\det(u1 + E) = u^N + \Phi_1 u^{N-1} + \cdots + \Phi_N$$

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We have

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This implies

$$\mathrm{Z}(\mathfrak{gl}_N) = \mathbb{C}\big[\varpi(\Phi_1), \dots, \varpi(\Phi_N)\big] = \mathbb{C}\big[\varpi(\Psi_1), \dots, \varpi(\Psi_N)\big].$$

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They act by scalar multiplication in $L(\lambda_1, \dots, \lambda_N)$.

Elementary shifted symmetric polynomials:

$$e_m^*(\lambda_1, \ldots, \lambda_N) = \sum_{i_1 < \cdots < i_m} \lambda_{i_1}(\lambda_{i_2} - 1) \ldots (\lambda_{i_m} - m + 1).$$

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Casimir elements in types B, C, D

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Consider the matrix

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with entries in $\mathrm{S}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{o}_N$ or $\mathfrak{g} = \mathfrak{sp}_N$.

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We let $N = 2n + 1$ for type B, and $N = 2n$ for types C and D.

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The symmetrized invariants act by scalar multiplication in the irreducible highest weight $\mathfrak{g}$-modules $L(\lambda_1, \dots, \lambda_n)$.

Theorem (2021). (i) For $\mathfrak{g} = \mathfrak{sp}_{2n}$ we have

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Remark. If m is odd, then the elements Φ_m, Ψ_m are understood as equal to zero.

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Given $\mu \in \mathfrak{g}^*$ and nonzero $z \in \mathbb{C}$, consider the homomorphism

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[L. Rybnikov 2006].

If $S \in \mathfrak{z}(\widehat{\mathfrak{g}})$ is of degree d, define $S_{(a)} \in \mathrm{U}(\mathfrak{g})$ by

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Conjecture [FFTL 2010]. $\mathrm{gr}\ \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$ for all $\mu \in \mathfrak{g}^*$.

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Moreover, if μ is regular, then the non-constant coefficients of each family of polynomials

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[A. Tarasov 2000, 2003; O. Yakimova and M. 2017].

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Introduce another Young diagram by

$$\Pi = \alpha^{(1)} + \dots + \alpha^{(r)},$$

the sum is taken by rows.

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$$\Gamma \quad = \quad \begin{array}{ccccc} \Phi_{N\,N-1} & \Phi_{N\,N-2} & \ldots & \Phi_{N\,1} & \Phi_{N\,0} \\ \Phi_{N-1\,N-2} & \Phi_{N-1\,N-3} & \ldots & \Phi_{N-1\,0} & \\ \ldots & \ldots & \ldots & & \\ \Phi_{2\,1} & \Phi_{2\,0} & & & \\ \Phi_{1\,0} & & & & \end{array}$$

The elements $\Phi_{\ell k}$ corresponding to the boxes of the skew diagram Γ/ϱ are free generators of the subalgebra $\mathcal{A}_\mu$.

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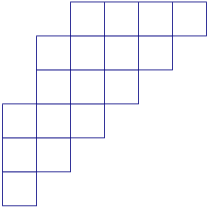

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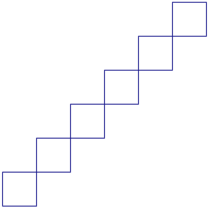

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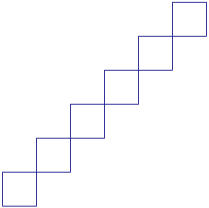
Example. If μ is a scalar matrix then $\varrho = (N-1, N-2, \dots, 1)$.

The skew diagram Γ/ϱ is



Thus $\mathcal{A}_\mu = \mathbb{C}\left[\Phi_{1\,0}, \dots, \Phi_{N\,0}\right] = \mathrm{Z}(\mathfrak{gl}_N)$.