Lecture 2

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- Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is the associative algebra over $\mathbb{C}$ with generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where $i,j=1,\dots,N$, and the defining relations

$$\big[t_{ij}^{(r)},t_{kl}^{(s)}\big]=\sum_{a=1}^{\min\{r,s\}}\Big(t_{kj}^{(a-1)}t_{il}^{(r+s-a)}-t_{kj}^{(r+s-a)}t_{il}^{(a-1)}\Big),$$

where $t_{ij}^{(0)}=\delta_{ij}$.

- Equivalently,

$$(u-v)\left[t_{ij}(u),t_{kl}(v)\right]=t_{kj}(u)\,t_{il}(v)-t_{kj}(v)\,t_{il}(u).$$

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$$(u-v)\left[t_{ij}(u), t_{kl}(v)\right] = t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u).$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]].$$

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$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]].$$

- Also, the defining relations take the form of RTT-relation

$$R(u - v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u - v).$$

- We have the evaluation homomorphism

$$\mathrm{ev} : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N), \qquad t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1},$$

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- and the embedding

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- In particular,

$$\big[E_{ij}, t_{kl}(u)\big] = \delta_{kj}\, t_{il}(u) - \delta_{il}\, t_{kj}(u).$$

- We have the automorphisms

$$T(u) \mapsto f(u)\, T(u),$$

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- and anti-automorphisms

$$\sigma_N : T(u) \mapsto T(-u),$$

$$t : T(u) \mapsto T^{t}(u),$$

$$\mathrm{S} : T(u) \mapsto T^{-1}(u).$$

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- The Poincaré–Birkhoff–Witt theorem holds:

  Given an arbitrary linear order on the set of generators $t_{ij}^{(r)}$, any element of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ can be uniquely written as a linear combination of ordered monomials in these generators.

- We noted in the proof that the associated graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N)$ is the algebra of polynomials in infinitely many variables $\bar{t}_{ij}^{(r)}$.

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For any $a \in A$ we have

$$\begin{aligned} a \cdot (v \otimes w) &= \Delta(a)(v \otimes w) \\ &= \Big(\sum a_{(1)} \otimes a_{(2)}\Big)(v \otimes w) = \sum a_{(1)} v \otimes a_{(2)} w. \end{aligned}$$

for any $v \in V$ and $w \in W$.

Example. If V and W are representations of a Lie algebra $\mathfrak{g}$, Example. If V and W are representations of a Lie algebra $\mathfrak{g}$, then $V \otimes W$ is also a representation with the action

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw, \qquad X \in \mathfrak{g}.$$

Example. If V and W are representations of a Lie algebra $\mathfrak{g}$, then $V \otimes W$ is also a representation with the action

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw, \qquad X \in \mathfrak{g}.$$

In fact, $\mathrm{U}(\mathfrak{g})$ is a Hopf algebra with the coproduct

$$\Delta : X \mapsto X \otimes 1 + 1 \otimes X, \qquad X \in \mathfrak{g},$$

the antipode $\mathrm{S} : X \mapsto -X$ and the counit $\varepsilon : X \to 0$.

Theorem. The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is a Hopf algebra with coproduct

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{N} t_{ik}(u) \otimes t_{kj}(u),$$

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Proof. We will verify the main axiom that

$$\Delta : \mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N)$$

is an algebra homomorphism.

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with the notation

$$T_{[1]}(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \otimes 1 \quad \text{and} \quad T_{[2]}(u) = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes t_{ij}(u).$$

We need to show that $\Delta(T(u))$ satisfies the RTT relation

$$R(u-v)\,T_{1[1]}(u)\,T_{1[2]}(u)\,T_{2[1]}(v)\,T_{2[2]}(v) = T_{2[1]}(v)\,T_{2[2]}(v)\,T_{1[1]}(u)\,T_{1[2]}(u)\,R(u-v)$$

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$$R(u-v)\,T_{1[1]}(u)\,T_{1[2]}(u)\,T_{2[1]}(v)\,T_{2[2]}(v) = T_{2[1]}(v)\,T_{2[2]}(v)\,T_{1[1]}(u)\,T_{1[2]}(u)\,R(u-v)$$

in the algebra

$$\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{gl}_N).$$

This follows by the *RTT* relation for $T(u)$ and by the observation that the elements $T_{1[2]}(u)$ and $T_{2[1]}(v)$ commute, as well as the elements $T_{1[1]}(u)$ and $T_{2[2]}(v)$. $\square$

Classical limit

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Introduce new generators of $\mathrm{Y}(\mathfrak{gl}_N)$ by setting

$$\widetilde{t}_{ij}^{(r)} = h^{r-1}\, t_{ij}^{(r)}, \qquad r \geqslant 1,$$

where h is a nonzero complex number.

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where h is a nonzero complex number.

The defining relations of the algebra $\mathrm{Y}_h(\mathfrak{gl}_N)$ take the form

$$\big[\widetilde{t}_{ij}^{(r)}, \widetilde{t}_{kl}^{(s)}\big] = \delta_{kj}\, \widetilde{t}_{il}^{(r+s-1)} - \delta_{il}\, \widetilde{t}_{kj}^{(r+s-1)}$$

$$+ h \sum_{a=2}^{\min\{r,s\}} \Big(\widetilde{t}_{kj}^{(a-1)} \widetilde{t}_{il}^{(r+s-a)} - \widetilde{t}_{kj}^{(r+s-a)} \widetilde{t}_{il}^{(a-1)}\Big).$$

Note that $\mathrm{Y}_0(\mathfrak{gl}_N) = \mathrm{U}(\mathfrak{gl}_N[x])$ via the identification

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For the coproduct we have

$$\Delta : \widetilde{t}_{ij}^{(r)} \mapsto \widetilde{t}_{ij}^{(r)} \otimes 1 + 1 \otimes \widetilde{t}_{ij}^{(r)} + h \sum_{k=1}^{N} \sum_{s=1}^{r-1} \widetilde{t}_{ik}^{(s)} \otimes \widetilde{t}_{kj}^{(r-s)}.$$

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Hence the Yangian is a deformation of $\mathrm{U}(\mathfrak{gl}_N[x])$

in the class of Hopf algebras.

Let Δ' be the opposite coproduct on $\mathrm{Y}(\mathfrak{gl}_N)$,

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In the algebra $\mathrm{Y}_h(\mathfrak{gl}_N)$ we have

$$\frac{\Delta\big(\widetilde{t}_{ij}^{\,(r)}\big) - \Delta'\big(\widetilde{t}_{ij}^{\,(r)}\big)}{h} = \sum_{k=1}^{N} \sum_{s=1}^{r-1} \widetilde{t}_{ik}^{\,(s)} \otimes \widetilde{t}_{kj}^{\,(r-s)} - \sum_{k=1}^{N} \sum_{s=1}^{r-1} \widetilde{t}_{kj}^{\,(s)} \otimes \widetilde{t}_{ik}^{\,(r-s)}.$$

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For $h = 0$ this coincides with the image $\delta\big(\widetilde{t}_{ij}^{\,(r)}\big) = \delta\big(E_{ij}\, x^{r-1}\big)$ of the cocommutator δ on $\mathfrak{gl}_N[x]$.

The cocommutator is the map

$$\delta : \mathfrak{gl}_N[x] \mapsto \mathfrak{gl}_N[x] \otimes \mathfrak{gl}_N[x] \cong (\mathfrak{gl}_N \otimes \mathfrak{gl}_N)[x, y],$$

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Remark. This is the starting point to define the Yangian $\mathrm{Y}(\mathfrak{a})$ associated with a simple Lie algebra $\mathfrak{a}$.

Quantum determinant

Quantum determinant

Direct definition. The quantum determinant of the matrix

$$T(u) = \begin{bmatrix} t_{11}(u) & t_{12}(u) & \dots & t_{1N}(u) \\ t_{21}(u) & t_{22}(u) & \dots & t_{2N}(u) \\ \dots & \dots & \dots & \dots \\ t_{N1}(u) & t_{N2}(u) & \dots & t_{NN}(u) \end{bmatrix}$$

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is defined as the series

$$\operatorname{qdet} T(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1)\,1}(u) \dots t_{p(N)\,N}(u - N + 1).$$

Exercise. (1) Show that for $N = 2$ we have

$$\begin{aligned}
\operatorname{qdet} T(u) &= t_{11}(u)\, t_{22}(u-1) - t_{21}(u)\, t_{12}(u-1) \\
&= t_{22}(u)\, t_{11}(u-1) - t_{12}(u)\, t_{21}(u-1) \\
&= t_{11}(u-1)\, t_{22}(u) - t_{12}(u-1)\, t_{21}(u) \\
&= t_{22}(u-1)\, t_{11}(u) - t_{21}(u-1)\, t_{12}(u).
\end{aligned}$$

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(2) Prove that the coefficients of the series $\operatorname{qdet} T(u)$ belong to the center of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$.

R-matrix construction of $\mathrm{qdet}\, T(u)$

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For any positive integer m consider the algebra

$$(\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathrm{Y}(\mathfrak{gl}_N).$$

For any $a \in \{1, \dots, m\}$ denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the a-th copy of the algebra $\mathrm{End}\,\mathbb{C}^N$ in the tensor product algebra.

R-matrix construction of $\operatorname{qdet} T(u)$

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For any $a \in \{1, \dots, m\}$ denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the a-th copy of the algebra $\operatorname{End} \mathbb{C}^N$ in the tensor product algebra. That is, $T_a(u)$ is a formal power series in u^{-1} given by

$$T_a(u) = \sum_{i,j=1}^{N} 1^{\otimes (a-1)} \otimes e_{ij} \otimes 1^{\otimes (m-a)} \otimes t_{ij}(u),$$

where 1 is the identity matrix.

If

$$C = \sum_{i,j,k,l=1}^{N} c_{ijkl}\, e_{ij} \otimes e_{kl} \in \operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N,$$

then for any two indices $a, b \in \{1, \ldots, m\}$ such that $a < b$, define the element C_{ab} of the algebra $(\operatorname{End}\mathbb{C}^N)^{\otimes m}$ by

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$$C_{ab}=\sum_{i,j,k,l=1}^{N} c_{ijkl}\, 1^{\otimes(a-1)}\otimes e_{ij}\otimes 1^{\otimes(b-a-1)}\otimes e_{kl}\otimes 1^{\otimes(m-b)}.$$

The tensor factors e_{ij} and e_{kl} belong to the a-th and b-th copies of $\operatorname{End}\mathbb{C}^N$, respectively.

For any $m \geqslant 2$ introduce the rational function $R(u_1, \dots, u_m)$ with values in the tensor product algebra $(\operatorname{End} \mathbb{C}^N)^{\otimes m}$ by

$$R(u_1, \dots, u_m) = (R_{m-1,m})(R_{m-2,m} R_{m-2,m-1}) \dots (R_{1m} \dots R_{12}),$$

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where $u_1, \dots, u_m$ are independent complex variables and we abbreviate

$$R_{ab} = R_{ab}(u_a - u_b) = 1 - \frac{P_{ab}}{u_a - u_b}.$$

Using the Yang–Baxter equation, we also get

$$R(u_1, \dots, u_m) = (R_{12} \dots R_{1m}) \dots (R_{m-2,m-1} R_{m-2,m})(R_{m-1,m})$$

$$= \prod_{a<b} \Big(1 - \frac{P_{ab}}{u_a - u_b}\Big)$$

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We used the observation that R_{ab} and R_{cd} commute, if the indices a, b, c, d are all distinct.

Applying the RTT relation repeatedly, we come

to the fundamental relation for the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$,

$$R(u_1,\dots,u_m)\,T_1(u_1)\dots T_m(u_m) = T_m(u_m)\dots T_1(u_1)\,R(u_1,\dots,u_m).$$

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$$=R_{23}\,T_2\,T_3\,T_1\,R_{13}\,R_{12}=T_3\,T_2\,T_1\,R_{23}\,R_{13}\,R_{12}.$$

Lemma. We have

$$R(u, u-1, \ldots, u-m+1) = A_m,$$

Lemma. We have

$$R(u, u-1, \ldots, u-m+1) = A_m,$$

the image of the anti-symmetrizer

$$a_m = \sum_{p \in \mathfrak{S}_m} \operatorname{sgn} p \cdot p \in \mathbb{C}[\mathfrak{S}_m].$$

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Remark. This is a particular case of the fusion procedure going back to [A. Jucys 1966].

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By the induction hypothesis we have for $m > 2$

$$R(u, u-1, \dots, u-m+1) = (R_{12} \dots R_{1m}) \, A'_{m-1},$$

where A'_{m-1} denotes the anti-symmetrizer over $\{2, \dots, m\}$.

Calculate

$$(R_{12}\dots R_{1m})\,A'_{m-1} = (1-P_{12})\left(1-\frac{P_{13}}{2}\right)\dots\left(1-\frac{P_{1m}}{m-1}\right)A'_{m-1}.$$

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Expand the product and write

$$P_{1i_1}\,P_{1i_2}\dots P_{1i_k}\,A'_{m-1} = P_{1i_k}\,P_{i_k i_1}\dots P_{i_k i_{k-1}}\,A'_{m-1}$$

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which equals

$$(-1)^{k-1}\,P_{1i_k}\,A'_{m-1}.$$

Therefore,

$$(R_{12} \ldots R_{1m})\, A'_{m-1} = (1 - \alpha_2\, P_{12} - \cdots - \alpha_m\, P_{1m})\, A'_{m-1}$$

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Finally, note that

$$(1-P_{12}-\cdots-P_{1m})\,A'_{m-1} = A_m. \qquad \square$$

Hence, by the Lemma and the fundamental relation

$$R(u_1, \dots, u_m)\, T_1(u_1) \dots T_m(u_m) = T_m(u_m) \dots T_1(u_1)\, R(u_1, \dots, u_m),$$

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Observe that if $m = N$, then the operator A_N on $(\mathbb{C}^N)^{\otimes N}$ has a one-dimensional image.

We have

$$A_N(e_{i_1} \otimes \ldots \otimes e_{i_N}) = 0$$

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are equal to $A_N d(u)$ for a series $d(u)$ with coefficients in $\mathrm{Y}(\mathfrak{gl}_N)$, Since $A_N^2 = N\,A_N$, both sides of the relation

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$$d(u) = 1 + d_1\,u^{-1} + d_2\,u^{-2} + \dots.$$

Definition. The series $d(u)$ is called the quantum determinant of the matrix $T(u)$ and denoted $\mathrm{qdet}\,T(u)$.

Proposition. For any permutation $q \in \mathfrak{S}_N$ we have

$$\begin{aligned}\operatorname{qdet} T(u) &= \operatorname{sgn} q \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1)\, q(1)}(u) \dots t_{p(N)\, q(N)}(u-N+1) \\ &= \operatorname{sgn} q \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{q(1)\, p(1)}(u-N+1) \dots t_{q(N)\, p(N)}(u).\end{aligned}$$

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In particular,

$$\begin{aligned}\operatorname{qdet} T(u) &= \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1)\,1}(u) \dots t_{p(N)\,N}(u - N + 1) \\ &= \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{1\,p(1)}(u - N + 1) \dots t_{N\,p(N)}(u).\end{aligned}$$

Proof. By definition,

$$A_N\, T_1(u) \dots T_N(u - N + 1) = A_N\, \mathrm{qdet}\, T(u).$$

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For the left hand side we get

$$A_N \sum_{i_1,\dots,i_N} t_{i_1\, q(1)}(u) \dots t_{i_N\, q(N)}(u - N + 1)(e_{i_1} \otimes \dots \otimes e_{i_N}),$$

proving the first formula. □

Assuming that $m \leqslant N$ is arbitrary, define

the $m \times m$ quantum minors $t^{\, a_1 \dots a_m}_{\, b_1 \dots b_m}(u)$ so that each side of

$$A_m \, T_1(u) \dots T_m(u - m + 1) = T_m(u - m + 1) \dots T_1(u) \, A_m$$

equals

$$\sum e_{a_1 b_1} \otimes \dots \otimes e_{a_m b_m} \otimes t^{\, a_1 \dots a_m}_{\, b_1 \dots b_m}(u),$$

summed over the indices $a_i, b_i \in \{1, \dots, N\}$.

Skew-symmetry properties: for any $p \in \mathfrak{S}_m$ we have

$$t^{\,a_{p(1)}\dots a_{p(m)}}_{\,b_1\dots b_m}(u) = \operatorname{sgn} p \cdot t^{\,a_1\dots a_m}_{\,b_1\dots b_m}(u)$$

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As with the quantum determinant, we have

$$\begin{aligned} t^{\,a_1\dots a_m}_{\,b_1\dots b_m}(u) &= \sum_{p\in\mathfrak{S}_m} \operatorname{sgn} p \cdot t_{a_{p(1)}b_1}(u)\dots t_{a_{p(m)}b_m}(u-m+1) \\ &= \sum_{p\in\mathfrak{S}_m} \operatorname{sgn} p \cdot t_{a_1 b_{p(1)}}(u-m+1)\dots t_{a_m b_{p(m)}}(u). \end{aligned}$$

Proposition. The images of quantum minors under the coproduct are given by

$$\Delta\big(t^{a_1\dots a_m}_{b_1\dots b_m}(u)\big) = \sum_{c_1<\dots<c_m} t^{a_1\dots a_m}_{c_1\dots c_m}(u)\otimes t^{c_1\dots c_m}_{b_1\dots b_m}(u),$$

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to get the element of the algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]^{\otimes 2}$:

$$A_m T_{1[1]}(u)T_{1[2]}(u)\dots T_{m[1]}(u-m+1)T_{m[2]}(u-m+1).$$

Write $A_m = \frac{1}{m!} A_m^2$, and starting from the expression

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apply the fundamental relation to get

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Taking the matrix elements gives the formula. □

Corollary. We have

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Proof. Since

$$\operatorname{qdet} T(u) = t^{\,1\dots N}_{\,1\dots N}(u),$$

this follows from the proposition. □

Center of $Y(\mathfrak{gl}_N)$

Center of $\mathrm{Y}(\mathfrak{gl}_N)$

Proposition. We have the relations

$$(u-v)\left[\, t_{kl}(u), t^{\,a_1\dots\, a_m}_{\,b_1\dots\, b_m}(v)\right] = \sum_{i=1}^{m} t_{a_i l}(u)\, t^{\,a_1\dots\, k\,\dots\, a_m}_{\,b_1\ \dots\ \ b_m}(v) - \sum_{i=1}^{m} t^{\,a_1\ \dots\ \ a_m}_{\,b_1\dots\, l\,\dots\, b_m}(v)\, t_{kb_i}(u)$$

where the indices k and l in the quantum minors replace a_i and b_i, respectively.

Proof. The fundamental relation yields

$$R(u, v, v-1, \dots, v-m+1)\, T_0(u)\, T_1(v) \dots T_m(v-m+1)$$

$$= T_m(v-m+1) \dots T_1(v)\, T_0(u)\, R(u, v, v-1, \dots, v-m+1).$$

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We have

$$R(u,v,v-1,\dots,v-m+1)=A_m\,R_{0m}(u-v+m-1)\dots R_{01}(u-v).$$

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$$R(u, v, v-1, \dots, v-m+1) = A_m \Big(1 - \frac{1}{u-v}(P_{01} + \dots + P_{0m})\Big).$$

Apply both sides to the vector $e_l \otimes e_{b_1} \otimes \dots \otimes e_{b_m}$ and compare the coefficients of the vector $e_k \otimes e_{a_1} \otimes \dots \otimes e_{a_m}$. □

Corollary. The coefficients $d_1, d_2, \dots$ of the series

$$\operatorname{qdet} T(u) = 1 + d_1\, u^{-1} + d_2\, u^{-2} + \dots$$

belong to the center $\mathrm{ZY}(\mathfrak{gl}_N)$ of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$.

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and so

$$\big[\, t_{kl}(u), \operatorname{qdet} T(v)\big] = 0,$$

as we wanted. $\square$

Theorem. The coefficients $d_1, d_2, \dots$ are algebraically independent and generate the center $\mathrm{ZY}(\mathfrak{gl}_N)$.

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Let $\widetilde{t}_{ij}^{(r)}$ be the image of $t_{ij}^{(r)}$ in the $(r-1)$ component of $\mathrm{gr}' \, \mathrm{Y}(\mathfrak{gl}_N)$.

Recalling the defining relations

$$\left[t_{ij}^{(r)}, t_{kl}^{(s)}\right] = \sum_{a=1}^{\min\{r,s\}} \left(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\right),$$

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Hence, the mapping

$$\mathrm{U}(\mathfrak{gl}_N[x]) \to \mathrm{gr}'\,\mathrm{Y}(\mathfrak{gl}_N), \qquad E_{ij}\, x^{r-1} \mapsto \widetilde{t}_{ij}^{(r)}$$

is an isomorphism.

Using the formula

$$\operatorname{qdet} T(u) = \sum_{p\in\mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1)\,1}(u)\dots t_{p(N)\,N}(u-N+1),$$

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we derive that

$$d_r = t_{11}^{(r)} + \dots + t_{NN}^{(r)} \quad + \quad \text{terms of smaller degree}.$$

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Hence, the image of d_r in the $(r-1)$-th component of $\operatorname{gr}' \mathrm{Y}(\mathfrak{gl}_N)$ can be identified with $I x^{r-1}$ where $I = E_{11} + \dots + E_{NN}$.

Using the formula

$$\operatorname{qdet} T(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1)\,1}(u) \dots t_{p(N)\,N}(u - N + 1),$$

we derive that

$$d_r = t_{11}^{(r)} + \dots + t_{NN}^{(r)} \quad + \quad \text{terms of smaller degree}.$$

Hence, the image of d_r in the $(r-1)$-th component of $\operatorname{gr}' \mathrm{Y}(\mathfrak{gl}_N)$ can be identified with $I x^{r-1}$ where $I = E_{11} + \dots + E_{NN}$.

However, the elements $I x^{r-1}$ are algebraically independent generators of the center of $\mathrm{U}(\mathfrak{gl}_N[x])$. $\square$