The Work Performed by a Transformation Semigroup

James East*

School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia

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Abstract

A partial transformation α on the finite set $\{1,\ldots,n\}$ moves an element i of its domain a distance of $|i - i\alpha|$ units. The *work* $w(\alpha)$ performed by α is defined to be the sum of all of these distances. In this article we derive a formula for the total work $w(S) = \sum_{\alpha \in S} w(\alpha)$ performed by a subset S of the partial transformation semigroup $\mathcal{PT}_n$. We then obtain explicit formulae for $w(S)$ when S is one of seven important subsemigroups of $\mathcal{PT}_n$: the partial transformation semigroup, the (full) transformation semigroup, the symmetric group, and the symmetric inverse semigroup, as well as their order-preserving submonoids. Each of these formulae gives rise to a formula for the average work $\overline{w}(S) = \frac{1}{|S|} w(S)$ performed by an element of S.

Keywords: Transformation semigroup, work.

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1 Introduction

Fix a positive integer n and write $\mathbf{n} = \{1,\ldots,n\}$. The *partial transformation semigroup on* $\mathbf{n}$, denoted $\mathcal{PT}_n$, is the semigroup of all functions (transformations) between subsets of $\mathbf{n}$. Such a function is called a *partial transformation on* $\mathbf{n}$. If $\alpha \in \mathcal{PT}_n$ we will write $\mathrm{dom}(\alpha)$ and $\mathrm{im}(\alpha)$ for the domain and image of α (respectively). The semigroup operation in $\mathcal{PT}_n$ is composition, although the semigroup structure of $\mathcal{PT}_n$ will not play any role in our investigations.

Let $\alpha \in \mathcal{PT}_n$ and $i \in \mathbf{n}$. We define the *work performed by* α *to move* i to be

$$w_i(\alpha) = \begin{cases} |i - i\alpha| & \text{if } i \in \mathrm{dom}(\alpha) \\ 0 & \text{otherwise,} \end{cases}$$

and we define the *(total) work performed by* α to be

$$w(\alpha) = \sum_{i \in \mathbf{n}} w_i(\alpha).$$

*jamese@maths.usyd.edu.au

If $S \subseteq \mathcal{PT}_n$ then we define the *(total) work performed by* S to be

$$w(S) = \sum_{\alpha \in S} w(\alpha).$$

We also write

$$\overline{w}(S) = \frac{1}{|S|} w(S)$$

for the *average* work performed by an element of S. The purpose of this article is to derive explicit formulae for $w(S)$ and $\overline{w}(S)$ when S is either $\mathcal{PT}_n$ or one of its subsemigroups

- $\mathcal{T}_n = \{\alpha \in \mathcal{PT}_n \mid \mathrm{dom}(\alpha) = \mathbf{n}\}$,
- $\mathcal{I}_n = \{\alpha \in \mathcal{PT}_n \mid \alpha \text{ is injective}\}$,
- $\mathcal{S}_n = \{\alpha \in \mathcal{T}_n \mid \alpha \text{ is injective}\}$,
- $\mathcal{O}_n = \{\alpha \in \mathcal{T}_n \mid \alpha \text{ is order-preserving}\}$,
- $\mathcal{PO}_n = \{\alpha \in \mathcal{PT}_n \mid \alpha \text{ is order-preserving}\}$, or
- $\mathcal{POI}_n = \{\alpha \in \mathcal{I}_n \mid \alpha \text{ is order-preserving}\}$.

The subsemigroups $\mathcal{T}_n$, $\mathcal{I}_n$, and $\mathcal{S}_n$ are known as the transformation semigroup, the symmetric inverse semigroup, and the symmetric group on $\mathbf{n}$ (respectively). A (partial) transformation $\alpha \in \mathcal{PT}_n$ is said to be *order-preserving* if $i\alpha < j\alpha$ whenever $i, j \in \mathrm{dom}(\alpha)$ and $i < j$.

All numbers in this article are assumed to be integers. Thus a statement such as "let $1 \leq i \leq 5$" should be read as "let i be an integer such that $1 \leq i \leq 5$". It will also be convenient to interpret a binomial coefficient $\binom{p}{q}$ to be 0 if $p < q$.

2 Example Calculations

Before moving on, let us calculate $w(S)$ and $\overline{w}(S)$ for a selection of semigroups $S \subseteq \mathcal{PT}_3$.

Example 1 (The symmetric group $\mathcal{S}_3$) The elements of the symmetric group $\mathcal{S}_3$ are pictured in Figure 1 below. For the moment, the reader should not be concerned with our use of colours.

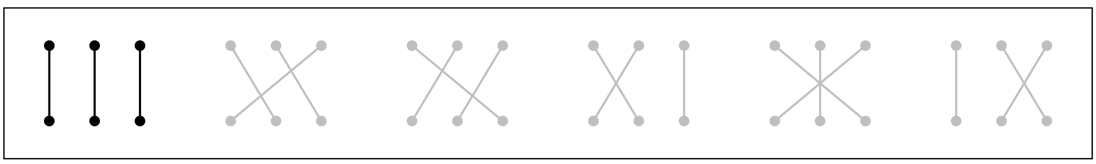

Figure 1: The elements of $\mathcal{S}_3$.

The pictures should be interpreted so that, for example, the right-most diagram represents the permutation which fixes 1 and interchanges 2 and 3. Adding up the work performed by each permutation (from left to right as arranged in Figure 1) gives

$$w(\mathcal{S}_3) = 0 + 4 + 4 + 2 + 4 + 2 = 16.$$

The average work performed by a permutation in $\mathcal{S}_3$ is $\overline{w}(\mathcal{S}_3) = \frac{16}{6} = 2\frac{2}{3}$.

Example 2 (The transformation semigroup $\mathcal{T}_3$) The transformation semigroup $\mathcal{T}_3$ consists of the six permutations pictured in Figure 1 together with the 21 non-invertible maps pictured in Figure 2 below. (The maps are arranged in an *egg-box* diagram; elements in the same row or column of a box have the same domain or image respectively.)

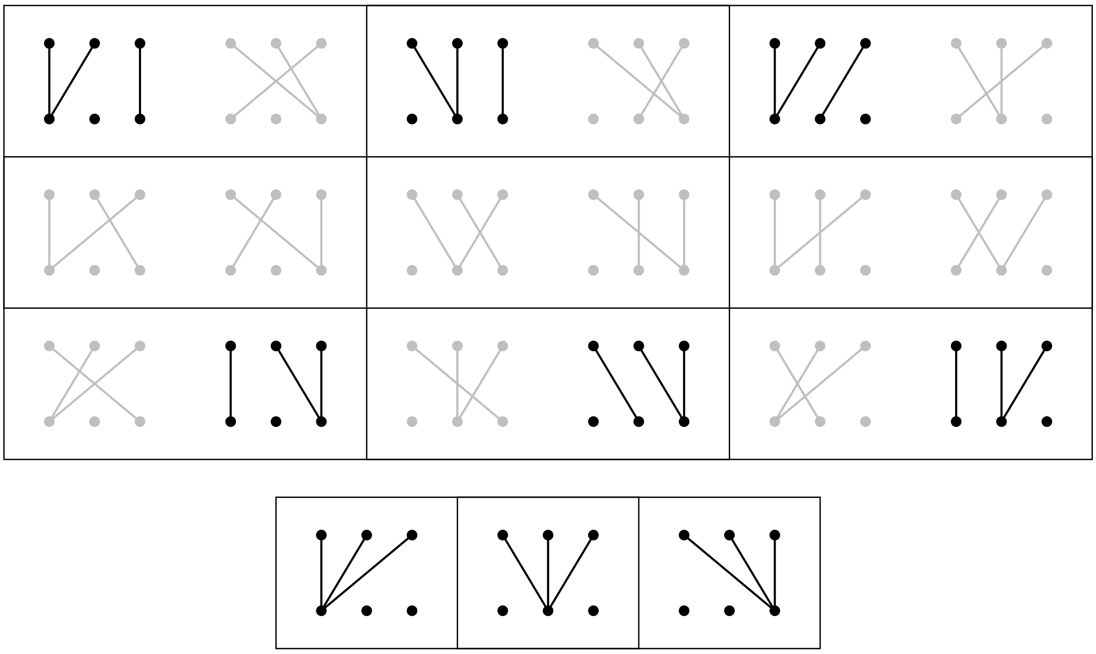

Figure 2: The remaining elements of $\mathcal{T}_3$.

One may then calculate that the total work performed by these 21 maps is 56 so that

$$w(\mathcal{T}_3) = 16 + 56 = 72 \qquad \text{and} \qquad \overline{w}(\mathcal{T}_3) = \tfrac{72}{27} = 2\tfrac{2}{3}.$$

The observant reader will have noticed that $\overline{w}(\mathcal{S}_3) = \overline{w}(\mathcal{T}_3)$. In fact, this is not a conincidence, but rather an special case of a more general phenomenon; in Section 4 we will see that $\overline{w}(\mathcal{S}_n) = \overline{w}(\mathcal{T}_n)$ for all n.

Example 3 (The semigroup $\mathcal{O}_3$) The semigroup $\mathcal{O}_3$ consists of all order-preserving transformations on $\{1, 2, 3\}$; these maps appear as the black diagrams in Figures 1 and 2. One may compute that

$$w(\mathcal{O}_3) = 16 \qquad \text{and} \qquad \overline{w}(\mathcal{O}_3) = \tfrac{16}{10} = 1\tfrac{3}{5}.$$

Example 4 (The semigroup $\mathcal{POI}_3$) The elements of the semigroup $\mathcal{POI}_3$ are pictured in Figure 3 below.

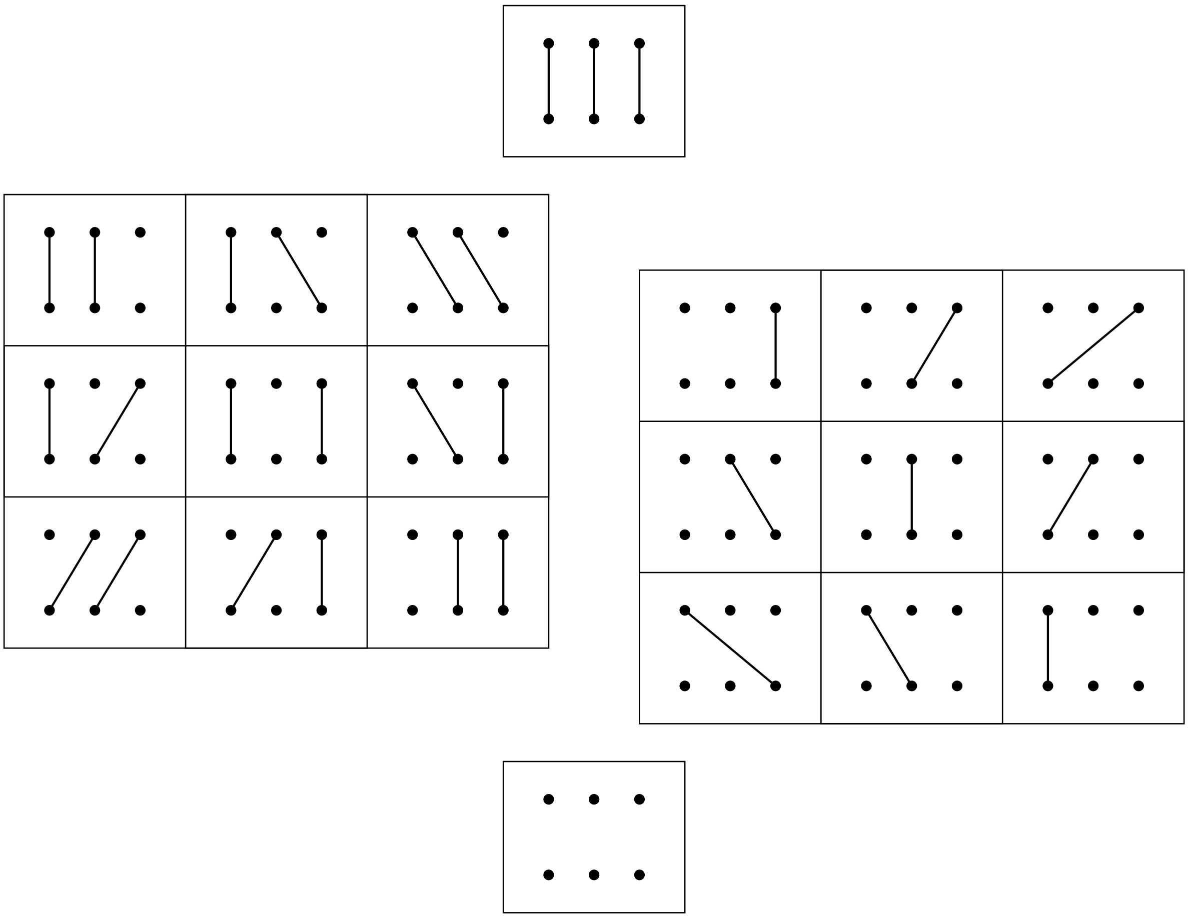

Figure 3: The elements of $\mathcal{POI}_3$.

One calculates that

$$w(\mathcal{POI}_3) = 16 \qquad \text{and} \qquad \overline{w}(\mathcal{O}_3) = \tfrac{16}{20} = \tfrac{4}{5},$$

illustrating another general phenomenon: $w(\mathcal{O}_n) = w(\mathcal{POI}_n)$ for all n.

3 General Calculations

Let $S \subseteq \mathcal{PT}_n$ and $i \in \mathbf{n}$. Write

$$w_i(S) = \sum_{\alpha \in S} w_i(\alpha)$$

for the total work performed by S in moving i. We then have

$$w(S) = \sum_{\alpha \in S} w(\alpha) = \sum_{\alpha \in S} \sum_{i \in \mathbf{n}} w_i(\alpha) = \sum_{i \in \mathbf{n}} \sum_{\alpha \in S} w_i(\alpha) = \sum_{i \in \mathbf{n}} w_i(S).$$

For $i, j \in \mathbf{n}$ let

$$M_{ij}(S) = \{\alpha \in S \mid i\alpha = j\}$$

be the set of all elements of S which move i to j, and write

$$m_{ij}(S) = \big|M_{ij}(S)\big|.$$

Note that $w_i(\alpha) = |i - j|$ for all $\alpha \in M_{ij}(S)$ and so

$$w_i(S) = \sum_{j \in \mathbf{n}} |i - j| m_{ij}(S).$$

Thus we have the following.

Lemma 1 *Let $S \subseteq \mathcal{PT}_n$. Then $w(S) = \sum_{i,j \in \mathbf{n}} |i - j| m_{ij}(S)$.*

4 Specific Calculations

We now consider the cases in which S is one of the semigroups $\mathcal{PT}_n$, $\mathcal{T}_n$, $\mathcal{I}_n$, $\mathcal{S}_n$, $\mathcal{O}_n$, $\mathcal{PO}_n$, or $\mathcal{POI}_n$. For each such S we calculate an explicit formula for the numbers $m_{ij}(S)$, and then apply Lemma 1 to obtain formulae for $w(S)$ and $\overline{w}(S)$. We consider each case separately, covering them roughly in order of difficulty.

It is a well-known fact that

$$\sum_{1 \le i < j \le n} |i - j| = \binom{n+1}{3}.$$[1]

It then follows that

$$\sum_{i,j \in \mathbf{n}} |i - j| = 2\binom{n+1}{3} = \frac{n^3 - n}{3}, \qquad (*)$$

a fact which will prove useful in several of the calculations performed below.

Before moving on we remark that although some of the combinatorial results of this section (particularly Lemmas 2, 4, and 6) may be well-known, the proofs given here are believed to be original. The reader is refered to the introduction of [2] for an excellent review of related articles.

4.1 The Symmetric Group $\mathcal{S}_n$

For any $i, j \in \mathbf{n}$ we have $M_{ij}(\mathcal{S}_n) = \{\alpha \in \mathcal{S}_n \mid i\alpha = j\}$, and it follows immediately that $m_{ij}(\mathcal{S}_n) = (n-1)!$. Thus, by Lemma 1 and $(*)$, the total work performed by the symmetric group $\mathcal{S}_n$ is

$$w(\mathcal{S}_n) = \sum_{i,j \in \mathbf{n}} |i - j|(n-1)! = \frac{n^3 - n}{3} \cdot (n-1)! = \frac{(n+1)!(n-1)}{3}.$$

[1]The number $\binom{n+1}{3}$ is sometimes referred to as the $(n-1)$th *tetrahedral number*; see for example [4] (Sequence A000292). The reader is reminded that we interpret $\binom{n+1}{3} = 0$ if $n = 1$.

The average work performed by an element of $\mathcal{S}_n$ is

$$\overline{w}(\mathcal{S}_n) = \frac{w(\mathcal{S}_n)}{|\mathcal{S}_n|} = \frac{(n+1)!(n-1)}{3 \cdot n!} = \frac{n^2-1}{3}.^2$$

4.2 The Transformation Semigroup $\mathcal{T}_n$

For any $i, j \in \mathbf{n}$ we have $M_{ij}(\mathcal{T}_n) = \{\alpha \in \mathcal{T}_n \mid i\alpha = j\}$, and it follows that $m_{ij}(\mathcal{T}_n) = n^{n-1}$. Thus, by Lemma 1 and $(*)$, we have

$$w(\mathcal{T}_n) = \sum_{i,j \in \mathbf{n}} |i-j| n^{n-1} = \frac{n^3-n}{3} \cdot n^{n-1} = \frac{n^n(n^2-1)}{3}.$$

We also have

$$\overline{w}(\mathcal{T}_n) = \frac{w(\mathcal{T}_n)}{|\mathcal{T}_n|} = \frac{n^n(n^2-1)}{3 \cdot n^n} = \frac{n^2-1}{3}$$

which, rather curiously, is the same as the average work performed by a permutation.

4.3 The Partial Transformation Semigroup

For any $i, j \in \mathbf{n}$ we have $M_{ij}(\mathcal{PT}_n) = \{\alpha \in \mathcal{PT}_n \mid i\alpha = j\}$, so that $m_{ij}(\mathcal{PT}_n) = (n+1)^{n-1}$. Thus, by Lemma 1 and $(*)$, we have

$$w(\mathcal{PT}_n) = \sum_{i,j \in \mathbf{n}} |i-j|(n+1)^{n-1} = \frac{n^3-n}{3} \cdot (n+1)^{n-1} = \frac{(n+1)^n(n^2-n)}{3},$$

and

$$\overline{w}(\mathcal{PT}_n) = \frac{w(\mathcal{PT}_n)}{|\mathcal{PT}_n|} = \frac{(n+1)^n(n^2-n)}{3 \cdot (n+1)^n} = \frac{n^2-n}{3}.$$

Although $\overline{w}(\mathcal{PT}_n) \neq \overline{w}(\mathcal{S}_n) = \overline{w}(\mathcal{T}_n)$, all three sequences are of course assymptotic to $\frac{n^3}{3}$.

4.4 The Symmetric Inverse Semigroup $\mathcal{I}_n$

For all $i, j \in \mathbf{n}$ we have $M_{ij}(\mathcal{I}_n) = \{\alpha \in \mathcal{I}_n \mid i\alpha = j\}$, and it follows that $m_{ij}(\mathcal{PT}_n) = |\mathcal{I}_{n-1}|$. Thus, by Lemma 1 and $(*)$, we have

$$w(\mathcal{I}_n) = \sum_{i,j \in \mathbf{n}} |i-j||\mathcal{I}_{n-1}| = 2\binom{n+1}{3}|\mathcal{I}_{n-1}| = \frac{n^3-n}{3} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 k!.$$

The average work performed by an element of $\mathcal{I}_n$ is

$$\overline{w}(\mathcal{I}_n) = \frac{w(\mathcal{I}_n)}{|\mathcal{I}_n|} = 2\binom{n+1}{3}\frac{|\mathcal{I}_{n-1}|}{|\mathcal{I}_n|} = \frac{(n^3-n)|\mathcal{I}_{n-1}|}{3|\mathcal{I}_n|}.$$

However, it does not seem easy to obtain a formula for $\overline{w}(\mathcal{I}_n)$ as simple as those obtained above for $\overline{w}(\mathcal{S}_n)$, $\overline{w}(\mathcal{T}_n)$, and $\overline{w}(\mathcal{PT}_n)$.

[2] This result may be found in [1] (in a slightly different form).

4.5 The Semigroup $\mathcal{POI}_n$

For $0 \leq p, q \leq n$ let $\mathcal{POI}_{p,q}$ denote the set of all order-preserving injective partial maps from $\mathbf{p}$ to $\mathbf{q}$.

Lemma 2 *Let $0 \leq p, q \leq n$. Then $|\mathcal{POI}_{p,q}| = \binom{p+q}{p} = \binom{p+q}{q}$.*

Proof Let $\mathbf{q}' = \{1', \ldots, q'\}$ be a set in one-one correspondence with $\mathbf{q}$, and put

$$\Omega = \big\{ A \subseteq \mathbf{p} \cup \mathbf{q}' \,\big|\, |A| = q \big\}.$$

For $A \in \Omega$ put $A_{\mathbf{p}} = A \cap \mathbf{p}$ and $A_{\mathbf{q}} = \{ i \in \mathbf{q} \,|\, i' \in A \}$, and define $\phi_A \in \mathcal{POI}_{p,q}$ by

$$\mathrm{dom}(\phi_A) = A_{\mathbf{p}} \qquad \text{and} \qquad \mathrm{im}(\phi_A) = \mathbf{q} \setminus A_{\mathbf{q}},$$

noting that $|A_{\mathbf{p}}| = |\mathbf{q} \setminus A_{\mathbf{q}}|$, and that an element of $\mathcal{POI}_{p,q}$ is completely determined by its domain and image. It is then easy to check that $A \mapsto \phi_A$ $(A \in \Omega)$ defines a bijection $\Omega \to \mathcal{POI}_{p,q}$ and the result follows since $|\Omega| = \binom{p+q}{q}$. $\square$

Lemma 3 *Let $i, j \in \mathbf{n}$. Then $m_{ij}(\mathcal{POI}_n) = \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$.*

Proof Let $\alpha \in M_{ij}(\mathcal{POI}_n)$. Then since $i\alpha = j$ and α is order-preserving, we see that $k\alpha < j$ whenever $k \in \mathrm{dom}(\alpha)$ and $k < i$. Thus, we may define a map $\lambda_\alpha \in \mathcal{POI}_{i-1,j-1}$ by

$$\mathrm{dom}(\lambda_\alpha) = \mathrm{dom}(\alpha) \cap \{1, \ldots, i-1\} \qquad \text{and} \qquad \mathrm{im}(\lambda_\alpha) = \mathrm{im}(\alpha) \cap \{1, \ldots, j-1\}.$$

Similarly, we have $k\alpha > j$ whenever $k \in \mathrm{dom}(\alpha)$ and $k > i$, and so we may also define a map $\rho_\alpha \in \mathcal{POI}_{n-i,n-j}$ by

$$\mathrm{dom}(\rho_\alpha) = \big\{ k - i \,\big|\, k \in \mathrm{dom}(\alpha)\,,\ k > i \big\} \qquad \text{and} \qquad \mathrm{im}(\rho_\alpha) = \big\{ k - j \,\big|\, k \in \mathrm{im}(\alpha)\,,\ k > j \big\}.$$

It is then easy to check that the map $\alpha \mapsto (\lambda_\alpha, \rho_\alpha)$ $(\alpha \in M_{ij}(\mathcal{POI}_n))$ defines a bijection $M_{ij}(\mathcal{POI}_n) \to \mathcal{POI}_{i-1,j-1} \times \mathcal{POI}_{n-i,n-j}$. The result now follows from Lemma 2. $\square$

It follows by Lemmas 1 and 3 that the total work performed by $\mathcal{POI}_n$ is

$$w(\mathcal{POI}_n) = \sum_{i,j \in \mathbf{n}} |i-j| \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}.$$

The average work performed by an element of $\mathcal{POI}_n$ is

$$\overline{w}(\mathcal{POI}_n) = \frac{w(\mathcal{POI}_n)}{|\mathcal{POI}_n|} = \frac{1}{\binom{2n}{n}} \sum_{i,j \in \mathbf{n}} |i-j| \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}.$$

4.6 The Semigroup $\mathcal{O}_n$

For $0 \le p \le n$ and $q \in \mathbf{n}$ let $\mathcal{O}_{p,q}$ denote the set of all order-preserving maps from $\mathbf{p}$ to $\mathbf{q}$.

Lemma 4 *Let $0 \le p \le n$ and $q \in \mathbf{n}$. Then $|\mathcal{O}_{p,q}| = \binom{p+q-1}{p} = \binom{p+q-1}{q-1}$.*

Proof If $p = 0$ then the result is trivial, so suppose that $p \ge 1$. Let $\mathbf{p}^\flat = \mathbf{p} \setminus \{p\}$, let $\mathbf{q}' = \{1', \dots, q'\}$, and put

$$\Omega = \big\{ A \subseteq \mathbf{p}^\flat \cup \mathbf{q}' \,\big|\, |A| = q-1 \big\}.$$

Let $A \in \Omega$ and put

$$A_{\mathbf{p}} = (A \cap \mathbf{p}^\flat) \cup \{p\} \qquad \text{and} \qquad A_{\mathbf{q}} = \{ i \in \mathbf{q} \,|\, i' \in A \}.$$

Suppose that $|A_{\mathbf{p}}| = k$. It then follows that $|\mathbf{q} \setminus A_{\mathbf{q}}| = k$ also, and so we may write

$$A_{\mathbf{p}} = \{x_1, \dots, x_k\} \qquad \text{and} \qquad \mathbf{q} \setminus A_{\mathbf{q}} = \{y_1, \dots, y_k\}$$

where $x_1 < \cdots < x_k$ and $y_1 < \cdots < y_k$. It will also be convenient to put $x_0 = 0$. Now define $\phi_A \in \mathcal{O}_{p,q}$, for $i \in \mathbf{p}$, by

$$i\phi_A = y_\ell \qquad \text{if } i \in \{x_{\ell-1}+1, \dots, x_\ell\}.$$

Then one may check that $A \mapsto \phi_A$ ($A \in \Omega$) defines a bijection $\Omega \to \mathcal{O}_{p,q}$. The result now follows since $|\Omega| = \binom{p+q-1}{q-1}$. $\square$

Lemma 5 *Let $i, j \in \mathbf{n}$. Then $m_{ij}(\mathcal{O}_n) = \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$.*

Proof Let $\alpha \in M_{ij}(\mathcal{O}_n)$. Then since $i\alpha = j$ and α is order-preserving, we see that $k\alpha \le j$ whenever $k \in \mathrm{dom}(\alpha)$ and $k < i$. Thus, we may define a map $\lambda_\alpha \in \mathcal{O}_{i-1,j}$ by

$$k\lambda_\alpha = k\alpha \qquad \text{for each } k \in \{1, \dots, i-1\}.$$

Similarly, we may also define a map $\rho_\alpha \in \mathcal{O}_{n-i,n-j+1}$ by

$$k\rho_\alpha = (k+i)\alpha - j + 1 \qquad \text{for each } k \in \{1, \dots, n-i\}.$$

Then one may check that the map $\alpha \mapsto (\lambda_\alpha, \rho_\alpha)$ ($\alpha \in M_{ij}(\mathcal{O}_n)$) defines a bijection $M_{ij}(\mathcal{O}_n) \to \mathcal{O}_{i-1,j} \times \mathcal{O}_{n-i,n-j+1}$. The result now follows from Lemma 4. $\square$

In particular, Lemma 5 demonstrates another curious fact; namely that $m_{ij}(\mathcal{O}_n) = m_{ij}(\mathcal{POI}_n)$ for all $i, j \in \mathbf{n}$. By Lemmas 1 and 5 we have

$$w(\mathcal{O}_n) = \sum_{i,j \in \mathbf{n}} |i-j| \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$$

which, of course, is the same as $w(\mathcal{POI}_n)$. However, since $|\mathcal{O}_n| = \binom{2n-1}{n} \ne |\mathcal{POI}_n|$, we see that $\overline{w}(\mathcal{O}_n) \ne \overline{w}(\mathcal{POI}_n)$. Rather, we have

$$\overline{w}(\mathcal{O}_n) = \frac{w(\mathcal{O}_n)}{|\mathcal{O}_n|} = \frac{1}{\binom{2n-1}{n}} \sum_{i,j \in \mathbf{n}} |i-j| \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}.$$

But, since $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, we do have the interesting relation $\overline{w}(\mathcal{O}_n) = 2\overline{w}(\mathcal{POI}_n)$; that is, an element of $\mathcal{O}_n$ works "twice as hard" as an element of $\mathcal{POI}_n$ on average.

4.7 The Semigroup $\mathcal{PO}_n$

For $0 \le p \le n$ and $q \in \mathbf{n}$ let $\mathcal{PO}_{p,q}$ denote the set of all order-preserving partial transformations from $\mathbf{p}$ to $\mathbf{q}$.

Lemma 6 *Let $0 \le p \le n$ and $q \in \mathbf{n}$. Then* $|\mathcal{PO}_{p,q}| = \sum_{k=0}^{p} \binom{p}{k}\binom{q+k-1}{k}$.

Proof For $A \subseteq \mathbf{p}$ write $\mathcal{PO}_{p,q}^{A} = \{\alpha \in \mathcal{PO}_{p,q} \,|\, \mathrm{dom}(\alpha) = A\}$. We then have the disjoint union

$$\mathcal{PO}_{p,q} = \dot{\bigcup_{A \subseteq \mathbf{p}}} \mathcal{PO}_{p,q}^{A}.$$

Now for any $0 \le k \le p$, there are $\binom{p}{k}$ subsets $A \subseteq \mathbf{p}$ for which $|A| = k$ and, for each such subset A, we have $\left|\mathcal{PO}_{p,q}^{A}\right| = |\mathcal{O}_{k,q}| = \binom{q+k-1}{k}$, the last equality following by Lemma 4. This shows that

$$|\mathcal{PO}_{p,q}| = \sum_{A \subseteq \mathbf{p}} \left|\mathcal{PO}_{p,q}^{A}\right| = \sum_{k=0}^{p} \binom{p}{k}\binom{q+k-1}{k},$$

and the proof is complete. □

Lemma 7 *Let $i, j \in \mathbf{n}$. Then*

$$m_{ij}(\mathcal{PO}_n) = \sum_{k,\ell=0}^{n} \binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}.$$

Proof A similar argument to that used in the proof of Lemma 5 shows that there is a bijection from $M_{ij}(\mathcal{PO}_n)$ to $\mathcal{PO}_{i-1,j} \times \mathcal{PO}_{n-i,n-j+1}$. It then follows by Lemma 6 that

$$\begin{aligned} m_{ij}(\mathcal{PO}_n) &= |\mathcal{PO}_{i-1,j}| \times |\mathcal{PO}_{n-i,n-j+1}| \\ &= \sum_{k=0}^{i-1} \binom{i-1}{k}\binom{j+k-1}{k} \sum_{\ell=0}^{n-i} \binom{n-i}{\ell}\binom{n-j+\ell}{\ell}. \end{aligned}$$

The upper limits on both sums may be changed to n in light of the convention regarding binomial coefficients explained at the end of Section 1. □

Thus, by Lemmas 1 and 7, we have

$$w(\mathcal{PO}_n) = \sum_{i,j=1}^{n} \sum_{k,\ell=0}^{n} |i-j| \binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}.$$

We also have

$$\overline{w}(\mathcal{PO}_n) = \frac{w(\mathcal{PO}_n)}{|\mathcal{PO}_n|} = \frac{\sum_{i,j=1}^{n} \sum_{k,\ell=0}^{n} |i-j| \binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}}{\sum_{m=0}^{n} \binom{n}{m}\binom{n+m-1}{m}}.$$

5 Summary of Results

We now collect the results obtained in the previous section. Tables 1 and 2 catalogue the formulae obtained for $w(S)$ and $\overline{w}(S)$, respectively, for the various semigroups S considered in Section 4.

S	Formula for $w(S)$
$\mathcal{S}_n$	$\frac{(n+1)!(n-1)}{3}$
$\mathcal{T}_n$	$\frac{n^n(n^2-1)}{3}$
$\mathcal{PT}_n$	$\frac{(n+1)^n(n^2-n)}{3}$
$\mathcal{I}_n$	$\frac{n^3-n}{3}\sum_{k=0}^{n-1}\binom{n-1}{k}^2 k!$
$\mathcal{POI}_n$	$\sum_{i,j=1}^{n} \lvert i-j\rvert \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$
$\mathcal{O}_n$	$\sum_{i,j=1}^{n} \lvert i-j\rvert \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$
$\mathcal{PO}_n$	$\sum_{i,j=1}^{n}\sum_{k,\ell=0}^{n} \lvert i-j\rvert \binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}$

Table 1: Formulae for the total work $w(S)$ performed by a semigroup $S \subseteq \mathcal{PT}_n$.

S	Formula for $\overline{w}(S)$
$\mathcal{S}_n$	$\frac{n^2-1}{3}$
$\mathcal{T}_n$	$\frac{n^2-1}{3}$
$\mathcal{PT}_n$	$\frac{n^2-n}{3}$
$\mathcal{I}_n$	$\frac{n^3-n}{3\sum_{\ell=0}^{n}\binom{n}{\ell}^2 \ell!}\sum_{k=0}^{n-1}\binom{n-1}{k}^2 k!$
$\mathcal{POI}_n$	$\frac{1}{\binom{2n}{n}}\sum_{i,j=1}^{n} \lvert i-j\rvert \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$
$\mathcal{O}_n$	$\frac{1}{\binom{2n-1}{n}}\sum_{i,j=1}^{n} \lvert i-j\rvert \binom{i+j-2}{i-1}\binom{2n-i-j}{n-i}$
$\mathcal{PO}_n$	$\frac{1}{\sum_{m=0}^{n}\binom{n}{m}\binom{n+m-1}{m}}\sum_{i,j=1}^{n}\sum_{k,\ell=0}^{n} \lvert i-j\rvert \binom{i-1}{k}\binom{j+k-1}{k}\binom{n-i}{\ell}\binom{n-j+\ell}{\ell}$

Table 2: Formulae for the average work $\overline{w}(S)$ performed by an element of a semigroup $S \subseteq \mathcal{PT}_n$.

Tables 3 and 4 catalogue calculated values of $w(S)$ and $\overline{w}(S)$ for values of n up to 10.

n	1	2	3	4	5	6	7	8	9	10
$w(\mathcal{S}_n)$	0	2	16	120	960	8400	80640	846720	9676800	119750400
$w(\mathcal{T}_n)$	0	4	72	1280	25000	544320	13176688	352321536	10331213040	330000000000
$w(\mathcal{PT}_n)$	0	6	128	2500	51840	1176490	29360128	803538792	24000000000	778122738030
$w(\mathcal{I}_n)$	0	4	56	680	8360	108220	1492624	21994896	346014960	5798797620
$w(\mathcal{POI}_n)$	0	2	16	96	512	2560	12288	57344	262144	1179648
$w(\mathcal{O}_n)$	0	2	16	96	512	2560	12288	57344	262144	1179648
$w(\mathcal{PO}_n)$	0	4	48	424	3312	24204	169632	1155152	7702944	50550932

Table 3: Calculated values of $w(S)$ for small values of n.

n	1	2	3	4	5	6	7	8	9	10
$\overline{w}(\mathcal{S}_n)$	0	1	$2\frac{2}{3}$	5	8	$11\frac{2}{3}$	16	21	$26\frac{2}{3}$	33
$\overline{w}(\mathcal{T}_n)$	0	1	$2\frac{2}{3}$	5	8	$11\frac{2}{3}$	16	21	$26\frac{2}{3}$	33
$\overline{w}(\mathcal{PT}_n)$	0	$\frac{2}{3}$	2	4	$6\frac{2}{3}$	10	14	$18\frac{2}{3}$	24	30
$\overline{w}(\mathcal{I}_n)$	0	0.5714	1.6471	3.2536	5.4075	8.1204	11.4009	15.2559	19.6911	24.7112
$\overline{w}(\mathcal{POI}_n)$	0	0.3333	0.8000	1.3714	2.0317	2.7706	3.5804	4.4556	5.3916	6.3848
$\overline{w}(\mathcal{O}_n)$	0	0.6667	1.6000	2.7429	4.0635	5.5411	7.1608	8.9113	10.7833	12.7697
$\overline{w}(\mathcal{PO}_n)$	0	0.5000	1.2632	2.2083	3.3054	4.5360	5.8871	7.3490	8.9139	10.5754

Table 4: Calculated values of $\overline{w}(S)$ for small values of n.

6 Concluding Remarks

The work presented in this article was inspired by a talk given by Tim Lavers in which a conjecture was described; namely that $w(\mathcal{O}_n) = (n-1)2^{2n-3}$. The second-last row of Table 3 (and some modest labour) shows that this is true if $n \leq 10$. Some further values of $w(\mathcal{O}_n)$ are provided in Table 5 below, giving further credibility to the conjecture. The

n	11	12	13	14	15	16	17	18
$w(\mathcal{O}_n)$	5242880	23068672	100663296	436207616	1879048192	8053063680	34359738368	146028888064

n	19	20	21	22	23	24
$w(\mathcal{O}_n)$	618475290624	2611340115968	10995116277760	46179488366592	193514046488576	809240558043136

Table 5: Further values of $w(\mathcal{O}_n)$.

author believes that Lavers has also verified the conjecture for some values of n; see [3] for more details. In light of the results of Sections 4.5 and 4.6, a verification of Lavers' conjecture amounts to a proof of the identity

$$\sum_{i,j=1}^{n} |i-j| \binom{i+j-2}{i-1} \binom{2n-i-j}{n-i} = (n-1)2^{2n-3}$$

for all $n \geq 1$. Replacing n by $n+1$, and substituting $p = i-1$ and $q = j-1$, the above identity takes the more pleasing form

$$\sum_{p,q=0}^{n} |p-q| \binom{p+q}{p} \binom{2n-p-q}{n-p} = n2^{2n-1}.$$

Finally we remark that some of the numbers $w(S)$ have been calculated before. Indeed, $w(\mathcal{S}_n)$ appears as Sequence A090672 in [4]; see also [1] where the quantity $\frac{1}{n}w(\mathcal{S}_n)$ was investigated in relation to "turbo coding". The numbers $w(\mathcal{O}_n) = w(\mathcal{POI}_n)$ as calculated above agree with the first 23 entries of Sequence A002699 of [4] which (not surprisingly[3]) is $n2^{2n-1}$. At the time of writing, the other sequences in Table 3 had not been listed in [4].

References

[1] D. Daly and P. Vojte. How Permutations Displace Points and Stretch Intervals. *Preprint*, `http://www.math.du.edu/data/preprints/m0508.pdf`.

[2] A. Laradji and A. Umar. Combinatorial Results for Semigroups of Order-Decreasing Partial Transformations. *J. Integer Seq.*, 7(3):14pp. (electronic), 2004.

[3] T. Lavers. On a Conjecture Concerning $\mathcal{O}_n$. *Preprint.*

[4] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. *published electronically at* `http://www.research.att.com/njas/sequences/`, 2005.

[3]What is surprising perhaps is that these numbers, $n2^{2n-1}$, are coefficients of "shifted Chebyshev polynomials".