A Recent Look at the Quantum Berezinian in the Yangian $Y(\mathfrak{gl}_{m|n})$

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Brundan and Kleshchev recently introduced a new family of presentations of the Yangian $Y(\mathfrak{gl}_n)$ associated to the general linear Lie algebra $\mathfrak{gl}_n$, and thus provided a fresh approach to its study. In this article, we would like to show how some of their ideas can be fruitfully extended to consider the Yangian $Y(\mathfrak{gl}_{m|n})$ associated to the Lie superalgebra $\mathfrak{gl}_{m|n}$. In particular, we give a new proof of the result by Nazarov that the quantum Berezinian is central.[1]

1 Definition of $Y(\mathfrak{gl}_{m|n})$

The Yangian $Y(\mathfrak{gl}_{m|n})$ is defined in [7] to be the $\mathbb{Z}_2$-graded associative algebra over $\mathbb{C}$ with generators $t_{ij}^{(r)}$ and certain relations described below. We define the formal power series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \dots,$$

and a matrix

$$T(u) = \sum_{i,j=1}^{m+n} t_{ij}(u) \otimes E_{ij}\ (-1)^{\bar{j}(\bar{i}+1)}, \tag{1}$$

where E_{ij} is the standard elementary matrix and $\bar{i}$ is the parity of the index i. In analogy with the usual Yangian $Y(\mathfrak{gl}_n)$ (see for example [2], [5], [6]), the defining relations are then expressed by the matrix product

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$

where

$$R(u-v) = 1 - \frac{1}{(u-v)}P_{12}$$

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and P_{12} is the permutation matrix:

$$P_{12} = \sum_{i,j=1}^{m+n} E_{ij} \otimes E_{ji} (-1)^{\overline{j}}.$$

Then we have the following equivalent form of the defining relations:

$$[t_{ij}(u),\ t_{kl}(v)] \ = \ \frac{(-1)^{\overline{i}\overline{j}+\overline{i}\overline{k}+\overline{j}\overline{k}}}{(u-v)}(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)).$$

Throughout this article we will observe the following notation for entries of the inverse matrix of $T(u)$:

$$T(u)^{-1} =: \left(t'_{ij}(u)\right)_{i,j=1}^{n}.$$

A straightforward calculation yields the following relation in $Y(\mathfrak{gl}_{m|n})$:

$$[t_{ij}(u), t'_{kl}(v)] = \frac{(-1)^{\overline{i}\overline{j}+\overline{i}\overline{k}+\overline{j}\overline{k}}}{(u-v)} \cdot (\ \delta_{kj} \sum_{s=1}^{m+n} t_{is}(u)t'_{sl}(v) - \ \delta_{il} \sum_{s=1}^{m+n} t'_{ks}(v)t_{sj}(u)). \qquad (2)$$

2 Gauss Decomposition of $T(u)$

In [1], the Drinfeld presentation is described in terms of the quasideterminants of Gelfand and Retakh ([3], [4]). In this article we make use of the analogous set of generators of the Yangian $Y(\mathfrak{gl}_{m|n})$. First we recall the definition of the quasideterminants and some conventional notation.

Definition 2.1. *Let X be a square matrix over a ring with identity such that its inverse matrix X^{-1} exists, and such that its jith entry is an invertible element of the ring. Then the ijth* quasideterminant *of X is defined by the formula*

$$|X|_{ij} = \left((X^{-1})_{ji}\right)^{-1}.$$

Equivalently, we may define quasideterminants inductively as follows. If $X = (x_{11})$ is a 1×1-matrix then there is only one quasideterminant of X; and this is $|X|_{11} = x_{11}$. For $n > 1$, we have

$$|X|_{ij} = x_{ij} - \sum_{k \neq i, l \neq j} x_{ik}(|X^{ij}|_{lk})^{-1} x_{lj},$$

where X^{ij} is the matrix obtained from X by removing both the ith row and the jth column. It is sometimes convenient to adopt the following alternative notation for the quasideterminants:

$$|X|_{ij} =: \begin{vmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ & \cdots & & \cdots & \\ x_{i1} & \cdots & \boxed{x_{ij}} & \cdots & x_{in} \\ & \cdots & & \cdots & \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{vmatrix}.$$

The matrix $T(u)$ defined in (1) has the following Gauss decomposition in terms of quasideterminants (by Theorem 4.96 in [3]; see §5 in [1]):

$$T(u) = F(u)D(u)E(u)$$

for unique matrices

$$\begin{aligned}
D(u) &= \begin{pmatrix} d_1(u) & & \cdots & 0 \\ & d_2(u) & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & d_{m+n}(u) \end{pmatrix}, \\
E(u) &= \begin{pmatrix} 1 & e_{12}(u) & \cdots & e_{1,m+n}(u) \\ & \ddots & & e_{2,m+n}(u) \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix}, \\
F(u) &= \begin{pmatrix} 1 & & \cdots & 0 \\ f_{21}(u) & \ddots & & \vdots \\ \vdots & & \ddots & \\ f_{m+n,1}(u) & f_{m+n,2}(u) & \cdots & 1 \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
d_i(u) &= \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\ \vdots & \ddots & & \vdots \\ t_{i1}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{ii}(u)} \end{vmatrix}, \\
e_{ij}(u) &= d_i(u)^{-1} \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,i}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,j}(u) \\ t_{i1}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{ij}(u)} \end{vmatrix}, \\
f_{ji}(u) &= \begin{vmatrix} t_{11}(u) & \cdots & t_{1,i-1}(u) & t_{1i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,i}(u) \\ t_{ji}(u) & \cdots & t_{j,i-1}(u) & \boxed{t_{ji}(u)} \end{vmatrix} d_i(u)^{-1}.
\end{aligned}$$

It is easy to recover each generating series $t_{ij}(u)$ by multiplying together and taking commutators of $d_i(u);\ 1 \le i \le m+n$, and $e_i(u) := e_{i,i+1}(u)$, $f_i(u) = f_{i+1,i}(u);\ 1 \le i < m+n$ (see §5 of [1]). Thus the Yangian $Y(\mathfrak{gl}_{m|n})$ is generated by the coefficients of the latter.

2.1 Some Useful Maps

Here we define some automorphisms of the Yangian $Y(\mathfrak{gl}_{m|n})$ and homomorphisms between Yangians, so that we may refer to them in the next section.

Let $\omega_{m|n} : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{m|n})$ be the automorphism defined by

$$\omega : T(u) \mapsto T(-u)^{-1}.$$

Let $\tau : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{m|n})$ be the automorphism defined by

$$\tau(t_{ij}(u)) = t_{ji}(-u) \times (-1)^{\bar{i}(\bar{j}+1)}.$$

Let $\rho_{m|n} : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{n|m})$ be the isomorphism defined by

$$\rho_{m|n}(t_{ij}(u)) = t_{m+n+1-i,m+n+1-j}(-u).$$

Let $\varphi_{m|n} : Y(\mathfrak{gl}_{m|n}) \hookrightarrow Y(\mathfrak{gl}_{m+k|n})$ be the inclusion which sends each generator $t_{ij}^{(r)} \in Y(\mathfrak{gl}_{m|n})$ to the generator $t_{k+i,k+j}^{(r)}$ in $Y(\mathfrak{gl}_{m+k|n})$.

Finally, let $\psi_k : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{m+k|n})$ be the injective homomorphism defined by

$$\psi_k = \omega_{m+k|n} \circ \varphi_{m|n} \circ \omega_{m|n}. \tag{3}$$

This last homomorphism is useful for studying quasideterminants so we discuss it in some detail with the following remarks.

Remark 2.1. *We can calculate $\psi_k(t_{ij}(u))$ explicitly for any $1 \leq i, j \leq m+n$ (see Lemma 4.2 of [1]) :*

$$\psi_k(t_{ij}(u)) = \begin{vmatrix} t_{11}(u) & \cdots & t_{1k}(u) & t_{1,k+j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{k1}(u) & \cdots & t_{kk}(u) & t_{k,k+j}(u) \\ t_{k+i,1}(u) & \cdots & t_{k+i,k}(u) & \boxed{t_{k+i,k+j}(u)} \end{vmatrix}.$$

In particular, this means that for $k \geq 1$, we have $\psi_k(d_1(u)) = d_{k+1}(u)$, $\psi_k(e_1(u)) = e_{k+1}(u)$, and $\psi_k(f_1(u)) = f_{k+1}(u)$.

Furthermore, by (3), we have for any $k, l \geq 1$ that $\psi_k \circ \psi_l = \psi_{k+l}$, so we may generalise this observation to give for instance $\psi_k(d_l(u)) = d_{k+l}(u)$.

Remark 2.2. *Notice that the map ψ_k sends $t'^{(r)}_{ij} \in Y(\mathfrak{gl}_{m|n})$ to the element $t'^{(r)}_{k+i,k+j}$ in $Y(\mathfrak{gl}_{m+k|n})$. Thus the subalgebra $\psi_k(Y(\mathfrak{gl}_{m|n}))$ is generated by the elements $\{t'^{(r)}_{k+s,k+t}\}_{s,t=1}^{n}$. Then, by (2), all elements of this subalgebra commute with those of the subalgebra generated by the elements $\{t_{ij}^{(r)}\}_{i,j=1}^{k}$.*

By Remark 2.1, this implies in particular that for any $i, j \geq 1$, the quasideterminants $d_i(u)$ and $d_j(v)$ commute.

3 The Quantum Berezinian

The quantum Berezinian was defined by Nazarov [7] and plays a similar role in the study of the Yangian $Y(\mathfrak{gl}_{m|n})$ as the quantum determinant does in the case of the Yangian $Y(\mathfrak{gl}_n)$ (see [5]).

Definition 3.1. *The quantum Berezinian is the following power series with coefficients in the Yangian $Y(\mathfrak{gl}_{m|n})$:*

$$\begin{aligned} b_{m|n}(u) \quad := \quad & \sum_{\rho \in S_m} \operatorname{sgn}(\tau)\, t_{\tau(1)1}(u) t_{\tau(2)2}(u-1) \cdots t_{\tau(m)m}(u-m+1) \\ & \times \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, t'_{m+1,m+\sigma(1)}(u-m+1) \cdots t'_{m+n,m+\sigma(n)}(u-m+n) \end{aligned}$$

The first part of this expression for $b_{m|n}(u)$ is quite special and so is given its own notation:

$$C_m(u) := \sum_{\tau \in S_m} \operatorname{sgn}(\tau) t_{\tau(1)1}(u) t_{\tau(2)2}(u-1) \cdots t_{\tau(m)m}(u-m+1).$$

It is clear that $C_m(u)$ is an element of the subalgebra of $Y(\mathfrak{gl}_{m|n})$ generated by the set $\{t_{ij}^{(r)}\}_{1 \le i,j \le m; r \ge 0}$. This subalgebra is isomorphic to the Yangian $Y(\mathfrak{gl}_\mathfrak{m})$ associated to the Lie algebra $\mathfrak{gl}_m$ by the inclusion $Y(\mathfrak{gl}_m) \to Y(\mathfrak{gl}_{m|n})$ which send each generator $t_{ij}^{(r)}$ in $Y(\mathfrak{gl}_\mathfrak{m})$ to the generator of the same name in $Y(\mathfrak{gl}_{m|n})$. Moreover, $C_m(u)$ is in fact the image under this map of the *quantum determinant* of the smaller Yangian $Y(\mathfrak{gl}_\mathfrak{m})$ (see [1], [5]). Then it is well known (see Theorem 2.32 in [6]) that we have the alternative expression:

$$C_m(u) = d_1(u) d_2(u-1) \cdots d_m(u-m+1).$$

We can extend this observation as follows:

Theorem 1. *We have the following alternative expression for the quantum Berezinian:*

$$\begin{aligned} b_{m|n}(u) \quad = \quad & d_1(u)\, d_2(u-1) \cdots d_m(u-m+1) \\ & \times d_{m+1}(u-m+1)^{-1} \cdots d_{m+n}(u-m+n)^{-1}. \end{aligned}$$

Proof. Notice that the second part of the expression for $b_{m|n}(u)$ in Definition 3.1 is the image under the isomorphism $\rho_{n|m} \circ \omega_{n|m} : Y(\mathfrak{gl}_{n|m}) \to Y(\mathfrak{gl}_{m|n})$ of

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, t_{n,\sigma(n)}(u-m+1) \cdots t_{2,\sigma(2)}(u-m+n-1)\, t_{1,\sigma(1)}(u+m-n) \qquad (4)$$

where in this expression (4) we are following the usual convention for denoting generators *in the Yangian* $Y(\mathfrak{gl}_{n|m})$. We recognise (by comparing with (8.3) of [1] for example) that the expression (4) is in fact $C_n(u-m+n)$, the image of the quantum determinant of $Y(\mathfrak{gl}_n)$ under the natural inclusion $Y(\mathfrak{gl}_n) \hookrightarrow Y(\mathfrak{gl}_{n|m})$. So in order to verify the claim we must calculate the image of $C_n(u-m+n)$ under this map explicitly in terms of our quasideterminants $d_i(v)$. Applying Proposition 1.6 of [4], we find that the image of $d_i(v)$ in $Y(\mathfrak{gl}_{n|m})$ is $(d_{m+n+1-i}(v))^{-1}$ in $Y(\mathfrak{gl}_{m|n})$. This gives the desired result. ∎

The following theorem is a result of Nazarov [7]. We give a new proof.

Theorem 2. *The coefficients of the quantum Berezinian (3.1) are central in the algebra* $Y(\mathfrak{gl}_{m|n})$.

Proof. By Remark 2.2, we already know that the quantum Berezinian $b_{m|n}(u)$ commutes with $d_i(v)$ for $1 \le i \le m+n$. In addition, if we know that the quantum Berezinian commutes with $e_i(v)$, then by applying the automorphism τ, we find that it also commutes with $f_i(-v)$. So our problem reduces to showing that $b_{m|n}(u)$ commutes with $e_i(v)$ for each i between 1 and $m+n-1$. We proceed by breaking this problem into three cases.

Case 1: $1 \le i \le m-1$. For $1 \le i \le m-1$, we have that $e_i(v)$ commutes with $C_m(u) = d_1(u) \cdots d_m(u-m+1)$ by Theorem 7.2 in [1]. On the other hand, $e_i(v)$ is an element of the subalgebra generated by $\{t_{jk}^{(r)}\}_{1\le j,k\le m}$ and thus by Remark 2.2 commutes with $d_{m+s}(u-m+s)^{-1} = t'_{m+s,m+s}(u-m+s)$ for $1 \le s \le n$.

Case 2: $m+1 \le i \le m+n-1$. Applying Propositions 1.6 and 1.4 of [4] in turn to $f_i(v)$, we find an alternative expression:

$$f_i(v) = - \begin{vmatrix} \boxed{t'_{i+1,i+1}(v)} & \cdots & t'_{i+1,m+n}(v) \\ \vdots & & \vdots \\ t'_{m+n,i+1}(v) & \cdots & t'_{m+n,m+n}(v) \end{vmatrix}^{-1} \cdot \begin{vmatrix} \boxed{t'_{i+1,i}(v)} & t'_{i+1,i+2}(v) & \cdots & t'_{i+1,m+n}(v) \\ t'_{i+2,i}(v) & t'_{i+2,i+2}(v) & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ t'_{m+n,i}(v) & t'_{m+n,i+2}(v) & \cdots & t'_{m+n,m+n}(v) \end{vmatrix}.$$

Thus, we find that for $m+1 \le i \le m+n-1$,

$$e_i(v) = \rho_{n|m} \circ \omega_{n|m}(-f_{m+n-i}(v)).$$

We apply this isomorphism to the results of Case 1 in the Yangian $Y(\mathfrak{gl}_{n|m})$. This shows that $e_i(v)$ commutes with the quantum Berezinian in the case where $m+1 \le i \le m+n-1$.

Case 3: i=m. We begin by considering the Yangian $Y(\mathfrak{gl}_{1|1})$. For this algebra we have $b_{1|1}(u) = d_1(u)d_2(u)^{-1}$ and we would like to show that this commutes with $e_1(v)$. So it will suffice to show

$$d_1(u)e_1(v)d_2(u) = d_2(u)e_1(v)d_1(u). \tag{5}$$

We have

$$\begin{pmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & t_{22}(u) \end{pmatrix} = \begin{pmatrix} d_1(u) & d_1(u)\, e_1(u) \\ f_1(u)d_1(u) & f_1(u)d_1(u)e_1(u) + d_2(u) \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} t'_{11}(v) & t'_{12}(v) \\ t'_{21}(v) & t'_{22}(v) \end{pmatrix} = \begin{pmatrix} d_1(v)^{-1} + e_1(v)d_2(v)^{-1}f_1(v) & -e_1(v)\, d_2(v)^{-1} \\ -d_2(v)^{-1}f_1(v) & d_2(v)^{-1} \end{pmatrix}. \tag{7}$$

An application of (2) gives

$$(u-v)[t_{11}(u),t'_{12}(v)] \ = \ t_{11}(u)t'_{12}(v)+t_{12}(u)t'_{22}(v).$$

Substituting in the expressions from (6) and (7) then cancelling $d_2(v)$, this gives

$$(u-v)[d_1(u),e_1(v)] \ = \ d_1(u)(e_1(v)-e_1(u)).$$

Similarly, by considering the commutator $[t_{12}(u),t'_{22}(v)]$, we derive the relation

$$(u-v)[d_2(u),e_1(v)] \ = \ d_2(u)(e_1(v)-e_1(u)).$$

We rewrite these relations to find

$$\begin{aligned}(u-v)e_1(v)d_1(u) &= (u-v-1)d_1(u)e_1(v)+d_1(u)e_1(u),\\ (u-v)e_1(v)d_2(u) &= (u-v-1)d_2(u)e_1(v)+d_2(u)e_1(u),\end{aligned}$$

and by considering these expressions we see that (5) holds.

Now we return our attention to the general Yangian $Y(\mathfrak{gl}_{m|n})$. By similar appeals to Remark 2.2 as in the first case, we see that $e_m(v)$ commutes with $d_1(u)\cdots d_{m-1}(u-m+2)$ and with $d_{m+2}(u-m+2)^{-1}\cdots\ d_{m+n}(u-\ m\ +\ n)^{-1}$. So we need only show that $e_m(v)$ commutes with $d_m(u-m+1)d_{m+1}(u-m+1)^{-1}$. This follows immediately when we apply the homomorphism ψ_{m-1} to the identity (5) in $Y(\mathfrak{gl}_{1|1})$. □

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