AN ISOPERIMETRIC INEQUALITY FOR THE SECOND EIGENVALUE OF THE LAPLACIAN WITH ROBIN BOUNDARY CONDITIONS

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Abstract. We prove that the second eigenvalue of the Laplacian with Robin boundary conditions is minimised amongst all bounded Lipschitz domains of fixed volume by the domain consisting of the disjoint union of two balls of equal volume.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain (not necessarily connected) and consider the eigenvalue problem for the Laplacian with Robin boundary condition

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\beta > 0$ is a constant and ν is the outer unit normal to Ω. This problem is often referred to as that of the elastically supported membrane. It is well known that, as in the case of Dirichlet boundary conditions, the associated operator on $L^2(\Omega)$ has compact resolvent, with the eigenvalues forming a sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$.

It has been shown in [1, 5] that the first eigenvalue $\lambda_1 = \lambda_1(\Omega)$ satisfies the isoperimetric, or Faber-Krahn, inequality $\lambda_1(\Omega) \geq \lambda_1(B)$, where B is a ball having the same volume as Ω. The goal of this short paper is to prove a similar inequality for $\lambda_2(\Omega)$.

Theorem 1. *The second eigenvalue $\lambda_2(\Omega)$ of (1) on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ satisfies $\lambda_2(\Omega) \geq \lambda_2(D)$, where D is a domain of the same volume as Ω consisting of the disjoint union of two balls of equal volume.*

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We defer the proof of Theorem 1 to Section 3 and first discuss some background issues and consequences.

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2. Observations and Remarks

We first wish to consider the Laplacian with Dirichlet boundary conditions. Not only is the minimising domain D the same for both Dirichlet and Robin boundary conditions, but our proof uses ideas from the Dirichlet case. For this reason we will give a brief sketch of the proof of the latter here. A more complete proof, together with further references, can be found in [10, Sec. 4].

Let φ denote an eigenfunction of the second Dirichlet eigenvalue, which we will call $\mu_2(\Omega)$. The idea is to consider the nodal domains $\Omega^+ := \{x \in \Omega : \varphi(x) > 0\}$ and $\Omega^- := \{x \in \Omega : \varphi(x) < 0\}$. Then φ is an eigenfunction of the Dirichlet Laplacian that does not change sign in Ω^+, so that $\mu_2(\Omega) = \mu_1(\Omega^+)$ (μ_1 being the first Dirichlet eigenvalue). Denoting by B^+ a ball of the same volume as Ω^+, by the usual Faber-Krahn inequality, $\mu_1(\Omega^+) \geq \mu_1(B^+)$, that is, $\mu_2(\Omega) \geq \mu_1(B^+)$. Similarly, if B^- is a ball of the same volume as Ω^-, then $\mu_2(\Omega) \geq \mu_1(B^-)$.

Hence $\mu_2(\Omega) \geq \max\{\mu_1(B^+), \mu_1(B^-)\}$. The latter is minimised if $B^+ = B^- =: B$ has half the volume of Ω. But D (defined in Theorem 1) can be written as the disjoint union of two copies of B, so that $\mu_2(D) = \mu_1(D) = \mu_1(B) \leq \mu_2(\Omega)$.

We would like to use a similar idea in the Robin case. Denoting an eigenfunction of $\lambda_2(\Omega)$ by ψ, we wish to describe $\lambda_2(\Omega)$ as the first eigenvalue of a problem on Ω^+ (and Ω^-) with mixed Robin-Dirichlet boundary conditions $\frac{\partial\psi}{\partial\nu} + \beta\psi = 0$ on $\partial\Omega^+ \cap \partial\Omega$ and $\psi = 0$ on $\partial\Omega^+ \cap \Omega$. This is greater than the first eigenvalue $\lambda_1(\Omega^+)$ of the pure Robin problem on Ω^+. By the Faber-Krahn inequality for Robin problems [1,5], $\lambda_1(\Omega^+) \geq \lambda_1(B^+)$, and we proceed as before.

However, there is a major complication. We cannot directly apply the inequality from [1,5] to Ω^+, Ω^- since that result is only valid for Lipschitz domains and in general Ω^+, Ω^- may not be this smooth. The problem is twofold.

First, we have no control over the behaviour of $\partial\Omega^+$, $\partial\Omega^-$ near where the nodal surface $\{x \in \Omega : \psi(x) = 0\}$ meets $\partial\Omega$. At such points x, supposing the boundary condition holds pointwise we have $\frac{\partial\psi}{\partial\nu}(x) = 0$. This is in general consistent with the possibility that $\nabla\psi(x) = 0$.

Second, even though the eigenfunction ψ will be C^∞ in Ω, this is not enough to guarantee that the nodal surface is a smooth manifold in the interior. Sard's Lemma (see [11, Ch. 3, Theorem 1.3]) does not suffice, since 0 may be in the null set of non-regular values of ψ.

We overcome these problems by constructing suitable approximations to the nodal domains. Note that we do not use Sard's Lemma or even the Courant-Hilbert Theorem [3, Ch. VI, Sec. 6].

Remark 2. (i) We emphasize that we do not require our domains to be connected. Although connectedness of Ω was implicitly assumed in the proof sketched above, and is explicitly assumed in Section 3, there is a standard and easy way to remove this assumption. Suppose that Theorem 1 holds for connected domains and that Ω is not connected. Then either $\lambda_2(\Omega) = \lambda_2(\Omega_0)$ for some connected component Ω_0 of Ω or there exist disjoint connected components Ω_1, Ω_2 of Ω such that $\lambda_1(\Omega) = \lambda_1(\Omega_1)$, $\lambda_2(\Omega) = \lambda_1(\Omega_2)$. In the former case Theorem 1 applied to Ω_0, together with the monotonicity of $\lambda_1(D) = \lambda_2(D)$ with respect to the volume of D, yield the result. In the latter case, we use a similar argument as in the proof that $\mu_2(\Omega) \geq \mu_2(D)$, with Ω_1 in place of Ω^+ and Ω_2 in place of Ω^-, to deduce $\lambda_1(\Omega_2) \geq \lambda_2(D)$. This argument works equally well for Dirichlet and Robin boundary conditions.
(ii) We might ask if there is a minimiser of λ_2 amongst all *connected* domains. In the Dirichlet case there is none: we can find a sequence of connected domains Ω_n with $\mu_2(\Omega_n) \to \mu_2(D)$, with D being the unique (overall) minimiser of μ_2 (see [10, Sec. 4]). A similar construction works in the Robin case (see Example 3), but we cannot yet finish the argument as we do not yet know if D is the unique minimiser of λ_2 (see Remark 4).

Example 3. We construct a sequence of connected domains Ω_n of fixed volume such that $\lambda_2(\Omega_n) \to \lambda_2(D)$ (see Figure 1). Our domains are almost identical to the "dumbbells" used in [10]. Start with $D = B_1 \cup B_2$ and join B_1 to B_2 with a cylinder C_n of total volume $\frac{1}{n}$. To keep the volume of Ω_n constant, remove part of B_1 and B_2 in a small neighbourhood U_n near where they meet C_n (as in Figure 1) in such a way that the resulting boundary is still smooth. It now follows from [4, Corollary 3.7] that $\lambda_2(\Omega_n) \to \lambda_2(D)$, since the U_n can be chosen in such a way that Assumption 3.2 of [4] is satisfied.

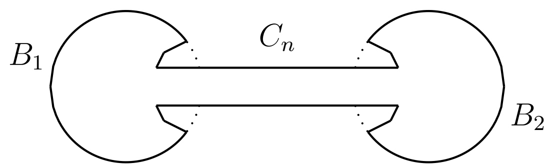

FIGURE 1. The domain Ω_n

Remark 4. We leave as an open problem the sharpness of the inequality. That is, is the domain D the *unique* minimiser of λ_2? It is for the Dirichlet Laplacian, at least up to sets of capacity zero and rigid transformations such as translations and rotations. Moreover, the inequality for the first eigenvalue of the Robin problem (1) is sharp, at least for C^2 domains (see [7, Theorem 1.1]). Our method is unlikely to yield a sharpness result as it uses approximation arguments.

3. Proof of Theorem 1

First we fix our notation. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. As noted in Remark 2(i) we may assume without loss of generality that Ω is connected. Its second eigenvalue $\lambda_2(\Omega)$ has an eigenfunction $\psi \in H^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega)$ (interior regularity is standard and continuity up to the boundary comes from combining [6, Corollary 5.5] with [12, Corollary 2.9]). Since Ω is connected, ψ changes sign in Ω, so that the open subsets $\Omega^+ = \{x \in \Omega : \psi(x) > 0\}$ and $\Omega^- = \{x \in \Omega : \psi(x) < 0\}$ are both nonempty. Set $\psi^+ := \max\{\psi, 0\}, \psi^- := \max\{-\psi, 0\}$. Then $\psi^+, \psi^- \in H^1(\Omega) \cap C(\overline{\Omega})$ and $\nabla\psi^+ \neq 0$ only on Ω^+, with a similar statement for $\nabla\psi^-$ (see [9, Lemma 7.6]). Henceforth $\lambda_1(V)$ will denote the first eigenvalue of the Robin problem (1) on the domain V. We will denote N-dimensional Lebesgue measure by $|\,.\,|$ and $N-1$-dimensional surface (Hausdorff) measure by σ.

Let B^+, B^- be balls having the same volume as Ω^+, Ω^- respectively. As sketched at the beginning of Section 2, to prove Theorem 1 it suffices to show $\lambda_2(\Omega) \geq \max\{\lambda_1(B^+), \lambda_1(B^-)\}$. Without loss of generality we only consider Ω^+ and prove $\lambda_2(\Omega) \geq \lambda_1(B^+)$.

The key idea is to attach a thin strip near $\partial\Omega$ to Ω^+ to avoid any problems when $\{x \in \Omega : \psi(x) = 0\}$ meets $\partial\Omega$. Fix $\varepsilon > 0$ and set $S_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$, where $\delta = \delta(\varepsilon) > 0$ is chosen such that $|S_\varepsilon| < \varepsilon$. Set $U_\varepsilon := \Omega^+ \cup S_\varepsilon$. Then $\partial\Omega \subset \partial U_\varepsilon$. Denote the rest of ∂U_ε by Γ_ε. Then Γ_ε is compactly contained in Ω, with $\operatorname{dist}(\partial\Omega, \Gamma_\varepsilon) \geq \delta$. Moreover, $|U_\varepsilon \setminus \Omega^+| \leq |S_\varepsilon| < \varepsilon$. Note however that Γ_ε may not be Lipschitz.

We consider the mixed problem on U_ε

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } U_\varepsilon, \\ \frac{\partial u}{\partial \nu} + \beta u &= 0 && \text{on } \partial\Omega, \\ u &= 0 && \text{on } \Gamma_\varepsilon. \end{aligned} \tag{2}$$

Denote by H_{U_ε} the space of weak solutions to (2). Then H_{U_ε} is given by the closure in $H^1(U_\varepsilon)$ (equivalently, in $H^1(\Omega)$) of $C_c^\infty(U_\varepsilon \cup \partial\Omega)$, the space of all $C^\infty(\overline{\Omega})$ functions with support compactly contained in $U_\varepsilon \cup \partial\Omega$. (Any element of H_{U_ε} may be considered an element of $H^1(\Omega)$ by extending it by zero outside U_ε.) The problem (2) then has a first eigenvalue, call it $\Lambda_1(U_\varepsilon)$, given by the variational formula

$$\Lambda_1(U_\varepsilon) = \inf_{\varphi \in H_{U_\varepsilon}} Q(\varphi, U_\varepsilon) := \inf_{\varphi \in H_{U_\varepsilon}} \frac{\int_{U_\varepsilon} |\nabla\varphi|^2\,dx + \int_{\partial\Omega} \beta\varphi^2\,d\sigma}{\int_{U_\varepsilon} \varphi^2\,dx}. \tag{3}$$

In fact for arbitrary open $V \subset \Omega$ with $\partial\Omega \subset \partial V$ and $\operatorname{dist}(\partial\Omega, \partial V \setminus \partial\Omega) > 0$, we may consider the problem (2) on V, with $\Lambda_1(V)$ and H_V given by the obvious analogues of $\Lambda_1(U_\varepsilon)$ and H_{U_ε}. We denote by $Q(\varphi, V)$ the Rayleigh quotient of (3) on V, for a given function $\varphi \in H_V$. Then we have the following important estimate for $\lambda_2(\Omega)$.

Lemma 5. *For any $\varepsilon > 0$, $\lambda_2(\Omega) \geq \Lambda_1(U_\varepsilon)$.*

The proof of Lemma 5 is based on the following characterisation of $\lambda_2(\Omega)$, combined with (3).

Lemma 6. *For any $\varepsilon > 0$, we have*

$$\lambda_2(\Omega) = \frac{\int_{U_\varepsilon} |\nabla\psi^+|^2\,dx + \int_{\partial\Omega} \beta(\psi^+)^2\,d\sigma}{\int_{U_\varepsilon} (\psi^+)^2\,dx}. \tag{4}$$

Proof. $\lambda_2(\Omega)$ satisfies

$$\int_\Omega \nabla\psi \cdot \nabla\varphi\,dx + \int_{\partial\Omega} \beta\psi\varphi\,d\sigma = \lambda_2(\Omega) \int_\Omega \psi\varphi\,dx$$

for all $\varphi \in H^1(\Omega)$. Choosing $\psi^+ \in H^1(\Omega) \cap C(\overline{\Omega})$ as a test function, we have $\nabla\psi \cdot \nabla\psi^+ = |\nabla\psi^+|^2$ in Ω (see for example [9, Lemma 7.6]) and $\psi\psi^+ = (\psi^+)^2$ pointwise in $\overline{\Omega}$. Since $\|\psi^+\|_2 \neq 0$,

$$\lambda_2(\Omega) = \frac{\int_\Omega |\nabla\psi^+|^2\,dx + \int_{\partial\Omega} \beta(\psi^+)^2\,d\sigma}{\int_\Omega (\psi^+)^2\,dx}.$$

But the integrands in the volume integrals are nonzero only if $x \in \Omega^+ \subset U_\varepsilon$. Hence (4) follows. $\square$

Proof of Lemma 5. By Lemma 6 and (3), we only have to prove that $\psi^+ \in H_{U_\varepsilon}$. Since H_{U_ε} is closed in the H^1 norm, it suffices to prove that there exist $u_n \in H_{U_\varepsilon}$ such that $u_n \to \psi^+$ in $H^1(U_\varepsilon)$.

Noting that $\psi^+ \in H^1(U_\varepsilon) \cap C(\overline{U}_\varepsilon)$ and $\psi^+ = 0$ on Γ_ε, our claim follows from a trivial modification of the proof of [2, Théorème IX.17] (see also Remarque 20 there). The only difference is that the approximating functions u_n there will have support compactly contained in $U_\varepsilon \cup \partial\Omega$ rather than U_ε (so that our limit function will lie in H_{U_ε} rather than $H_0^1(U_\varepsilon)$). $\square$

Next, since Γ_ε may not be smooth, we approximate U_ε by a suitable sequence of smooth domains $U_n \subset U_\varepsilon$ such that $\Lambda_1(U_n) \to \Lambda_1(U_\varepsilon)$.

Lemma 7. *There exists a sequence of Lipschitz domains $U_n \subset U_\varepsilon$ such that $\partial\Omega \subset \partial U_n$ and $\operatorname{dist}(\partial\Omega, \partial U_n \setminus \partial\Omega) > 0$ for all n, $|U_\varepsilon \setminus U_n| \to 0$ and $\Lambda_1(U_n) \to \Lambda_1(U_\varepsilon)$ as $n \to \infty$.*

Proof. The existence of the U_n converging in volume is a standard result (see for example [8, Ch. V, Theorem 4.20]); note that since $\partial\Omega$ and Γ_ε are separated, this is equivalent to having a sequence of the form $\mathbb{R}^N \setminus (\Omega \setminus U_n)$ converge to $\mathbb{R}^N \setminus (\Omega \setminus U_\varepsilon)$. Moreover, the U_n can be chosen such that $\{x \in U_\varepsilon : \operatorname{dist}(x, \Gamma_\varepsilon) > \frac{1}{n}\} \subset U_n$.

Comparing the characterisation (3) for U_n and U_ε, and since $H_{U_n} \subset H_{U_\varepsilon}$, we have $\Lambda_1(U_n) \geq \Lambda_1(U_\varepsilon)$. So we only have to prove that

$$\limsup_{n\to\infty} \Lambda_1(U_n) \leq \Lambda_1(U_\varepsilon).$$

Fix $\delta > 0$ and choose $\varphi \in H_{U_\varepsilon}$ such that the Rayleigh quotient $Q(\varphi, U_\varepsilon) < \Lambda_1(U_\varepsilon) + \delta$. By density, we may assume $\varphi \in C_c^\infty(U_\varepsilon \cup \partial\Omega)$.

Then $\operatorname{supp}\varphi$ and Γ_ε are compact and $\operatorname{dist}(\operatorname{supp}\varphi, \Gamma_\varepsilon) > 0$. In particular, $\operatorname{dist}(\operatorname{supp}\varphi, \Gamma_\varepsilon) > \frac{1}{n}$ for all $n \in \mathbb{N}$ large enough. By choice of the U_n it follows that $\operatorname{supp}\varphi$ is compactly contained in $U_n \cup \partial\Omega$ for n large enough.

In particular, $\varphi \in C_c^\infty(U_n \cup \partial\Omega)$, whence $Q(\varphi, U_\varepsilon) = Q(\varphi, U_n) \geq \Lambda_1(U_n)$, for all n large enough (where we have used $\partial\Omega \subset \partial U_n$ to get $Q(\varphi, U_\varepsilon) = Q(\varphi, U_n)$). That is, for any $\delta > 0$, $\Lambda_1(U_n) < \Lambda_1(U_\varepsilon) + \delta$ for all n large enough. □

Choose a sequence of U_n as in Lemma 7 and consider the Robin problem (1) on U_n. Since ∂U_n is Lipschitz for each n, we may use the Faber-Krahn inequality for Robin problems and then pass to the limit. So let B_n be a ball of the same volume as U_n. Then $\lambda_1(U_n) \geq \lambda_1(B_n)$ for all n, by [5, Theorem 1.1]. Moreover, by a direct comparison of variational formulae and since $H_{U_n} \subset H^1(U_n)$, $\Lambda_1(U_n) \geq \lambda_1(U_n)$.

Now let B_ε be a ball having the same volume as U_ε. As $n \to \infty$ and $|U_n| \to |U_\varepsilon|$, we have $|B_n| \to |B_\varepsilon|$. Since the first eigenvalue of (1) on the ball depends continuously on the size of the ball, $\lambda_1(B_n) \to \lambda_1(B_\varepsilon)$. By Lemma 7, $\Lambda_1(U_n) \to \Lambda_1(U_\varepsilon)$. Putting this all together we have

$$\Lambda_1(U_\varepsilon) \leftarrow \Lambda_1(U_n) \geq \lambda_1(U_n) \geq \lambda_1(B_n) \to \lambda_1(B_\varepsilon),$$

that is, $\Lambda_1(U_\varepsilon) \geq \lambda_1(B_\varepsilon)$.

Finally, let $\varepsilon \to 0$. Since $|U_\varepsilon| \to |\Omega^+|$, we have $|B_\varepsilon| \to |B^+|$ and so $\lambda_1(B_\varepsilon) \to \lambda_1(B^+)$. Also, since $\lambda_2(\Omega) \geq \Lambda_1(U_\varepsilon)$ by Lemma 5, we conclude $\lambda_2(\Omega) \geq \lambda_1(B^+)$. In light of our earlier comments, this completes the proof of Theorem 1.

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