EXCURSIONS TO AND FROM SEMANTIC OBLIVION

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This report offers some recent experiences from the author's teaching that highlight the fragility of language and mathematical formalism in communicating ideas and concepts. Syntactic reasoning is at the superficial end of the spectrum, "skating on the surface", involving formal manipulation of symbols, simple rules and substitutions. Semantic reasoning is deeper, "diving down towards the seabed", drawing conclusions from underlying meanings and heuristics. The author argues that illuminating the tension and interplay between these complementary modes of reasoning, and creating heightened awareness, may enhance approaches to successful learning, improve morale and attitudes, and lead to more robust outcomes and willingness to engage in challenging mathematical activities.

INTRODUCTION AND CONTEXT

It is common to experience frustration or feel demoralised when calculations or mathematical arguments inevitably go awry, make little or no sense, or appear to lack relevance or significance. It is unfortunate that what in fact may be natural states of incomprehension, or apparent "chaotic mindlessness", can become painful, have negative connotations and a tendency to undermine confidence and put students off mathematics, even permanently. People are not stupid simply because they cannot comprehend explanations, even when delivered with care and diligence by an experienced teacher. They are not hopeless mathematical thinkers just because they become lost or "frozen" in attempting to create their own mathematical solutions or arguments, even after feeling that they have already achieved a reasonable degree of comprehension. Mathematics is inherently difficult and the creative processes that lead to successful communication are fragile and easily corrupted. Language is the medium of communication, and, through *syntax* (which includes grammar and formal rules of manipulation and deduction), *semantics* (which includes meaning and any underlying ideas or heuristics) may be conveyed in various degrees of approximation. The nexus between syntax and semantics is poorly understood, and there has been a plethora of attempts to explore it (going back to Frege (1892), and see, for example, Chomsky (1957), Heim and Kratzer (1998) and a discussion of the *syntax-semantics interface* in Escribano (1999)) and even develop a specialised theory of formal languages that intertwine the two through the notion of a *syntactic congruence*, a little more about which is explained below.

Wigner (1960), in a physics context, writes about the "miracle of the appropriateness of the language of mathematics" in an influential paper whose title refers to the "unreasonable effectiveness of mathematics in the natural sciences". But this apparent appropriateness or effectiveness comes at a considerable expense: it is easy

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to overlook hundreds or (in the case of calculus) thousands of years of mathematical evolution. What might seem natural or inevitable in hindsight is the result of many minds tilling the mathematical soil and making incremental contributions and adding flashes of inspiration to a creative effort that has been refined over many generations. Whilst we may look to certain landmarks or individuals as unlocking keys to the development of mathematics, one should not ignore the historical context and framework in which discoveries are made. As Newton famously remarked: "If I have seen further, it is because I have stood on the shoulders of giants." The metaphor of standing on the shoulders of giants dates back to the twelfth century (see, for example, Merton (1965)), and is intended to pay tribute to an historical continuum. In the author's opinion, the historical context of mathematics should have as much prominence in the classroom as the mathematics itself, at least in introductory phases.

It is too easy (and especially dangerous in the context of teaching weak or inexperienced students) to take modern mathematical notation and terminology for granted and lose sight of the significance of a variety of conceptual advances that seem to us now quite trivial, but, in their day, were substantial breakthroughs. A symbol for zero was introduced in about 800 AD, and, up until about the sixteenth century, solutions to quadratic equations took many lines and several cases to write down and explain (see Stillwell (1989) for an excellent historical account of this and the many variations, and Fitzgerald (2010) for a beautiful anecdote of the role of zero in putting him off mathematics at a young age). Once modern mathematical language had established itself, combined with the axiomatic method initiated by Euclid (see, for example, Artmann (1999)), it seemed inevitable that leading mathematicians, such as Hilbert, would ask whether mathematics could be reduced to formal manipulation of expressions and axioms (see, for example, Ewald (1996)). Gödel (1931) essentially proved that interesting mathematics could not be trivialised in this way (his celebrated Incompleteness Theorem, which is really the basis of all undecidability results, including, for example, the unsolvability of the word problem). His discoveries tell us that, in a sense that can be made precise, there is an unbridgeable gulf between syntax and semantics. It is no surprise that students of mathematics, and practising mathematicians, want to minimise effort by using, wherever possible, syntactic or "formulaic" methods. These methods however are inherently fragile and inexperienced students frequently come unstuck. The established mathematician, by contrast, has a wealth of semantic knowledge and experience, combined with well lubricated technique, and is able to use syntax expertly to move quickly and economically through series of deductions and just use semantics at a few pivotal points in a calculation or extended argument. The author contends that stimulating awareness of these two contrasting modes of reasoning should enhance learning. Including the terminology "syntactic reasoning" and "semantic reasoning" in classroom practice and parlance may assist in highlighting levels of depth and degrees of importance of certain ideas or techniques.

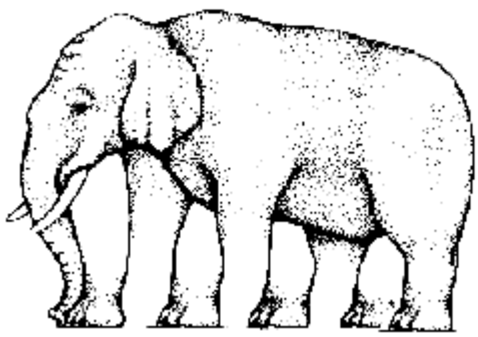
At this point, we remark about the origin of this terminology from the theory of formal languages, used for example in theoretical computer science (see, for example, Eilenberg (1974)). Let Σ be an alphabet and denote by Σ^* the collection of all *words* over Σ, by which we mean formal strings of symbols from Σ. A *formal language L* is just a subset of Σ^*. Two words v and w from Σ^* are called *syntactically congruent* with respect to L, and we write $v \sim_L w$, if substituting one for the other in any given context does not affect membership of L, that is,

$$(\forall s, t \in \Sigma^*) \quad svt \in L \quad \Leftrightarrow \quad swt \in L\ .$$

This captures precisely the idea of v and w being equivalent "synonyms" with respect to L. For example, if L comprises all well-formed sentences in English, then all nouns become (syntactically) congruent, all verbs become congruent, but a verb and a noun will not be congruent. It is not difficult to modify L (for example, by only including fruit in the vocabulary of nouns used to make sentences) so that all names of fruit become congruent, all non-fruit are congruent, but a fruit and non-fruit are not congruent with respect to L, and then modify L again to distinguish, say, apples and oranges up to congruence. Modifying the language changes the syntactic congruence $\sim_L$ on words from Σ^*. It then becomes fruitful to collectively study so-called *streams* or *varieties* of languages (see Eilenberg (1974)). The syntactic congruence classes with respect to a fixed L form a monoid under concatenation, denoted by $M_L = \Sigma^*/\sim_L$ and called the *syntactic monoid* of L. The relationships between formal languages and syntactic monoids are well studied and lead to an elegant and rich theory. For example, a language is regular (built from singletons using boolean operations, concatenation and star) if and only if it is recognised by a finite state automaton, and this occurs if and only if its syntactic monoid is finite. Algebraic properties of the monoid M_L may be regarded as closely related to the underlying semantics of L, regardless of how L is described in terms of syntax or grammar. In this way syntax leads to semantics. We will not pursue this any further here, but the point is that the idea of simple substitution of words in context provides a test for syntactic congruence. Whenever we perform mathematics by making a simple substitution, disregarding meaning, we use *syntactic reasoning*. Whenever we make a mathematical deduction using underlying heuristics or meaning, we are applying *semantic reasoning*. It becomes very interesting when errors creep in, making incorrect substitutions or applying invalid heuristics. In a certain sense, all of the examples discussed in the remainder of this paper are pathological. But studying pathology is illuminating and strengthens our understanding of everyday phenomena, just as, for example, Oliver Sacks (1995) draws our attention to extreme examples of behaviour in neuropsychology, or a mathematician tests the boundaries of his or her theory using counterexamples (see, for example, Gelbaum and Olmsted (1990)). Errors and misconceptions are interesting and revealing, not just of a student's current state of knowledge or understanding, but of the process of thinking itself, and strategies for tackling difficult or sophisticated mathematical problems. Rather than regarding the tension between syntactic and semantic reasoning as a nuisance or source of frustration, one can exploit the differences to create opportunities to enhance learning and expose weaknesses or gaps in understanding. Almost always, in the author's experience, errors in reasoning tell us more than we imagined and their resolution makes us more robust and creative in the long run.

EXCURSIONS TO AND FROM OBLIVION

The following figure of an elephant (downloaded from CoolOptical Illusions (2009)) with an indeterminate number of legs is a fine illustration of how slight perturbations of syntax (in this case the way feet are joined to legs in an outline of an elephant) can make an elementary question such as "How many legs does an elephant have?" difficult, if not impossibly difficult to answer, or even meaningless.



One could speculate how the artist came up with this figure. Possibly it was an intentional variation of the famous trident illusion, or it could be just that the artist misplaced the drawing of one of the hind feet, because of a suitable gap (all gaps are syntactically congruent!) and then proceeded to fill in some of the other gaps, and then realised the error leads to a pleasant illusion. This is an artificial example, but our students may have no warning, in natural contexts, to help them recognise when something we tell them, or something they do themselves, is slightly "out of tune", or when a seemingly innocuous question has not been properly formulated. The associated anxiety and feelings of helplessness can be anything but pleasant, and further compounded by exam or (the equivalent of) stage fright.

Introducing calculus recently to students of agriculture, the author set an assignment, separated into parts, that explains the well-known *Rule of 70.* This rule tells an investor that approximately $70/i$ years is required to double an investment compounded at i % annually. For example, at 1% and 7% interest, one expects the investment to double in value after 70 and 10 years respectively. Students were asked to manipulate the equation

$$2P = P\,(1 + i/100)^t$$

where P dollars is the principal invested and t the number of years for it to double. After one particular class, the author was visited by a contingent of students in distress, unable to progress through an early step of the assignment, which was to eliminate P from both sides of this equation. The leader of the contingent was well spoken and articulate, and said "We have taken P away from both sides, to get

$$P = (1 + i/100)^t$$

and do not know what to do now.” The author replied: “Are you sure? What did you do to the left-hand side?” Student: “We took *P* away.” Author: “Hmmm...” The student elaborated: “If you take one *P* away from two *P*’s you get one *P*. We don’t know what to do with the *P* that’s left.” The group, about half a dozen students, nodded in perplexed unison. Author: “But that is subtraction. What did you do on the right-hand side?” The penny dropped and they suddenly realised that they had confused division and subtraction together in the idea of “taking away”. The error was syntactic, in terms of formally manipulating symbols. However, there was probably a semantic component in being tempted to use subtraction: it is a common heuristic when introducing abstract variables such as *P*’s and Q’s, to think of them like apples and oranges. If you have two oranges and take one orange away, you are left with one orange. This is a powerful heuristic and may have infected the syntax of this simple first step (even though it is less effort just to cross the *P* out). This incident is also interesting, psychologically, because there were several students in the group asking for help, and it seems surprising that not one of them noticed the error, and all were eager for assistance. Magicians and other performers exploit the fact that it is remarkably easy to fool or distract a large number of people. The group misconception here was unintentional, but the student spokesman probably had led the discussion prior to the group visiting the author, and he was confident and well-spoken. The correct cancellation of *P* leads to the equation

$$2 = (1 + i/100)^t$$

and then the next steps of the assignment required students to express t in terms of i,

$$t = \ln 2 / \ln(1 + i/100) ,$$

and finally use a tangent approximation (equivalent to the linear term of a Taylor series) to make the substitution

$$\ln(1 + i/100) \approx i/100 ,$$

and so deduce the *Rule of 70*:

$$t \approx \ln 2 / (i/100) = 100 \ln 2/i \approx 70/i .$$

The extended exercise involved formal manipulation of equations (or approximations) and substitutions, so was particularly ripe for syntactic errors. Other students in the class found the fraction $i/100$ especially problematic and there were many stumbles and errors, particularly when differentiating and using the Chain Rule. The author believes that successful technique and understanding involving *fractions* lead to one of the key *threshold concepts* in mathematics, in the sense of Meyer and Land (2005). Many obstacles to teaching and learning introductory calculus would evaporate if students had already successfully passed through the crucial “fraction portal”. Unfortunately this has such a “primary school”, and therefore derogatory, connotation in a tertiary setting, that we don’t pay enough attention to it at university. (It seems ironic to the author that he also delivers honours courses in which talented and gifted students at university learn about fields and modules of fractions, without any hint of stigma, involving precisely the primary school concepts that are required for a successful introduction to calculus.)

In the final exam for the same unit of study as the previous assignment example, students were asked to perform some very routine differentiations. Asked to find the derivative y' when

$$y = \frac{1}{3x} ,$$

one student wrote the following:

$$y = \frac{1}{3x} = 3x^{-1}, \quad y' = -3x^{-2} = \frac{1}{-3x^2} .$$

This is an interesting and strangely beautiful answer and one can speculate about the student's reasoning or thought processes. The author suspects (but is not absolutely sure) that, under exam pressure, this answer was produced hastily and with at best superficial and formulaic attention to detail. If one could have been a fly on the wall and asked the student at the time what he or she was thinking, most probably the phenomenon would have been interrupted and the answer spoilt. The final expression for y' is correct, but the steps in the reasoning, if interpreted literally, become semantic nonsense. Most probably the student used syntactic substitutions in which the two errors cancelled out. If the student had bracketed correctly,

$$y = \frac{1}{3x} = (3x)^{-1} ,$$

the most natural next step would involve the Chain Rule. It seems reasonable to assume the Chain Rule did not enter the student's mind and that only the formula for differentiating x^{-1} was consciously applied. If this is the case, then a very slight modification to the answer would deserve full marks:

$$y = \frac{1}{3x} = 3^{-1}x^{-1}, \quad y' = -3^{-1}x^{-2} = \frac{1}{-3x^2} .$$

This then would be quite a sophisticated answer, deliberately avoiding the Chain Rule, and providing evidence of understanding of fractions, their manipulation and exponential notation. The author intends, in the future, to ask inexperienced students to analyse this example, and variations, as exercises. This example is interesting also in challenging the marker to appropriately and fairly assess the answer. Does it deserve 0, 0.5, 1 or perhaps even 1.5 marks out of 2? (It is not completely correct, regardless of how one interprets the student's reasoning.) Does one reward or penalise syntax or semantics? Under what circumstances does one have priority over the other? Should this be in the consciousness of the person designing the assessment?

These examples so far involve students who are inexperienced or weak at mathematics. The author believes however that the tension between syntax and semantics is a universal phenomenon concerning human communication and affects all of us, regardless of our experience or ability. The author set a difficult assignment question in abstract algebra for talented and gifted honours students:

Assignment Exercise: Prove <u>*Blah blah blah.*</u>

The author received an elaborate answer from a talented student that separated into two halves connected by an "isthmus":

$$(1 - e)m + em \ = \ 1 \ .$$

Each half was meticulously correct and involved sophisticated ideas and reasoning from the course, with good technique and correct semantics. The isthmus however involved an exceedingly simple "syntactic" cancellation error, typical of mistakes all of us make when routinely simplifying algebraic expressions. The brackets should have been expanded before cancelling, and the left-hand side becomes m, not 1, in which case the whole argument, as it turns out, unravels and falls apart. (The 1 on the right-hand side turned out to be crucial for the second half of the solution offered.) However there was a problem with the student's semantics, because at the end of the second half, he claimed to have proved *Super Duper Blah blah blah*, which in fact is false (only *Blah blah blah* is true). If he had thought a little about examples from the course, he would have realised that he must have fallen into error, and then searched through and found it and abandoned this particular solution. In fact, the density of his writing meant that the isthmus was buried somewhere in the middle of the argument and difficult to locate. The author awarded this answer 6/10, removing marks for the error and failure to realise that the final conclusion *Super Duper Blah blah blah* was absurd. If this answer were an arithmetic calculation, for example, and it was not obvious from the context of the problem that the final answer was incorrect, then the error could be regarded as very slight indeed and the student might get 9.5/10. This was interesting also because the entire honours class was asked to peer review each other's work. All of the student's peers commented that this particular answer was worthless (0/10) on the basis that the conclusion was ridiculous. Because of the density and length of the answer, none of the peers appeared to have the energy or inclination to try to locate the error or read and verify the careful reasoning in each of the halves surrounding the isthmus. This kind of example highlights the difficulties inherent in distinguishing between syntactic and semantic reasoning, the relative worth of industrious mathematical activity that leads to dead ends, and subtle issues about assessment and feedback.

We will finish with a story involving the author (and published in Easdown (1985)). Many years ago the author's wife asked him to put the kettle on to make a cup of tea. He did so willingly, and a few minutes later the apartment rapidly filled with smoke followed by an explosion of flames. He had taken an electric kettle, filled it with water and placed it on an electric plate on the stove and turned the stove on (instead of plugging the kettle in to a power point and turning on the switch in the wall). His mind was distracted for some reason, and he managed to get inequivalent operations mixed up, made a simple syntactic error involving electrical equipment and nearly burnt the house down. His error was not dissimilar to the students confusing multiplication and subtraction in an earlier example, though with potentially much more catastrophic consequences! One of the wonderful aspects of mathematics is that spectacular errors can be harmless and exquisite adventures that take place in the mind. We have much to learn from them and they are to be celebrated.

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