Growth of Rees quotients of free inverse semigroups defined by small numbers of relators

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Abstract: We study the asymptotic behaviour of a finitely presented Rees quotient $S = \mathrm{Inv}\langle A \mid c_i = 0 \ (i = 1, \ldots, k) \rangle$ of a free inverse semigroup over a finite alphabet A. It is shown that if the semigroup S has polynomial growth then S is monogenic (with zero) or $k \geq 3$. The three relator case is fully characterised, yielding a sequence of two-generated three-relator semigroups whose Gelfand-Kirillov dimensions form an infinite set, namely $\{4, 5, 6, \ldots\}$. The results are applied to give a best possible lower bound, in terms of the size of the generating set, on the number of relators required to guarantee polynomial growth of a finitely presented Rees quotient, assuming no generator is nilpotent. A natural operator is introduced, from the class of all finitely presented inverse semigroups to the class of finitely presented Rees quotients of free inverse semigroups, and applied to deduce information about inverse semigroup presentations with one or many relations. It follows quickly from Magnus' Freiheitssatz for one-relator groups that every inverse semigroup $\Pi = \mathrm{Inv}\langle a_1, \ldots, a_n \mid C = D \rangle$ has exponential growth if $n > 2$. It is shown that the growth of Π is also exponential if $n = 2$ and the Munn trees of both defining words C and D contain more than one edge.

1. Introduction

Inverse semigroups were introduced in the 1950s independently by Preston [23] [24] [25] and Wagner [35] [36], though their origins have been traced to much earlier times by the scholarly work of Lawson [16] in the context of partial symmetry and ordered groupoids. As a class of algebraic structures, inverse semigroups fall between groups and semigroups, but have their own distinct flavour and techniques that do not obviously lift from either of these classes. This is especially apparent in the description of elements and their multiplication in free inverse semigroups over an alphabet, combining both concatenation of words (as in free semigroups) and word reduction (as in free groups), but also other nonobvious ingredients. A beautiful and elegant solution to the word problem for free inverse semigroups was provided by Munn in a seminal paper [19], using birooted word trees (the so-called *Munn trees*), which are directed graphs whose edges are labelled by letters from an alphabet, with certain restrictions and initial and terminal vertices. Munn trees are exceedingly easy to visualise and manipulate, and provide the foundation for building the sophisticated techniques required for this paper. Of course, all inverse semigroups are quotients of free inverse semigroups, so an attempt to fully understand the behaviour of Rees quotients is a natural first step.

Denote the class of finitely presented Rees quotients of free inverse semigroups by $\mathfrak{M}_{FI}$ (defined carefully in terms of presentations in the next section). In [30] the authors initiated the study of growth of semigroups from $\mathfrak{M}_{FI}$ and the relationship with satisfiability of identities. Growth was shown to be polynomial or exponential and an algorithmic criterion given to recognise the type of growth. This work was continued in [31], refining and introducing new algorithmic criteria, culminating in the proof that polynomial growth occurs if and only if the semigroup in question has bounded height (in the sense of Shirshov [27]), and finding another equivalent condition in terms of the geometric structure of nilpotent elements.

Lau [13] [14] proved the rationality of the growth series of semigroups from $\mathfrak{M}_{FI}$ having polynomial growth. In [15], he showed that the Gelfand-Kirillov dimension in those cases may be any integer $n \geq 3$, where n is the degree of growth of some semigroup with $n-2$ generators and $f(n)$ relators, where f is a quadratic function.

The second author in [29] established that the set of finite Gelfand-Kirillov dimensions of arbitrary infinite Rees quotients of free inverse semigroups is $\{3\} \cup [4, \infty)$.

In [8], the authors proved that the semigroup $S = \mathrm{Inv}\langle a, b \mid ab = 0\rangle$ is, up to isomorphism, the unique principal Rees quotient of a free inverse semigroup that is not trivial or monogenic and satisfies a nontrivial identity in signature with involution. In fact, the semigroup S has exponential growth, as do all other non-monogenic one-relator semigroups from $\mathfrak{M}_{FI}$ (see [8]). The present article may be considered a continuation of the work in [8], by studying semigroups from $\mathfrak{M}_{FI}$ that can be defined by small numbers of relators, though the results focus on polynomial growth and move in several different directions. This present article also combines and unifies, in one place, many of the criteria and main results from [30] [31], with a variety of illustrations and applications.

In Section 2, all of the main definitions and ingredients are gathered together. The methods rely heavily on a graphical technique that is a modification of an idea due to Ufnarovsky [33] [34] (see also [20, Chapter 24]) in the setting of monomial algebras. This idea has wide applicability and arises in other settings (see, for example, De Bruijn [7] and [17, Chapter 1] where the terminology *De Bruijn graph* is introduced). A related construction is used by Gilman [9] for calculating degrees of growth and solving a word problem in a class of groups and monoids given by certain finite presentations. Properties of one of the three-relator non-monogenic inverse semigroups having polynomial growth and belonging to $\mathfrak{M}_{FI}$ are analysed in this section.

Section 3 is a short analysis, using elementary combinatorial properties of words, to characterise free generation in free inverse semigroups by a pair of reduced words. This gives a useful necessary condition for presentations of semigroup from $\mathfrak{M}_{FI}$ having polynomial growth which is applied repeatedly for sieve procedures in the following sections. Section 4 is also short and introduces an important general class of three relator semigroups, where the third relator is described in terms of a parameter γ. This class of examples, together with the semigroup T discussed in Section 2, characterise three relator semigroups from our class that have polynomial growth. This characterisation is the topic of Section 6, which is the longest and most detailed section of the paper. This analysis is preceded, in Section 5, by one of our main theorems that tells us that the presence of at least three relators is a necessary condition for polynomial growth for semigroups from our class.

In Section 7 we return to the class described in Section 4, and exhibit a sequence of three relator semigroups and prove that the set of their Gelfand-Kirillov dimensions is infinite and consists of every integer value greater than or equal to four. This is in contrast with Lau's examples [15] in which the number of generators and relators is nonfixed and increases together with the Gelfand-Kirillov dimension. We prove further, in the general three relator case, assuming a natural irredundancy condition on the presentations, that only integer values greater than or equal to four arise as Gelfand-Kirillov dimensions (by excluding the value three). These results are of topical interest in light of recent activity exhibiting and calculating irrational or non-integer Gelfand-Kirillov dimensions in a variety of settings. Belov and Ivanov [3] [4] constructed the first examples of finitely presented semigroups having non-integer Gelfand-Kirillov dimension. Bartholdi and Reznykov [1] gave an example of a semigroup with irrational Gelfand-Kirillov dimension associated with Mealy automata with two nontrivial states over a two-letter alphabet and satisfying the periodicity identity $x^4 = x^6$. Sidki [32] found an example of a nil semigroup satisfying the identity $x^5 = 0$, generated by two functionally recursive matrices over the integers and having the same Gelfand-Kirillov dimension as the Bartholdi-Reznykov semigroup.

In Section 8, an operator $\mathcal{Z}$ is introduced that takes a finitely presented inverse semigroup Π and produces a homomorphic image $\mathcal{Z}(\Pi)$ within the class of finitely presented Rees quotients of free inverse semigroups. As an application of one of our main theorems, we deduce that, under a mild constraint on word trees, an inverse semigroup defined by one relation has exponential growth. This constraint however is necessary, because of another paper [28] by the second author where he exhibits a one relator nonmonogenic inverse semigroup with polynomial growth. In Section 9, the operator $\mathcal{Z}$ is employed again to deduce further information about finitely presented inverse semigroups with many relations. The underlying result is a quadratic lower bound, in terms of the number of generators, on the number of relations required to guarantee polynomial growth for a finitely presented Rees quotient of a free inverse semigroup in which none of the generators is nilpotent.

2. Preliminaries

We assume familiarity with the basic definitions and elementary results from the theory of semigroups, which can be found in any of [5], [10], [11] or [21]. Throughout let A be a finite alphabet containing at least two letters and put

$$B = A \cup A^{-1}$$

where the elements of A^{-1} are formal inverses of corresponding elements of A and vice-versa (so A and A^{-1} are disjoint and any a in A may also be denoted by $(a^{-1})^{-1}$). Let k be a positive integer and suppose that $c_1, \ldots, c_k \in B^+$. Consider the inverse semigroup S with zero given by the following finite presentation:

$$S \;=\; \mathrm{Inv}\langle\, A \mid c_i = 0 \;\text{ for }\; i = 1 \ldots, k \,\rangle\,. \qquad (*)$$

In this paper we only consider presentations within the class of inverse semigroups. Because presentations of the form $(*)$ occur so often in this paper we abbreviate the notation to write

$$S \;=\; \langle\, A \mid c_i = 0 \;\text{ for }\; i = 1 \ldots, k \,\rangle\,.$$

The words $c_1, \ldots, c_k$ are called (*zero*) *relators*. Observe that S may be regarded as (isomorphic to) the Rees quotient of the free inverse semigroup FI_A generated by A with respect to the ideal generated by the relators. The class of finitely presented inverse semigroups with zero defined by presentations $(*)$ may now be formally referred to as $\mathfrak{M}_{FI}$.

The *content* of a word $w \in B^*$, denoted by $\mathrm{content}(w)$, is the set of letters from A which appear in w or w^{-1}. If $w_1, \ldots, w_n \in B^*$ then denote by $(w_1, \ldots, w_n)$ the subsemigroup of B^* generated by $w_1, \ldots, w_n$, which we may regard as a subset of FI_A or of S in context. In contrast, denote by $\mathrm{Inv}\langle w_1, \ldots, w_n\rangle$ the inverse subsemigroup of S generated by $w_1, \ldots, w_n$. We use the symbol $\underline{\circ}$ to denote literal equality of words, that is, $w_1 \,\underline{\circ}\, w_2$ means that words w_1 and w_2 coincide letter by letter. If $v, w \in B^*$ and $x \,\underline{\circ}\, xvy$ for some $x, y \in B^*$ then we call v a *subword* (or *factor*) of w. The number of letters in a word w is denoted by $|w|$. Recall that w is *reduced* if w does not contain xx^{-1} as a subword for any letter $x \in B$, and that w is *cyclically reduced* if w and w^2 are both reduced (whence all powers of w are reduced).

Reference to Green's relation $\mathcal{J}$ throughout will be with respect to FI_A. Call a word u a *divisor* of a word v if the equation $v = sut$ holds in FI_A for some $s, t \in B^*$. For any set X of words, put

$$\mathrm{div}\, X \;=\; \{\,\text{divisors of words from } X\,\}\;,$$

elements of which are referred to simply as *divisors of* X. Recall that elements of FI_A may be regarded as birooted word trees (introduced for the first time in [19] and referred to also as *Munn trees*), the terminology and theory of which are explained in [10] (see also [30, Section 2]). As in [30], denote the word tree of a word w over B by $T(w)$. Two words are $\mathcal{J}$-related if and only if their word trees are identical. If u and v are words, then $T(u)$ is a subtree of $T(v)$ if and only if u divides v. A *chain* is a word tree that is either a single vertex (the word tree of the empty word) or one in which all vertices have degree 2 except for two leaves (at the respective ends of the chain) that have degree 1. The chain that consists of a single vertex is called *empty*. Recall also that an element s of a semigroup S with zero is *nilpotent* if some power of s is zero.

When writing about or using presentations of the form $(*)$ in the text of this paper, we make the following underlying assumptions:

(i) The alphabet A is finite and $|A| \geq 2$.
(ii) The number k of relators is at least one.
(iii) No relator is $\mathcal{J}$-equivalent to a single letter from A. (In particular, this guarantees that S is not a free monogenic inverse semigroup with zero.)
(iv) No relator $\mathcal{J}$-divides any other relator in the presentation for S. (If this were not the case then we could delete a relator without changing the Rees quotient.)
(v) At least one relator is $\mathcal{J}$-equivalent to a reduced word.

These assumptions may be referred to collectively as the *irredundancy* of the presentation. A useful consequence of (iii) is that every relator contains a reduced subword of length 2 and no relator can have the form aa^{-1} or $a^{-1}a$ for $a \in A$. Condition (v) is included, because if it failed then there would exist at least two letters $a, b \in A$ that generate a noncyclic free subsemigroup (see remarks following Theorem 2.1 of [31]), so that the growth of S would become exponential for a trivial reason, and the presentation would not be interesting from our point of view.

Remark 2.1. The following fact is used implicitly in some of the arguments in the paper, where we exchange some letters with their formal inverses as generators. Suppose that S is given by the presentation $(*)$ and write $A = \{a_1, a_2, \ldots, a_n\}$ where $|A| = n$. Put $A' = \{a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}\}$ where $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$. Then, interpreting formal inversion of generators in the usual way, it is clear that

$$B \;=\; A' \cup (A')^{-1}$$

and

$$S \;=\; \mathrm{Inv}\langle\, A' \mid c_i = 0 \;\text{ for }\; i = 1 \ldots, k \,\rangle\,.$$

We also use (implicitly) the facts that word inversion and word reversal both induce anti-isomorphisms from S to its dual semigroup.

We recall now some basic definitions and facts about growth of semigroups. Consider a semigroup S generated by a finite subset X. The *length* $\ell(t)$ of an element $t \in S$ (with respect to X) is the least number of factors in all representations of t as a product of elements of X, and

$$g_S(m) = \big|\,\{\, t \in S \mid \ell(t) \leq m \,\}\,\big|$$

is called the *growth function* of S. Recall that S has *polynomial growth* if there exist natural numbers q and d such that $g_S(m) \leq qm^d$ for all natural numbers m, and *exponential growth* if there exists a real number $\alpha > 1$ such that $g_S(m) \geq$

α^m for all sufficiently large m. These definitions and notions of growth apply also to subsets of S in an obvious way. Throughout this paper, we may implicitly assume that zero lies in the generating set of any semigroup with zero (so that zero always has length one). If $t \in S$ and the equation $t = w$ holds in S, where w is a product of $\ell(t)$ elements of X regarded as a word over the alphabet X, then we call the word w a *geodesic* for t (with respect to X).

The control of geodesics is necessary in estimating growth. In our context of Rees quotients, it then becomes important to locate geodesics in free inverse semigroups. Any given element w of FI_A may also be expressed as

$$w = u_1 u_1^{-1} u_2 u_2^{-1} \dots u_r u_r^{-1} \overline{w}$$

for some nonnegative integer r and reduced words $u_1, \dots, u_r, \overline{w}$. If r is as small as possible, so that no u_i can be an initial segment of u_j for $i \neq j$, then the previous expression for w is called the *Schein (left) canonical form of* w (see [26]), which is unique up to order of idempotents. Typically the Schein canonical form is a long way from being geodesic. In fact a geodesic $\widetilde{w}$ for w in FI_A has the form

$$\widetilde{w} \circeq u_0 e_1 u_1 e_2 \dots e_n u_n$$

where $n \geq 0$ is the number of idempotents in the Schein canonical form,

$$\overline{w} \circeq u_0 u_1 \dots u_n \qquad \text{and} \qquad \overline{e_1} \circeq \dots \circeq \overline{e_n} \circeq 1$$

(so that $e_1, \dots e_n$ represent idempotents). Geodesics in free inverse monoids were considered by Choffrut [6, Section 3.2], and also by Poliakova and Schein in [22], where they gave a rewriting system that reduces an arbitrary word in the free inverse semigroup to its geodesic form (which is unique up to the representation of idempotents). To form each e_i, one systematically traces the labels of edges of the branches of the word tree that emanate from the geodesic chain of $T(w)$ labelled by $\overline{w}$ at the vertex pointed to by the last letter of u_{i-1} (or the initial vertex if $i = 1$ and u_0 is empty). The word e_i is not unique unless there is exactly one branch emanating from that vertex and that branch is a chain, because one may arbitrarily choose the order of multiple subbranches to trace from any given vertex. The number of occurrences of a given letter in e_i however is unique. In all of our examples, the geodesics will be clear from context and the reader is encouraged to draw diagrams of the associated Munn trees.

Now consider the case that S has polynomial growth with respect to a finite generating set X. In this case, the limit

$$GKdim(S) \;=\; \lim_{m \to \infty} \sup \frac{\log g_S(m)}{\log m}$$

is finite and called the *Gelfand-Kirillov dimension* of S (see [2] and [12]). In all the cases S considered in this paper with polynomial growth, we can find positive real numbers α and β and a positive integer k such that, for all sufficiently large m,

$$\alpha m^k \ \le\ g_S(m)\ \le\ \beta m^k\ ,$$

and then it will follow immediately that $GKdim(S) = k$.

We recall briefly the classical notion of *bounded height* (introduced originally by Shirshov [27]). Let X be a subset of a semigroup S. Consistent with our notation introduced earlier, denote by (X) the subsemigroup of S generated by X. If $s \in (X)$ can be expressed as a product

$$s \ =\ h_1^{\alpha_1}\dots h_k^{\alpha_k}$$

for some $h_1,\dots,h_k \in X$ and positive integers $\alpha_1,\dots,\alpha_k$, and k is as small as possible, then we say the *height of s with respect to X* is k. We say that a subset K of S has *height bounded by k* if there exists a finite subset X of K such that $K \subseteq (X)$ and the height of elements of K with respect to X is at most k. One of the main results of [31] is that a semigroup from the class $\mathfrak{M}_{FI}$ has polynomial growth if and only if it has bounded height.

We recall shortly the definition of *Ufnarovsky graph* $\Gamma = \Gamma_S$ *of* S (depending on the presentation of S), which is the key tool introduced in [30] and modified slightly in [31]. Example 2.3 below is both illustrative in assisting the reader to digest the construction and also in seeing how various criteria can be applied that guarantee polynomial growth.

Consider an irredundant presentation of the form $(*)$ for a semigroup S from the class $\mathfrak{M}_{FI}$. Put $d+1 = \max\{\, l(c_i)\,|\, i=1,\dots,k\,\}$ and

$$\overline{d}+1 = \max\{\, l(c)\,|\, c \text{ is a reduced word } \mathcal{J}\text{−equivalent to some relator}\,\}\ .$$

Note that $\overline{d}$ exists by condition (v) of irredundancy of the presentation, and may be calculated easily by inspecting word trees of relators. By condition (iii) of irredundancy, no word is $\mathcal{J}$-equivalent to a single letter, so $d \ge \overline{d} \ge 1$. Vertices of $\Gamma = \Gamma_S$ are defined to be reduced words of length $\overline{d}$ which are nonzero in S. If v_1 and v_2 are vertices then a directed edge from v_1 to v_2 is defined in Γ if there exist letters $g, h \in A \cup A^{-1}$ such that $v_1 g$ is a reduced word which is nonzero in S and $v_1 g \ \underline{\circ}\ h v_2$. We regard the letter g as a label for this edge. Paths in Γ may then be labelled by reduced words which are nonzero in S. Conversely if $w \ \underline{\circ}\ vu \ \underline{\circ}\ u'v'$ is any nonzero reduced word where v and v' have length $\overline{d}$ then u labels a path in Γ emanating from v and terminating at v'.

By a *cycle* in Γ we mean a path that starts and finishes at the same vertex. By a *loop at a vertex* v we mean a cycle that begins at v using no other vertex more than once. (Note that our use of the word *loop* is slightly non-standard, but is concise, captures precisely the underlying geometric idea, and is consistent with the use made in earlier papers on which our results depend.) Recall from [30, Section 3] that (z, P) is an *adjacent pair* if z is a reduced word that labels a loop in Γ at a vertex v and P is a letter labelling an edge that emanates from v and terminates outside the loop. Combining Theorems 2.1, 3.3 and 4.3 of [31] and Lemma 3.2 of [30], we have the following criteria for polynomial growth:

Theorem 2.2. *Let S be given by an irredundant presentation $(*)$. Then the following conditions are equivalent:*

(a) *S has polynomial growth.*

(b) *S does not contain any noncyclic free subsemigroups.*

(c) *The set of reduced words that are nonzero in S has bounded height and all reduced words that are not cyclically reduced are nilpotent with index of nilpotency $\leq d+1$.*

(d) (i) *Γ_S has no vertex contained in different cycles; and*

(ii) *if (z, P) is an adjacent pair in Γ_S then*

$$z^{d+1}PP^{-1}z^{d+1} = 0 \quad \textit{in} \quad S\,.$$

A sufficient condition for polynomial growth (which becomes necessary if every relator is $\mathcal{J}$-related to a reduced word) is

(e) (i) *Γ_S has no vertex contained in different cycles; and*

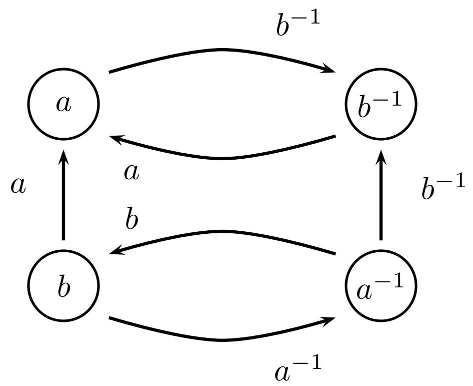
(ii) *if (z, P) is any adjacent pair then (z^{-1}, P) is not adjacent.*

The following example is one of the main ingredients in the full characterisation (Theorem 6.1 below) of three relator irredundant presentations with polynomial growth.

Example 2.3. As noted in the Introduction, the semigroup $S = \langle\, a, b \mid ab = 0\,\rangle$ was studied thoroughly in [8]. The following homomorphic image of S turns out to have polynomial growth:

$$T \;=\; \langle\, a, b \mid a^2 = b^2 = ab = 0\,\rangle\;.$$

Clearly, the presentation of T is irredundant. In terms of the notation preceding Theorem 2.2, $d = \overline{d} = 1$ and the Ufnarovsky graph Γ_T becomes



Our apparatus now gives us several ways to see why the growth of T must be polynomial. Clearly no vertex is contained in different cycles of the graph. The adjacent pairs are precisely

$$(ba^{-1}, b^{-1}) \quad \text{and} \quad (a^{-1}b, a)\ .$$

We may observe that

$$(ba^{-1})^2 b^{-1} b (ba^{-1})^2 \ = \ (a^{-1}b)^2 a a^{-1} (a^{-1}b)^2 \ = \ 0$$

in T, because b^2 is a factor of the first word and a^{-2} is a factor of the second. Thus T has polynomial growth by condition (d) of Theorem 2.2. Alternatively we may observe directly from the graph that

$$(ab^{-1}, b^{-1}) \quad \text{and} \quad (b^{-1}a, a)$$

are not adjacent pairs. Hence condition (e) of Theorem 2.2 also implies that the growth of T is polynomial. To apply condition (c) directly, to illustrate yet another alternative, we need to find reduced words that are nonzero in T. These are precisely the labels of nonempty paths in Γ_T. Together with the empty word these comprise the following regular language that can be read easily from the graph:

$$(1 \cup b)(a^{-1}b)^*(1 \cup a(b^{-1}a)^*(1 \cup b^{-1})) \cup (1 \cup a^{-1})(ba^{-1})^*(1 \cup b^{-1}(ab^{-1})^*(1 \cup a))$$

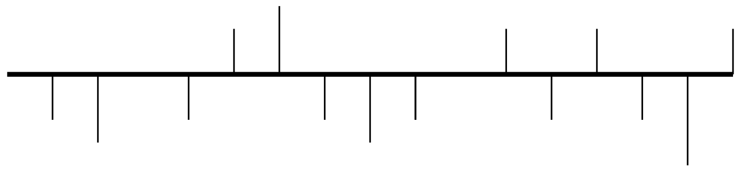
Because there are no nested stars, and the regular expression is the union of two pieces, each with four factors, the set of reduced words has height bounded by 4. By inspection, the reduced words that are not cyclically reduced are precisely conjugates of $a^{\pm 1}$ and $b^{\pm 1}$, so these are nilpotent in T, since their squares are clearly zero in T. Hence by condition (c) of Theorem 2.2, we see again that T has polynomial growth.

In order to calculate the Gelfand-Kirillov dimension of T, we need to be able to control geodesic representatives for elements of T. Because of the presence of idempotents, it is not enough to consider just the set of reduced words (which in fact has Gelfand-Kirillov dimension 2 as a regular language, because of the presence of at most two cycles in a row (see for example [33], [34])). The relations $a^2 = b^2 = 0$ prevent any vertex of a Munn tree of a word in T having degree more than 2. Hence all word trees of nonzero elements are chains, and we may adapt the technique used in [29] (see, for example, the preamble leading up to Lemma 3.1 of [29]). If w is a word over the alphabet B such that the word tree $T(w)$ is a chain with m edges, then the length of w in FI_A is at most $2m$ and there are $\frac{(m+1)(m+2)}{2}$ different divisors of w in FI_A having the same word tree. So, every reduced word v of length m is associated with some subset that consists of a quadratic function in m distinct nonzero elements of the semigroup T, and different reduced words are associated with disjoint subsets. By inspection, if v is reduced and nonzero in T then v or v^{-1} may be written in one of the following forms:

$$b^\alpha (a^{-1}b)^\beta\ ,\ b^\alpha (a^{-1}b)^\beta a (b^{-1}a)^\gamma b^{-\delta}$$

for some nonnegative integers β, γ and $\alpha, \delta \in \{0, 1\}$. It follows quickly that, for large integers ℓ, the number of reduced words of length ℓ that are nonzero in T is bounded above and below by quadratic functions of ℓ. It now follows, by the preceding observation, that $g_T(m)$ is bounded above and below, for large m, by polynomial functions of m of degree 4. Hence, $GKdim(T) = 4$.

The following terminology and notation will be useful in understanding the proof of Lemma 3.1 below, and again when investigating Gelfand-Kirillov dimensions in Section 7. If w is a word over B then we say w *has the whisker property* if the word tree $T(w)$ contains an underlying chain $\mathcal{C}$ of vertices such that each leaf vertex of $T(w)$ is connected to $\mathcal{C}$ by a chain of edges, degrees of vertices are only allowed to be 1, 2 or 3, and vertices of degree 3 only occur on $\mathcal{C}$. These connecting chains are called *whiskers* and are themselves word trees of reduced words.



The definition allows for a whisker located at a leaf of $\mathcal{C}$ to be empty, in which case that leaf of $\mathcal{C}$ also becomes a leaf of $T(w)$. In the diagram above the thick line is intended to represent the underlying chain $\mathcal{C}$ and the thin lines represent the whiskers. On this diagram, one of the leaves of the tree is at one end of $\mathcal{C}$, and at the other end there is a nonempty whisker attached. Whiskers may be of varying length. However, if $u_i \overset{\circ}{=} w^{-1}v_iw$ are reduced words that are not cyclically reduced, for $i = 1, \ldots, n$, and the word $v_1 \ldots v_k$ is reduced, then $u_1 \ldots u_n$ has the whisker property with respect to the chain $\mathcal{C} = T(v_1 \ldots v_n)$, and the whiskers are, in this case, identical copies of the chain $T(w)$.

3. Free Semigroup Generation by a Pair of Reduced Words

In this section we provide a useful characterisation of free semigroup generation in FI_A by a pair of two reduced words. The corollary that follows is used extensively throughout this paper. If w is a nonempty reduced word then we write

$$w \overset{\circ}{=} w_P w_C w_P^{-1}$$

where w_P and w_C are unique reduced words such that w_C is nonempty and cyclically reduced.

LEMMA 3.1. *Let $u, v \in B^*$ be reduced words. Then u and v are a pair of free generators for a subsemigroup of FI_A if and only if*

(a) $u_P \mathrel{\not\overline{\underline{\circ}}} v_P$; *or*

(b) $u_P \mathrel{\overline{\underline{\circ}}} v_P$ *(possibly empty) and u_C and v_C are not powers of the same reduced word in B^*; or*

(c) $u_P \mathrel{\overline{\underline{\circ}}} v_P$ *is a nonempty word and u_C and v_C are different positive powers of the same reduced word in B^*.*

PROOF. Suppose first that u and v are free generators for a subsemigroup of FI_A and that neither (a) nor (b) holds. Then $u_P \mathrel{\overline{\underline{\circ}}} v_P$ and $u_C \mathrel{\overline{\underline{\circ}}} w^k$ and $v_C \mathrel{\overline{\underline{\circ}}} w^\ell$ for some nonzero integers k and ℓ. If k and ℓ have different signs then, without loss of generality, we may suppose $k > 0$ and $\ell < 0$, and clearly

$$\overline{v^k u^{-\ell}} \;\mathrel{\overline{\underline{\circ}}}\; \overline{u_P w^{k\ell} u_P^{-1} u_P w^{-k\ell} u_P^{-1}} \;\mathrel{\overline{\underline{\circ}}}\; 1\,,$$

so that $v^k u^{-\ell}$ is idempotent in FI_A, contradicting that u and v are free generators. Hence k and ℓ have the same sign, so that u_C and v_C are both positive powers of w or w^{-1}. In particular u_C and v_C commute, so it follows immediately that u_P is nonempty. Clearly $u_C \mathrel{\not\overline{\underline{\circ}}} v_C$, so (c) is proved.

Suppose now that (a) or (b) holds. Clearly u and v are not powers of the same word evaluated in the free group FG_A generated by A, so they generate a free subgroup of FG_A of rank two. But this subgroup contains a morphic image of the subsemigroup (u, v) of FI_A. This implies that (u, v) is freely generated by u and v, and we are done.

Suppose finally that (c) holds, so

$$u \mathrel{\overline{\underline{\circ}}} u_P c^k u_P^{-1} \quad \text{and} \quad v \mathrel{\overline{\underline{\circ}}} u_P c^\ell u_P^{-1}$$

for some reduced word c and positive integers k and ℓ with $k \neq \ell$. If $W(x, y)$ is a positive word over the alphabet $\{x, y\}$ then clearly the word $W(u, v)$ over B has the whisker property with respect to some chain $T(c^N)$ for some positive integer N and whiskers that are chains all of the form $T(u_P)$. By inspecting the distances between successive whiskers, it is immediate that if $W_1(u, v)$ and $W_2(u, v)$ are two different positive words over the alphabet $\{x, y\}$, then the Munn trees of $W_1(u, v)$ and $W_2(u, v)$ cannot be equal. This shows that u and v are free generators of (u, v) in FI_A, completing the proof of the lemma.

Because a word is zero in a semigroup S with presentation $(*)$ if and only if some relator divides it, Lemma 3.1 yields the following useful corollary:

COROLLARY 3.2. *Suppose that S is given by a presentation $(*)$. If S contains no noncyclic free subsemigroup and $u, v \in B^*$ are reduced words such that either* (a), (b) *or* (c) *holds in the previous lemma, then some relator used in the presentation for S is a divisor of (u, v).*

4. Key Class of Three Relator Inverse Semigroups

The following proposition introduces a class of three relator Rees quotients of free inverse semigroups over an alphabet with two letters, where the first two relators place severe restrictions on the shape of nonzero reduced words. The third relator, described in terms of a parameter γ, is just enough, in combination with the first two relators, to guarantee polynomial growth. This class is important later in providing three relator examples with Gelfand-Kirillov dimensions taking all integer values greater than or equal to four.

Proposition 4.1. *Consider the Rees quotient*

$$S_C \;=\; \langle\, a, b \mid ab = a^{-1}b = C = 0 \,\rangle \,,$$

where C is some nonempty word over the alphabet $\Sigma = \{a^{\pm 1}, b^{\pm 1}\}$ that is not $\mathcal{J}$-equivalent to a single letter. For each positive integer γ, write $w_\gamma \;\underline{\circ}\; a^\gamma b^{-1} b a^\gamma$. Then S_C has polynomial growth if and only if C divides w_γ for some positive integer γ.

Proof. Put $S = S_C$. Suppose first that C divides w_γ for some positive γ. Put

$$S_0 = \langle\, a, b \mid ab = a^{-1}b = 0 \,\rangle \,.$$

Since $a^\theta b = b^{-1} a^\theta = 0$ in S_0 for any integer θ, the reduced words that are nonzero in S_0 have the form $b^\alpha a^{\pm\beta} b^{-\delta}$ where α, β and δ are nonnegative integers, not all zero. The reduced words that are nonzero in S are also nonzero in S_0 and, by what we have just observed, these form a set of height bounded by three with respect to Σ. Let v be any reduced word that is not cyclically reduced and nonzero in S. Then $v \;\underline{\circ}\; b^\alpha a^{\pm\beta} b^{-\delta}$ for some positive α, β and δ. If $\alpha > \delta$ then

$$\overline{v^2} \;\;\underline{\circ}\;\; b^\alpha a^{\pm\beta} b^{\alpha-\delta} a^{\pm\beta} b^{-\delta} \;,$$

and it is clear that v^2 is zero in S. Similarly, if $\alpha < \delta$, then v^{-2} is zero in S, whence v^2 is zero in S. If $\alpha = \delta$ then, by inspection, w_γ divides v^γ, whence C divides v^γ, so that v^γ is zero in S. In all cases, v is nilpotent with index of nilpotency $\leq d+1$ for the relevant d. Hence, S has polynomial growth by Theorem 2.2.

Suppose conversely that S has polynomial growth. If C divides $a^\gamma b^{-1}$ for some positive integer γ, then certainly C divides $a^\gamma b^{-1} b a^\gamma$, and we are done. Suppose then that C does not divide $a^\gamma b^{-1}$ for all positive γ. Put $d+1 = \ell(C)$. Then none of C, ab or $a^{-1}b$ divide the reduced words a^d or $a^{d-1}b^{-1}$. These two reduced words are of length d so form vertices of the Ufnarovsky graph Γ_S. The letter a labels a loop at the vertex a^d and the letter b^{-1} labels an edge emanating from a^d and terminating at the vertex $a^{d-1}b^{-1}$, so that (a, b^{-1}) is an adjacent pair. By Theorem 2.2(d), w_{d+1} is zero in S. Hence one of the relators ab, $a^{-1}b$ or C must divide w_{d+1}. But neither ab nor $a^{-1}b$ divide w_{d+1}, so that C divides w_{d+1}. This completes the proof of the proposition.

5. Necessary Conditions on Numbers of Generators

In this section we prove the (sharp) lower bound of three for the number of relators in all irrendundant presentations of semigroups in the class $\mathfrak{M}_{FI}$ having polynomial growth. We begin by proving a lemma in the case where there are more than two generators.

Lemma 5.1. *Let S be given by an irredundant presentation $(*)$ with k relators and suppose that S contains no noncyclic free subsemigroups. If the alphabet A has more than two letters then $k > 3$.*

Proof. Suppose that a_1, a_2, a_3 are distinct letters from A. By Corollary 3.2, there exist relators

$$c_r \in \operatorname{div}(a_1, a_2)\,, \quad c_s \in \operatorname{div}(a_1, a_3)\,, \quad c_t \in \operatorname{div}(a_2, a_3)\,, \quad c_u \in \operatorname{div}(a_1 a_2^{-1}, a_1 a_3^{-1})\,.$$

Since c_u is not $\mathcal{J}$-equivalent to a single letter, the pattern of mixed exponents guarantees that c_u cannot be any of c_r, c_s or c_t. If c_r, c_s and c_t are all distinct then $k > 3$ and we are done. Suppose without loss of generality that $c_r = c_s$ so that

$$\operatorname{content}(c_r) \;=\; \{a_1\}\,.$$

By Corollary 3.2, there exist relators

$$c_m \in \operatorname{div}(a_2 a_1 a_2, a_2)\,, \quad c_n \in \operatorname{div}(a_3 a_1 a_3, a_3)\,.$$

Then, because of the shape of the words $a_2 a_1 a_2$ and $a_3 a_1 a_3$, and because neither c_m nor c_n are $\mathcal{J}$-equivalent to a_1 (part of the irredundancy assumption about the presentation of S), we have that

$$\{a_2\} \subseteq \operatorname{content}(c_m) \subseteq \{a_1, a_2\} \qquad \text{and} \qquad \{a_3\} \subseteq \operatorname{content}(c_n) \subseteq \{a_1, a_3\}\,,$$

so that certainly c_r, c_m and c_n are all distinct. For the same reason as before, c_u is different from c_m and c_n, so again $k > 3$, and the lemma is proved.

Theorem 5.2. *If S is given by an irredundant presentation $(*)$ with k relators and S contains no noncyclic free subsemigroups then $k \geq 3$.*

Proof. Suppose that S contains no noncyclic free subsemigroups. Since $|A| \geq 2$ (part of irredundancy of the presentation), it suffices, by Lemma 5.1, to consider the case when $|A| = 2$, so that $A = \{a, b\}$ for some distinct letters a and b.

Suppose first that a^2 is a divisor of c_1. By Corollary 3.2, there exist relators

$$c_r \in \operatorname{div}(ab, ab^{-1})\,, \qquad c_s \in \operatorname{div}(a^{-1}b, b)\,, \qquad c_t \in \operatorname{div}(ab, b)\,.$$

By inspection, from the shape of the words in the subsemigroups (ab, ab^{-1}), $(a^{-1}b, b)$ and (ab, b), certainly a^2 cannot divide c_r, c_s or c_t, so c_1 must be different from each of c_r, c_s and c_t. For the same reason, b^2 cannot divide c_r. Since c_r is not $\mathcal{J}$-equivalent to a single letter (again part of the irredundancy of the presentation), it follows that

$$\operatorname{content}(c_r) \;=\; \{a, b\}\,.$$

If $c_s \neq c_t$ then $k \geq 3$ and we are done. Suppose now that $c_s = c_t$, so

$$c_s \in \operatorname{div}(a^{-1}b, b) \cap \operatorname{div}(ab, b)\,.$$

By inspection, any reduced word of length two in $\operatorname{div}(a^{-1}b, b) \cap \operatorname{div}(ab, b)$ must be b^2 or b^{-2}. It follows that $\operatorname{content}(c_s) = \{b\}$. By comparing contents, we thus see that $c_r \neq c_s$. Hence again $k \geq 3$ and we are done.

Similarly, if b^2 is a divisor of c_1 then the statement of the theorem is proved.

Suppose now that neither a^2 nor b^2 is a divisor of c_1. Since c_1 is not $\mathcal{J}$-equivalent to a single letter, every reduced subword of c_1 of length 2 has the form $a^\varepsilon b^\delta$ or $b^\varepsilon a^\delta$ for some $\varepsilon, \delta \in \{\pm 1\}$. Without loss of generality, by interchanging letters or replacing letters with their inverses, if necessary, we may suppose that ab is a divisor of c_1. By Corollary 3.2, there exist relators

$$c_m \in \operatorname{div}(a^{-1}b, b)\,, \qquad c_n \in \operatorname{div}(ab^{-1}, a)\,.$$

The pattern of mixed signs of exponents in the words $a^{-1}b$ and ab^{-1} guarantees that ab is not a divisor of either c_m or c_n. If $c_m \neq c_n$ then c_1, c_m, c_n are distinct relators (because ab is a divisor of c_1), so that $k \geq 3$ and we are done. We may suppose then that $c_m = c_n$, so

$$c_m \in \operatorname{div}(a^{-1}b, b) \cap \operatorname{div}(ab^{-1}, a)\,.$$

By inspection, any reduced word of length two in $\operatorname{div}(a^{-1}b, b) \cap \operatorname{div}(ab^{-1}, a)$ must be $a^{-1}b$, ab^{-1}, $b^{-1}a$, or ba^{-1}. It follows, since c_m is not $\mathcal{J}$-related to a single letter, that $a^{-1}b$ or ab^{-1} divides c_m. Suppose that $k < 3$. Then $k = 2$ and

$$S_0 \;=\; \langle\, a, b \mid ab = a^{-1}b = 0 \,\rangle \quad \text{or} \quad S_0' \;=\; \langle\, a, b \mid ab = ab^{-1} = 0 \,\rangle$$

is a homomorphic image of S. Note, however, that

$$S_0' \;=\; \langle\, b^{-1}, a^{-1} \mid b^{-1}a^{-1} = ba^{-1} = 0 \,\rangle \;\cong\; \langle\, a, b \mid ab = a^{-1}b = 0 \,\rangle \cong\; S_0\,.$$

Hence S_0 is a homomorphic image of S. But S_0 does not have polynomial growth, by Proposition 4.1. Thus S also does not have polynomial growth, and so contains a noncyclic free subsemigroup, by Theorem 2.2(b), a contradiction. Thus $k \geq 3$ and the proof of the theorem is complete.

6. Growth and Three Relators

By Theorem 5.2, a non-monogenic semigroup from our class with an irredundant presentation must have at least three relators to have polynomial growth. In this section we investigate fully the minimal case of three relators. Suppose throughout this section that

$$S \;=\; \langle A \mid c_1 = c_2 = c_3 = 0 \rangle \qquad\qquad (**)$$

is an irredundant presentation for some words c_1, c_2, c_3. We characterise those semigroups S for which polynomial growth occurs, by supplementing the class of semigroups described in Proposition 4.1 and their duals by the semigroup T.

Theorem 6.1. *A semigroup S with presentation $(**)$ has polynomial growth if and only if S is isomorphic or anti-isomorphic to*

$$T \;=\; \langle\, a, b \mid a^2 = b^2 = ab = 0 \,\rangle$$

or to

$$S_C \;=\; \langle\, a, b \mid ab = a^{-1}b = C = 0 \,\rangle$$

where C is not $\mathcal{J}$-related to a single letter and C divides $w_\gamma \;\underline{\circ}\; a^\gamma b^{-1} b a^\gamma$ for some positive integer γ.

Proof. The "if" direction is verified in Example 2.3, for T, and in the proof of Proposition 4.1, for each S_C. We now prove the "only if" direction.

Suppose that S has polynomial growth. Hence no homomorphic or anti-homomorphic image of S can possess a noncyclic free subsemigroup. By Lemma 5.1, $|A| = 2$, so we can write $A = \{a, b\}$. Without loss of generality, it suffices to consider Cases (i), (ii) and (iii) below.

Case (i): Suppose that a^2 divides c_1 and b^2 divides c_2. Our initial task is to control the shape of the relator c_3, and to do this first observe that

$$\langle\, a, b \mid a^2 = b^2 = c_3 = 0 \,\rangle$$

is a homormorphic image of S. By Corollary 3.2,

$$\begin{aligned} c_3 \;\in\; & \operatorname{div}(a^{-1}b, aba^{-1}b) \cap \operatorname{div}(a^{-1}b^{-1}, ab^{-1}a^{-1}b^{-1}) \\ & \cap \operatorname{div}(b^{-1}a, bab^{-1}a) \cap \operatorname{div}(b^{-1}a^{-1}, ba^{-1}b^{-1}a^{-1}) \,. \end{aligned}$$

By irredundancy of the presentation and inspection of word trees, $T(c_3) = T(w)$ where w is some word of length at least two that alternates between letters from $X = \{a^{\pm 1}\}$ and $Y = \{b^{\pm 1}\}$, beginning either with a letter from X or a letter from Y. By inspection, no such word of length three can lie in the intersection of sets of divisors displayed above, so w has length two. It follows that c_3 is $\mathcal{J}$-related to one of ab, ab^{-1}, $a^{-1}b$ or $a^{-1}b^{-1}$. Note also that a^{-2} divides c_1 and b^{-2} divides c_2. Without loss of generality, by Remark 2.1, interchanging letters with their formal inverses, if necessary, as generators of S, we may suppose that c_3 is $\mathcal{J}$-related to ab, without disturbing our underlying assumption that a^2 divides c_1 and b^2 divides c_2. Thus

$$S_1 = \langle\, a, b \mid c_1 = b^2 = ab = 0\,\rangle \qquad \text{and} \qquad S_2 = \langle\, a, b \mid a^2 = c_2 = ab = 0\,\rangle$$

become homomorphic images of S, so these semigroups do not contain noncyclic free subsemigroups. Observe that $b^2, ab \notin \operatorname{div}(b^{-1}a, b^{-1}a^2)$, so that, by Corollary 3.2 applied to S_1,

$$c_1 \in \operatorname{div}(b^{-1}a, b^{-1}a^2)\,.$$

We will show that the word tree $T(c_1)$ has two edges. Suppose by way of contradiction that $T(c_1)$ has at least three edges. By inspection, since a^2 divides c_1, either a^2b^{-1} or $b^{-1}a^2$ must divide c_1. Hence either

$$S_3 = \langle\, a, b \mid a^2b^{-1} = b^2 = ab = 0\,\rangle \quad \text{or} \quad S_4 = \langle\, a, b \mid b^{-1}a^2 = b^2 = ab = 0\,\rangle$$

must become a homomorphic image of S, and neither can contain a noncyclic free subsemigroup. By inspection, $(a^{-2}ba)^3$ and $(bab^{-1})^3$ are nonzero in S_3 and S_4 respectively. Hence, by Theorem 2.2 (noting that $d + 1 = 3$ in each case), each of S_3 and S_4 contains a noncyclic free subsemigroup, a contradiction. Hence $T(c_1)$ has two edges, so $c_1 \,\mathcal{J}\, a^2$. A similar argument, considering S_2 above, yields $c_2 \,\mathcal{J}\, b^2$. All of this suffices to prove that S is isomorphic to T, completing the analysis of Case (i).

Case (ii): Suppose that a^2 divides c_1 but b^2 does not divide c_2 or c_3. Then

$$\langle\, a, b \mid a^2 = c_2 = c_3 = 0\,\rangle$$

is a homomorphic image of S, so, by Corollary 3.2, and without loss of generality,

$$c_2 \in \operatorname{div}(ab, ab^2) \quad \text{and} \quad c_3 \in \operatorname{div}(ba^{-1}, b^2a^{-1})\,.$$

It follows quickly, by irredundancy, that c_2 is divided by ab or ba, and c_3 is divided by ba^{-1} or $a^{-1}b$.

We will prove that the word tree $T(c_2)$ has two edges. Suppose by way of contradiction that $T(c_2)$ has at least three edges. Then, because b^2 does not divide c_2, either aba or bab divides c_2, so that one of the following inverse semigroups is a homomorphic image of S and therefore does not contain a noncyclic free subsemigroup:

$$S_5 = \langle\, a, b \mid a^2 = aba = ba^{-1} = 0 \rangle\ , \quad S_6 = \langle\, a, b \mid a^2 = bab = ba^{-1} = 0 \rangle\ ,$$

$$S_7 = \langle\, a, b \mid a^2 = aba = a^{-1}b = 0\, \rangle \quad \text{or} \quad S_8 = \langle\, a, b \mid a^2 = bab = a^{-1}b = 0\, \rangle\, .$$

Note that S_7 and S_8 are anti-isomorphic to S_5 and S_6 respectively. By inspection, $(a^{-1}ba)^3$ is nonzero in both S_5 and S_6. By Theorem 2.2, each of S_5 and S_6, and hence also each of S_7 and S_8, contains a noncyclic free subsemigroup, a contradiction.

Hence $T(c_2)$ has two edges. Repeating similar steps of the previous argument shows also that $T(c_3)$ has two edges. Thus c_2 is $\mathcal{J}$-equivalent to ab or ba, and c_3 is $\mathcal{J}$-equivalent to ba^{-1} or $a^{-1}b$. It follows that S is isomorphic or anti-isomorphic to one of the following, where the word C is either c_1 or the reversal of c_1:

$$S_9 = \langle\, a, b \mid C = ab = ba^{-1} = 0\, \rangle \quad \text{or} \quad S_{10} = \langle\, a, b \mid C = ab = a^{-1}b = 0\, \rangle\, .$$

Suppose first that S is isomorphic to S_9 or its dual. In particular, S_9 has polynomial growth. Observe that a^2 divides C (since a^2 divides c_1) and that $S_9 = \langle\, a, b \mid C = b^{-1}a^{-1} = ba^{-1} = 0\, \rangle$. Renaming the letters in Proposition 4.1, we obtain that C divides $b^\delta a a^{-1} b^\delta$ for some positive integer δ. Hence a^2 also divides $b^\delta a a^{-1} b^\delta$, which is impossible.

This shows that S is isomorphic to S_{10} or its dual. It follows immediately from Proposition 4.1 that C divides $w_\gamma \overset{\circ}{=} a^\gamma b^{-1} b a^\gamma$ for some positive integer γ, so S is isomorphic to S_C or its dual, completing the analysis of Case (ii).

Case (iii): Suppose that both a^2 and b^2 do not divide each of c_1, c_2 and c_3. By irredundancy, each of c_1, c_2 and c_3 cannot have the form aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$, and must have a reduced subword of length two from amongst

$$ab\, ,\ ab^{-1}\, ,\ a^{-1}b\, ,\ a^{-1}b^{-1}\, ,\ ba\, ,\ ba^{-1}\, ,\ b^{-1}a\, ,\ b^{-1}a^{-1}\ .$$

Thus, simplifying this list by removing one from each pair of mutual inverses, each of c_1, c_2 and c_3 must has a divisor from amongst

$$ab\, ,\ ab^{-1},\ a^{-1}b\, ,\ ba\ .$$

By Corollary 3.2, without loss of generality,

$$c_1 \in \operatorname{div}(ab, ba) \quad \text{and} \quad c \in \operatorname{div}(ab^{-1}, b^{-1}a)$$

where $c \in \{c_1, c_2, c_3\}$. Observe that the only reduced words of length two that lie in $\operatorname{div}(ab, ba) \cap \operatorname{div}(ab^{-1}, b^{-1}a)$ are $a^{\pm 2}$ and $b^{\pm 2}$. If $c = c_1$ then this implies that either a^2 or b^2 divides c_1, which contradicts our original assumption. Hence $c \neq c_1$. Without loss of generality, then,

$$c_2 \in \operatorname{div}(ab^{-1}, b^{-1}a) .$$

Because a^2 and b^2 do not divide c_1 and c_2 it follows from above that $T(c_1) = T(v)$ where v is a word that alternates between a and b, and $T(c_2) = T(w)$ where w is a word that alternates between a and b^{-1}.

We will prove that the word tree $T(c_1)$ has two edges. Suppose by way of contradiction that $T(c_1)$ has at least three edges. Then c_1 is divided by aba or bab. But ab^{-1} or $b^{-1}a$ divides c_2, so that at least one of the following is a homomorphic image of S:

$$H_1 = \langle\, a, b \mid aba = ab^{-1} = c_3 = 0 \,\rangle\,, \quad H_2 = \langle\, a, b \mid aba = b^{-1}a = c_3 = 0 \,\rangle\,,$$

$$H_3 = \langle\, a, b \mid bab = ab^{-1} = c_3 = 0 \,\rangle\,, \quad H_4 = \langle\, a, b \mid bab = b^{-1}a = c_3 = 0 \,\rangle\,.$$

Suppose first that H_1 is a homomorphic image of S. Then the semigroup

$$\langle\, a, b \mid ab = ab^{-1} = c_3 = 0 \,\rangle \;=\; \langle\, a, b \mid b^{-1}a^{-1} = ba^{-1} = c_3 = 0 \,\rangle$$

is also a homomorphic image of S, so has polynomial growth. By Proposition 4.1, after renaming letters, the relator c_3 must divide the word $b^{\gamma}aa^{-1}b^{\gamma}$ for some positive integer γ. Since b^2 does not divide c_3 we get that c_3 divides ba or $a^{-1}b$. By irredundancy of the presentation for H_1, c_3 must be $\mathcal{J}$-equivalent to $a^{-1}b$. Hence we may rewrite the presentation:

$$H_1 = \langle\, a, b \mid aba = ab^{-1} = a^{-1}b = 0 \,\rangle\,.$$

By inspection, none of the relators in this new presentation for H_1 divides a word in the subsemigroup (bab, ba^2b). By Corollary 3.2, H_1 must contain a noncyclic free subsemigroup, contradicting that H_1 has polynomial growth. Hence H_1 cannot be a homomorphic image of S. By similar arguments, H_2, H_3 and H_4 cannot be homomorphic images of S. This yields a contradiction and completes the proof that $T(c_1)$ has exactly two edges.

By a similar argument, $T(c_2)$ has exactly two edges. Thus c_1 is $\mathcal{J}$-equivalent to ab or ba and c_2 is $\mathcal{J}$-equivalent to ab^{-1} or $b^{-1}a$, so that S is isomorphic to one of the following:

$$L_1 = \langle\, a, b \mid ab = ab^{-1} = c_3 = 0 \,\rangle\,, \quad L_2 = \langle\, a, b \mid ab = b^{-1}a = c_3 = 0 \,\rangle\,,$$

$$L_3 = \langle\, a, b \mid ba = ab^{-1} = c_3 = 0 \,\rangle\,, \quad L_4 = \langle\, a, b \mid ba = b^{-1}a = c_3 = 0 \,\rangle\,.$$

As before, c_3 must have a divisor from amongst ab, ab^{-1}, $a^{-1}b$, ba.

Suppose first that S is isomorphic to L_1. We may rewrite the presentation of L_1 as follows:

$$L_1 = \langle\, a, b \mid b^{-1}a^{-1} = ba^{-1} = c_3 = 0 \,\rangle\, .$$

By Proposition 4.1, the relator c_3 divides $b^\gamma aa^{-1}b^\gamma$ for some positive integer γ. By irredundancy and the fact that b^2 does not divide c, we have that c_3 is $\mathcal{J}$-equivalent to $a^{-1}b$ or ba. Hence S is isomorphic to one of the following:

$$L_{1,1} = \langle\, a, b \mid ab = ab^{-1} = a^{-1}b = 0 \,\rangle\, , \quad L_{1,2} = \langle\, a, b \mid ab = ab^{-1} = ba = 0 \,\rangle\, .$$

Similar arguments in the cases that S is isomorphic to L_2, L_3 and L_4 lead to the following possibilities for S:

$$L_{2,1} = \langle\, a, b \mid ab = b^{-1}a = ab^{-1} = 0 \,\rangle\, , \quad L_{2,2} = \langle\, a, b \mid ab = b^{-1}a = ba = 0 \,\rangle\, ,$$

$$L_{3,1} = \langle\, a, b \mid ba = ab^{-1} = a^{-1}b = 0 \,\rangle\, , \quad L_{3,2} = \langle\, a, b \mid ba = ab^{-1} = ab = 0 \,\rangle\, ,$$

$$L_{4,1} = \langle\, a, b \mid ba = b^{-1}a = ab^{-1} = 0 \,\rangle\, , \quad L_{4,2} = \langle\, a, b \mid ba = b^{-1}a = ab = 0 \,\rangle\, .$$

It is straightforward now to check that each $L_{i,j}$ listed above is isomorphic to

$$\langle\, a, b \mid ab = a^{-1}b = ab^{-1} = 0 \,\rangle\, .$$

This is an instance of S_C, where $C \,\underline{\circ}\, ab^{-1}$ divides $w_1 \,\underline{\circ}\, ab^{-1}ba$, completing the analysis of Case (iii).

This completes the proof of Theorem 6.1.

7. Gelfand-Kirillov Dimensions of Three-Relator Rees Quotients of Free Inverse Semigroups

In this section, we apply our earlier results to prove that the Gelfand-Kirillov dimensions of semigroups with polynomial growth given by irredundant presentations and three relators take precisely all integer values greater than or equal to four. We begin by exhibiting a sequence of semigroups where these values are achieved.

Throughout, let a and b be distinct letters and $\gamma \geq 2$ an integer. Put

$$S_{a^\gamma} = \langle\, a, b \mid ab = a^{-1}b = a^\gamma = 0 \,\rangle\, .$$

These are special cases of the semigroups having polynomial growth that were catalogued in Section 4. In this section we prove that the Gelfand-Kirillov dimension of S_{a^γ} is $\gamma + 2$. This implies (and it is also straightforward to check directly) that the semigroups S_{a^γ} are pairwise nonisomorphic. Thus we will have exhibited Rees quotients of free inverse semigroups with just two generators and three relators that have polynomial growth but whose Gelfand-Kirillov dimensions take every positive integer value ≥ 4. Notice that the free monogenic inverse semigroup has Gelfand-Kirillov dimension 3. However, the proof of Theorem 7.1 below shows that dimension 3 is avoided by all irredundant presentations involving three relators.

The relations $ab = a^{-1}b = 0$ guarantee that the word trees of nonzero elements in S_{a^γ} all have the whisker property: the underlying chains are labelled by nonnegative powers of a, and the whiskers are labelled by nonnegative powers of b. The whiskers may be of arbitrary length, but the relation $a^\gamma = 0$ restricts the length of the underlying chain, so that it has at most $\gamma - 1$ edges labelled by a.

Let $\mathbb{N}$ denote the set of natural numbers, which we take to include zero. For any integer $k \geq 1$, and $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{N}^k$, put

$$w_{\mathbf{x}} \ \underline{\circ} \ b^{-x_1}b^{x_1}ab^{-x_2}b^{x_2}a \ldots b^{-x_{k-1}}b^{x_{k-1}}ab^{-x_k}b^{x_k} .$$

Then, the word tree $T(w_{\mathbf{x}})$ has the whisker property with respect to an underlying chain labelled by a^{k-1}. Further, every word that is nonzero in S_{a^γ} is $\mathcal{J}$-equivalent to $w_{\mathbf{x}}$ for some $k \leq \gamma$ and $\mathbf{x} \in \mathbb{N}^k$.

Let $J_{w_{\mathbf{x}}}$ denote the $\mathcal{J}$-class of $w_{\mathbf{x}}$. Then, following the method in [29] (Proposition 3.2 and its preamble), a word v lies in $J_{w_{\mathbf{x}}}$ if and only if v or v^{-1} has a geodesic representative of the form

$$w_{p,s} \ \underline{\circ} \ \overline{p}^{-1} w_{\mathbf{x}} \overline{s}^{-1}$$

for some prefix p and suffix s of $w_{\mathbf{x}}$ such that $|p| + |s| \leq |w_{\mathbf{x}}|$. Thus if $v \in J_{w_{\mathbf{x}}}$ then

$$\ell(v) \ \leq \ 2|w_{\mathbf{x}}| \ = \ 4(x_1 + \ldots + x_k) + 2k - 2 .$$

Moreover, varying any of the components of $\mathbf{x}$, the reduced prefix $\overline{p}$ or the reduced suffix $\overline{s}$ of $w_{\mathbf{x}}$ changes the Munn tree that the word $w_{p,s}$ represents. For large positive integers m, put

$$X \ = \ \left\{ w_{p,s} \in J_{w_{\mathbf{x}}} \ \middle| \ \mathbf{x} \in \mathbb{N}^\gamma \ , \ \frac{m}{10\gamma} \leq x_i \leq \frac{m}{5\gamma} \ \text{ for } i = 1, \ldots, \gamma \right\} .$$

Firstly, observe that the length of an element of X is bounded by

$$4(x_1 + \ldots + x_\gamma) + 2\gamma - 2 \ \leq \ \frac{4}{5}m + 2\gamma \ \leq \ m .$$

Secondly, observe that the size of X is bounded coarsely below by

$$\left(\left\lfloor \frac{m}{5\gamma} \right\rfloor - \left\lceil \frac{m}{10\gamma} \right\rceil \right)^\gamma \left\lceil \frac{m}{10\gamma} \right\rceil^2 \ \geq \ \alpha m^{\gamma+2}$$

for some positive α. This proves that, for large m,

$$g_{S_{a^\gamma}}(m) \ \geq \alpha m^{\gamma+2} .$$

On the other hand, if v is nonzero in S_{a^γ} and $\ell(v) \leq m$, then one of the equations

$$v \;=\; w_{p,s} \qquad \text{or} \qquad v^{-1} \;=\; w_{p,s}$$

holds in $FI_{\{a,b\}}$ for some $k \leq \gamma$ and $\mathbf{x} \in \mathbb{N}^k$ such that

$$x_1, \ldots, x_k \;\leq\; m \;,$$

and for some prefix p and suffix s of $w_{\mathbf{x}}$ such that $|p|+|s| \leq |w_{\mathbf{x}}|$. Certainly, under these conditions, $|w_{\mathbf{x}}| \leq m$, so a coarse upper bound for the number of nonzero elements of length at most m is

$$2\left(\sum_{k=0}^{\gamma}(m+1)^k\right)(m+1)^2 \;\leq\; \beta m^{\gamma+2}$$

for some positive β. Thus, for large m,

$$\alpha m^{\gamma+2} \;\leq\; g_{S_{a^\gamma}}(m) \;\leq\; \beta m^{\gamma+2} \;,$$

from which it follows immediately that the Gelfand-Kirillov dimension of S_{a^γ} is $\gamma + 2$.

Theorem 7.1. *The set of Gelfand-Kirillov dimensions of Rees quotients of free inverse semigroups having polynomial growth and given by an irredundant presentation with three relators is* $\{4, 5, 6, \ldots\}$.

Proof. Let S be a Rees quotient of a free inverse semigroup having polynomial growth and given by an irredundant presentation. By the earlier calculations in this section, all integer dimensions greater than or equal to 4 arise. By Theorem 5.4 of [15], the Gelfand-Kirillov dimensions of semigroups from $\mathfrak{M}_{FI}$ are always integers, so, to complete the proof, it suffices to show that $GKdim(S)$ is at least 4. If S is isomorphic to $T = \langle a^2 = b^2 = ab = 0\rangle$, then $GFdim(S) = 4$, by Example 2.3, and we are done. Without loss of generality, by Theorem 6.1, we may suppose S is isomorphic to $S_C = \langle a, b \mid ab = a^{-1}b = C = 0\rangle$, where C is not $\mathcal{J}$-related to a single letter and divides $w_\gamma \;\underline{\circ}\; a^\gamma b^{-1} b a^\gamma$ for some positive integer γ. Hence, at least one of the words ab^{-1}, ba or a^δ (for some $\delta > 1$) divides C. It follows that at least one of S_{a^2} or the semigroups $L_{2,1}$ or $L_{2,2}$ introduced at the end of Section 6 is a homomorphic image of S_C. As we noted in Section 6, the semigroups $L_{2,1}$ and $L_{2,2}$ are isomorphic. It is clear that the word tree of any nonzero element in $L_{2,1}$ is a chain. By [29], the Gelfand-Kirillov dimension of $L_{2,1}$ is two more than the Gelfand-Kirillov dimension of the subset H of its reduced nonzero words. Since

$$H \;=\; \{\; b^\alpha a^\beta \mid \alpha, \beta \geq 0 \;\text{ and }\; \alpha + \beta > 0 \;\}$$

we have that $GKdim(H) = 2$, and hence $GKdim(L_{2,1}) = 4$. Therefore, in view of the fact that $GKdim(S_{a^2}) = 4$ also, we have, in all cases, that $GKdim(S)$ is at least 4. This completes the proof of the theorem.

8. An Application

In this section we introduce an operator $\mathcal{Z}$ that takes a general inverse semigroup presentation Π and produces a homomorphic image $\mathcal{Z}(\Pi)$ in the class $\mathfrak{M}_{FI}$. We then apply one of our earlier theorems to obtain information in the case of one relation. A further application of this operator appears in Section 9.

Consider the following inverse semigroup given by an inverse semigroup presentation (not necessarily with zero):

$$\Pi \;=\; \mathrm{Inv}\langle A \mid C_i = D_i \quad \text{for} \quad i = 1, \ldots, k\rangle$$

where A is our usual alphabet and all C_i, D_i are words over $B = A \cup A^{-1}$. Form the following closely associated inverse semigroup with zero from our class $\mathfrak{M}_{FI}$, which is (isomorphic to) a Rees quotient of Π via the natural homomorphism extending the identity map on A:

$$\mathcal{Z}(\Pi) \;=\; \langle A \mid C_i = 0\,,\; D_i = 0 \quad \text{for} \quad i = 1, \ldots, k\rangle\ .$$

Clearly, if $\mathcal{Z}(\Pi)$ has exponential growth then so does Π, and if $\mathcal{Z}(\Pi)$ contains a non-monogenic free subsemigroup then so does Π.

Theorem 8.1. *Let Π be the inverse semigroup given by the presentation*

$$\Pi \;=\; \mathrm{Inv}\langle\, A \mid C = D\,\rangle$$

defined by one relation where C and D are words over B. If A has at least three letters, or A has exactly two letters and the word trees $T(C)$ and $T(D)$ both contain more than one edge, then Π contains a non-monogenic free subsemigroup, so in particular has exponential growth.

Proof. Suppose first that A has at least three letters. Consider the group G given by the same presentation as Π but in the class of all groups. In particular, G is a homomorphic image of Π. Furthermore, G can be defined by the group presentation

$$\langle A \mid W = 1\rangle$$

where W is a cyclically reduced word that is conjugate in the free group to CD^{-1}. If W is empty then G is a non-monogenic free group. If W is nonempty then, by Magnus' Freiheitssatz [18], we can find at least two elements of A that freely generate a free subgroup of G. In either case, the same two elements of A freely generate a non-monogenic free subsemigroup of Π.

Suppose now that A has exactly two letters and the word trees $T(C)$ and $T(D)$ both contain more than one edge. Then the two relators in the presentation for $\mathcal{Z}(\Pi)$ cannot be $\mathcal{J}$-related to single letters. If the presentation for $\mathcal{Z}(\Pi)$ is not irredundant (see the Preliminaries) then either condition (iv) or condition (v) of irredundancy fails. If (v) fails then there would exist at least two letters from A that generate a non-monogenic free subsemigroup. If (iv) fails then one of C or D is a $\mathcal{J}$-divisor of the other and may be deleted from the presentation without altering the semigroup. Thus we may suppose that we obtain an irredundant presentation for $\mathcal{Z}(\Pi)$ that has at most two relators. Then $\mathcal{Z}(\Pi)$ must contain a noncyclic free subsemigroup, by Theorem 5.2, and the proof of the theorem is complete.

The recent example in [28] demonstrates that the hypothesis that $T(C)$ and $T(D)$ contain more than one edge cannot be removed in the statement of the previous theorem.

9. Number of Relators and Examples

In our final main result, we relate the size of the generating set to a lower bound for the number of relators when an inverse semigroup from $\mathfrak{M}_{FI}$ has polynomial growth and the generators are not nilpotent. We use the operator $\mathcal{Z}$ introduced in the previous section to apply this result to obtain information about inverse semigroup presentations with many relations.

Theorem 9.1. *Suppose that $S = \langle A \mid c_i = 0 \quad \text{for} \quad i = 1, \ldots k\rangle$ has polynomial growth, where A is an alphabet of size $n \geq 2$, and none of the generators (elements of A) are nilpotent. Then $k \geq \frac{3}{2}n(n-1)$.*

Proof. Write $A = \{a_1, \ldots, a_n\}$. The hypotheses clearly imply that $k \geq 1$ and no relator is $\mathcal{J}$-equivalent to a single letter. If the presentation is not irredundant (because one relator $\mathcal{J}$-divides another) then we may delete relators until the presentation becomes irredundant. It suffices then to prove the statement of the theorem assuming the presentation for S is irredundant. For the time being, fix $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and put

$$S_{i,j} \;=\; \mathrm{Inv}\langle a_i, a_j\rangle \;,$$

regarded as an inverse subsemigroup of S. Certainly then $S_{i,j}$ has polynomial growth. Also put $\mathcal{C} = \{c_1, \ldots, c_k\}$ and

$$\mathcal{D} \;=\; \{c \in \mathcal{C} \mid \mathrm{content}(c) \subseteq \{a_i, a_j\}\} \;.$$

If $0 \notin S_{i,j}$ then $\mathcal{D}$ is empty, and then $S_{i,j}$ becomes a free nonmonogenic inverse semigroup, contradicting polynomial growth. Hence $0 \in S_{i,j}$. Since we are inside the Rees quotient of FI_A by the ideal generated by the members of $\mathcal{C}$, this implies that $\mathcal{D}$ is nonempty, say of size ℓ. We can write $\mathcal{D} = \{d_1, \ldots, d_\ell\}$. Now consider the semigroup $\widehat{S}_{i,j}$ given by the following irredundant presentation, as an inverse semigroup with zero:

$$\widehat{S}_{i,j} \;=\; \mathrm{Inv}\langle a_i, a_j \mid d_1 = \ldots = d_\ell = 0\rangle \;.$$

The nonzero multiplication of elements inside $S_{i,j}$ may be identified with the same multiplication regarded as elements of $FI_{\{a_i,a_j\}}$, and a product of words becomes zero in $S_{i,j}$ precisely when a relator from $\mathcal{D}$ divides it. Hence the natural identification of nonzero elements of $S_{i,j}$ with elements of $FI_{\{a_i,a_j\}}$ induces an isomorphism from $S_{i,j}$ to $\widehat{S}_{i,j}$, regarding the latter as a Rees quotient. Therefore $\widehat{S}_{i,j}$ has polynomial growth. By Theorem 5.2, $\ell \geq 3$. Since no generator of S is nilpotent, each of d_1, d_2, d_3 has content $\{a_i, a_j\}$. Put

$$u_{i,j} \mathrel{\underline{\overline{\circ}}} d_1\,, \quad v_{i,j} \mathrel{\underline{\overline{\circ}}} d_2\,, \quad w_{i,j} \mathrel{\underline{\overline{\circ}}} d_3\,.$$

We can now allow i and j to vary. The set

$$\{\, u_{i,j}, v_{i,j}, w_{i,j} \mid i,j \in \{1,\ldots,n\}\,,\ i \neq j\,\}$$

has size $\frac{3}{2}n(n-1)$ and the theorem is proved.

Corollary 9.2. *Consider the inverse semigroup*

$$\Pi \;=\; \mathrm{Inv}\langle\, A \mid C_i = D_i \ \text{ for } \ i = 1,\ldots,k\,\rangle$$

defined by k relations where all C_i, D_i are words over B. Suppose that A is an alphabet of size $n \geq 2$, C_i and D_i are not $\mathcal{J}$-related to a power of a single letter for each i, and Π does not contain a noncyclic free subsemigroup. Then $k \geq \frac{3}{4}n(n-1)$.

Proof. Certainly $\mathcal{Z}(\Pi)$ does not contain a noncyclic free subsemigroup, so by Theorem 2.2 (b), $\mathcal{Z}(\Pi)$ has polynomial growth. Therefore, by Theorem 9.1, $2k \geq \frac{3}{2}n(n-1)$, and the corollary follows.

Example 9.3. Let $n \geq 2$ and consider the semigroup

$$S \;=\; \langle\, a_1,\ldots,a_n \mid a_ia_j = a_ja_i = a_ia_j^{-1} = 0 \ \text{ for } \ i,j \in \{1,\ldots n\}\,,\ i < j\,\rangle\,.$$

Then none of the generators are nilpotent and the number of relators is $\frac{3}{2}n(n-1)$. One can check easily that the only adjacent pairs in the Ufnarovsky graph of the presentation are (a_i^{-1}, a_j) where $i < j$, yet

$$a_i^{-2}a_ja_j^{-1}a_i^{-2} \;=\; 0$$

in S. Hence, by Theorem 2.2, S has polynomial growth. This shows that the bound in the previous theorem is best possible.

Example 9.4. Let $n \geq 3$ and consider the semigroup

$$S \;=\; \langle\, a_1,\ldots,a_n \mid a_i^2 = a_2^{-1}a_1 = a_ja_i = a_j^{-1}a_i = 0 \ \text{ for all } \ i \geq 1, j \geq 3, j > i\,\rangle\,.$$

One may check easily that there are only four loops in the graph Γ_S, labelled by a_1a_2, a_2a_1, $a_2^{-1}a_1^{-1}$ and $a_1^{-1}a_2^{-1}$ respectively, and that the following is a complete list of adjacent pairs:

$$(a_1a_2, a_1^{-1})\,, (a_2a_1, a_2^{-1})\,,\ (a_1a_2, a_k^{\pm 1})\,,\ (a_2a_1, a_k^{\pm 1})$$

where $k \geq 3$. Since none of

$$(a_2^{-1}a_1^{-1}, a_1^{-1})\,, (a_1^{-1}a_2^{-1}, a_2^{-1})\,,\ (a_2^{-1}a_1^{-1}, a_k^{\pm 1})\,,\ (a_1^{-1}a_2^{-1}, a_k^{\pm 1})$$

are adjacent pairs, it follows from Theorem 2.2(e) that S has polynomial growth. Note that the number of relators is $n^2 - 1$, which is less than $\frac{3}{2}n(n-1)$, but of course the generators are nilpotent.

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