# Supplement to "Nonlinear regressions with nonstationary time series"

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This supplement gives the proofs of Lemmas 8.3, 8.6–8.7 (see Section 9), the proofs of Theorems 2.5, 4.1–4.3 (see Section 10), a unit root testing for empirical data (Section 11) and additional simulation results (see Section 12) in the official publication. Results (formulae) cited are along the lines of the official publication.

## 9 Proofs of Lemmas 8.3, 8.6 and 8.7

**Proof of Lemma 8.3.** The proofs of (35) and (36) see Theorem 2.1 of Chen (1999). To prove (37), we first impose an additional assumption that $g_2(x) = g_1^2(x)$. Denote

$$\Delta_n^2 = a(n)^{-1}\sum_{t=1}^{n} g_1^2(x_t), \quad Z_{nt} = a(n)^{-1/2} g_1(x_t)\Delta_n^{-1}, \quad \text{and} \quad W_n = \sum_{t=1}^{n} Z_{nt}u_k. \tag{57}$$

Recalling that $E(u_t|\mathcal{F}_{nt}) = 0$, it is readily seen that given $\{x_1, ..., x_n\}$, $\{Z_{nt}u_t, \mathcal{F}_{nt}\}_{t=1}^n$ forms a martingale difference sequence. The result (37) will follow if we prove,

$$\sup_x \big|P\big(W_n \le x \,|\, x_1, ..., x_n\big) - \Phi(x)\big| \to_P 0. \tag{58}$$

Indeed, by noting that $\Delta_n^2$ is measurable with respect to $\sigma(x_1, ...x_n)$, we have, for any $\alpha, \gamma \in R$,

$$\begin{aligned} &\big|E\big[e^{i\alpha W_n + i\beta\Delta_n^2}\big] - e^{-\frac{1}{2}\alpha^2}\, E\big[e^{i\beta\tau_5\Pi_\gamma}\big]\big| \\ &\le E\Big|E\big(e^{i\alpha W_n}\,|\, x_1, ..., x_n\big) - e^{-\frac{1}{2}\alpha^2}\Big| + e^{-\frac{1}{2}\alpha^2}\Big|Ee^{i\gamma\Delta_n^2} - Ee^{i\gamma\tau_5\Pi_\beta}\Big| \to 0, \end{aligned}$$

by dominated convergence theorem, due to (58) and $\Delta_n^2 \to_D \tau_5\Pi_\beta$ (see, e.g., Theorem 2.3 of Chen (1999)). This implies that

$$\{W_n, \Delta_n^2\} \to_D \{N, \tau_5\Pi_\beta\},$$

where $N$ is a standard normal random variable independent of $\Pi_\beta$. Hence, by continuous mapping theorem, we have

$$\Big\{a(n)^{-1/2}\sum_{t=1}^n g_1(x_t)u_t,\, a(n)^{-1}\sum_{t=1}^n g_1^2(x_t)\Big\} = \Big\{\Delta_n W_n, \Delta_n^2\Big\} \to_D \{\tau_5^{1/2} N \Pi_\beta^{1/2}, \tau_5 \Pi_\beta\},$$

which implies the required (37).

We now prove (58). By Theorem 3.9 ((3.75) there) in Hall and Heyde (1980) with $\delta = q/2 - 1$ that

$$\sup_x \big|P\big(W_n \le x \,|\, x_1, ..., x_n\big) - \Phi(x)\big| \le A(\delta)\mathcal{L}_n^{1/(1+q)} \quad a.s.,$$

where $A(\delta)$ is a constant depending only on $\delta$ and $q > 2$, and (set $\mathcal{F}_n^* = \sigma(x_1, ..., x_n)$)

$$\mathcal{L}_n = \Delta_n^{-q}\sum_{k=1}^n |Z_{nk}|^q E(|u_k|^q|\mathcal{F}_n^*) + E\Big[\big|\Delta_n^{-2}\sum_{k=1}^n Z_{nk}^2[E(u_k^2|\mathcal{F}_{nk}) - 1]\big|^{q/2}\Big|\, \mathcal{F}_n^*\Big].$$

Recall from Assumption 3.2 (iv) and the fact that $\Delta_n^2 = \sum_{k=1}^n Z_{nk}^2$, we have,

$$E\Big[\big|\Delta_n^{-2}\sum_{k=1}^n Z_{nk}^2[E(u_k^2|\mathcal{F}_{nk}) - 1]\big|^{q/2}\Big|\, \mathcal{F}_n^*\Big] \to_P 0,$$

by dominated convergence theorem. Hence, routine calculations show that

$$\mathcal{L}_n \le C\, \Delta_n^{-(q-2)}\, a(n)^{-(q-2)/2} + o_P(1) = o_P(1),$$

because $\Delta_n^{-2} = O_P(1)$ by (35) and $q > 2$. This proves (58), which implies that (37) holds true with $g_2(x) = g_1^2(x)$. Finally, note that, for any $a$, $b \in R$,

$$a(n)^{-1}\sum_{t=1}^n \big\{a g_1^2(x_t) + b g_2(x_t)\big\} \to_D \int_{-\infty}^{\infty} \big[a g_1^2(s) + b g_2(s)\big]\pi(ds)\,\Pi_\beta,$$

due to Theorem 2.3 of Chen (1999), which implies that

$$\Big\{a(n)^{-1}\sum_{t=1}^n g_1^2(x_t),\, a(n)^{-1}\sum_{t=1}^n g_2(x_t)\Big\} \to_D \Big\{\int_{-\infty}^{\infty} g_1^2(s)\pi(ds)\,\Pi_\beta, \int_{-\infty}^{\infty} g_2(s)\pi(ds)\,\Pi_\beta, \Big\}.$$

Hence, by continuous mapping theorem,

$$\frac{\sum_{t=1}^n g_1^2(x_t)}{\sum_{t=1}^n g_2(x_t)} \to_P \int_{-\infty}^{\infty} g_1^2(s)\,\pi(ds) \Big/ \int_{-\infty}^{\infty} g_2(s)\,\pi(ds).$$

This shows that (37) is still true with general $g_2(x)$. $\square$

**Proof of Lemma 8.6**. We only prove (42). Others are similar and the details are omitted. First note that, by Assumption 3.2 (i) and (iii),

$$\begin{aligned}\sup_{\theta\in\Theta}|f_i(x_t,\theta)| &\le \sup_{\theta\in\Theta}|\dot f_i(x_t,\theta)-\dot f_i(x_t,\theta_0)|+|\dot f_i(x_t,\theta_0)|\\ &\le \sup_{\theta\in\Theta}h(\|\theta-\theta_0\|)T(x_t)+|\dot f_i(x_t,\theta_0)|\le C.\end{aligned}$$

It follows that

$$\begin{aligned}&\big|\dot f_i(x_t,\theta)\,\dot f_j(x_t,\theta)-\dot f_i(x_t,\theta_0)\,\dot f_j(x_t,\theta_0)\big|\\ \le\;& \big|\dot f_i(x_t,\theta)\big|\big|\dot f_j(x_t,\theta)-\dot f_j(x_t,\theta_0)\big|+\big|\dot f_j(x_t,\theta_0)\big|\big|\dot f_i(x_t,\theta)-\dot f_i(x_t,\theta_0)\big|\\ \le\;& C\big|\dot f_j(x_t,\theta)-\dot f_j(x_t,\theta_0)\big|+C_1\big|\dot f_i(x_t,\theta)-\dot f_i(x_t,\theta_0)\big|.\end{aligned}$$

Therefore, by recalling (33), the result (42) follows from an application of Lemma 8.1 with $\kappa_n^2=n/d_n$. □

**Proof of Lemma 8.7.** Recall $\max_{1\le t\le n}|x_t|/d_n=O_P(1)$. Without loss of generality, we assume $\max_{1\le t\le n}|x_t|/d_n\le K_0$ for some $K_0>0$. It follows from Assumption 3.4 and $d_n\to\infty$ that, for any $1\le i\le m$ and $\theta\in\Theta$,

$$\begin{aligned}\big|\dot f_i(x_t,\theta)\big| &\le \dot v_i(d_n)\big(|\dot h_i(x_t/d_n)|+o(1)\,T_{1\dot f_i}(x_t/d_n)\big),\\ \big|\dot f_i(x_t,\theta)-\dot f_i(x_t,\theta_0)\big| &\le A_{\dot f_i}(||\theta-\theta_0||)\dot v_i(d_n)T_{1\dot f_i}(x_t/d_n).\end{aligned}$$

This implies that

$$\begin{aligned}&\sup_{\theta\in\mathcal{N}_\delta(\theta_0)}\sum_{t=1}^n\big|\dot f_i(x_t,\theta)\,\dot f_j(x_t,\theta)-\dot f_i(x_t,\theta_0)\,\dot f_j(x_t,\theta_0)\big|\\ \le\;& 2\,(A_{\dot f_i}(\delta)+A_{\dot f_j}(\delta))\,\dot v_i(d_n)\dot v_j(d_n)\sum_{t=1}^n\big(|\dot h_i(x_t/d_n)|+T_{1\dot f_i}(x_t/d_n)\big).\end{aligned}\tag{59}$$

(45) now follows from $A_{\dot f_i}(\delta)\to 0$ as $\delta\to 0$ and

$$\frac{1}{n}\sum_{t=1}^n\big[|\dot h_i(x_t/d_n)|+T_{1\dot f_i}(x_t/d_n)\big]\to_D\int_0^1\big[|\dot h_i(G(t))|+T_{1\dot f_i}(G(t))\big]dt=O_P(1),$$

due to Assumptions 3.3 (iii) and 3.4 (iii).

The proof of (46) is similar and hence the details are omitted. As for (47), by noting

$$\big|\sum_{t=1}^n\ddot f_{ij}(x_t,\theta)\,u_t\big|\le\big|\sum_{t=1}^n\ddot f_{ij}(x_t,\theta_0)\,u_t\big|+\sum_{t=1}^n|\ddot f_{ij}(x_t,\theta)-\ddot f_{ij}(x_t,\theta_0)|\,|u_t|,$$

the result can be proved by using the similar arguments as in (31) and (32) of Lemma 8.1. □

# 10 Proofs of Theorems 2.5, 4.1–4.3

**Proof of Theorem 2.5.** Let $\Theta_0 = \{\|\theta-\theta_0\| \geq \delta\}$ where $\delta>0$ is a constant. By virtue of Lemma 8.9, it suffices to prove that, for any $\eta, M_0>0$, there exist a $n_0>0$ such that, for all $n>n_0$,

$$P\Big(n^{-1}\inf_{\theta\in\Theta_0} D_n(\theta,\theta_0) > M_0\Big) > 1-\eta. \tag{60}$$

To prove (60), first note that $\sum_{t=1}^n u_t^2/n \leq M_0$ in probability, for some $M_0>0$, due to Assumption 2.2 (i). This, together with Cauchy-Schwarz Inequality, yields that

$$\begin{aligned}
& n^{-1}D_n(\theta,\theta_0)\\
&= \frac{1}{n}\sum_{t=1}^n (f(x_t,\theta)-f(x_t,\theta_0))^2 - \frac{2}{n}\sum_{t=1}^n (f(x_t,\theta)-f(x_t,\theta_0))u_t\\
&\geq \frac{1}{n}\sum_{t=1}^n (f(x_t,\theta)-f(x_t,\theta_0))^2 - \frac{2}{n}\Big(\sum_{t=1}^n (f(x_t,\theta)-f(x_t,\theta_0))^2\Big)^{1/2}\Big(\sum_{t=1}^n u_t^2\Big)^{1/2}\\
&\geq M_n(\theta,\theta_0)\Big[1-\frac{2\sqrt{M_0+o_P(1)}}{M_n(\theta,\theta_0)^{1/2}}\Big],
\end{aligned} \tag{61}$$

where $M_n(\theta,\theta_0)=\frac{1}{n}\sum_{t=1}^n (f(x_t,\theta)-f(x_t,\theta_0))^2$. Hence, for any equivalent process $x_k^*$ of $x_k$ (i.e., $x_k^* =_D x_k, 1\leq k\leq n, n\geq 1$, where $=_D$ denotes equivalence in distribution), we have

$$P\Big(n^{-1}\inf_{\theta\in\Theta_0} D_n(\theta,\theta_0) > M_0\Big) \geq P\Big(\inf_{\theta\in\Theta_0} M_n^*(\theta,\theta_0)\Big[1-\frac{2\sqrt{M_0+o_P(1)}}{M_n^*(\theta,\theta_0)^{1/2}}\Big] > M_0\Big), \tag{62}$$

where $M_n^*(\theta,\theta_0)=\frac{1}{n}\sum_{t=1}^n (f(x_t^*,\theta)-f(x_t^*,\theta_0))^2$.

Recalling $x_{[nt]}/d_n \to_D G(t)$ on $D[0,1]$ and $G(t)$ is a continuous Gaussian process, by the so-called Skorohod-Dudley-Wichura representation theorem (e.g., Shorack and Wellner, 1986, p. 49, Remark 2), we can choose an equivalent process $x_k^*$ of $x_k$ so that

$$\sup_{0\leq t\leq 1} |x_{[nt]}^*/d_n - G(t)| \;=\; o_P(1). \tag{63}$$

For this equivalent process $x_t^*$, it follows from the structure of $f(x,\theta)$ that

$$\begin{aligned}
m(d_n,\theta)^2 &:= \frac{1}{nv(d_n,\theta)^2}\sum_{t=1}^n f(x_t^*,\theta)^2\\
&= \frac{1}{n}\sum_{t=1}^n h(x_t^*/d_n,\theta)^2 + o_P(1)\\
&= \int_0^1 h(x_{[ns]}^*/d_n,\theta)^2 ds + o_P(1)\\
&\to_P \int_0^1 h(G(s),\theta)^2 ds =: m(\theta)^2,
\end{aligned} \tag{64}$$

uniformly in $\theta \in \Theta$. Due to (64), the same argument as in the proof of Theorem 4.3 in PP yields that

$$\inf_{\theta\in\Theta_0} M_n^*(\theta,\theta_0) \to \infty, \quad \text{in probability,}$$

which, together with (62), implies (60). □

**Proof of Theorem 4.1.** We first establish the consistency result. The proof goes along the same line as in Theorem 2.1. It suffices to show that for any fixed $\pi_0 \in \Theta \cap \mathcal{N}^c$,

$$\sup_{\theta\in\mathcal{N}_\delta(\pi_0)} \frac{d_n}{n}\sum_{t=1}^n |[f(x_t,\theta) - f(x_t,\pi_0)]u_t| \to_P 0, \tag{65}$$

as $\delta \to 0$ uniformly for all large $n$, and

$$\sum_{t=1}^n (f(x_t,\pi_0) - f(x_t,\theta_0))u_t = o_P(n/d_n). \tag{66}$$

By (38) in Lemma 8.4 with $g_1(x) = |T(x)|$, (65) follows from

$$\begin{aligned}\sup_{\theta\in\mathcal{N}_\delta(\pi_0)} \frac{d_n}{n}\sum_{t=1}^n |[f(x_t,\theta) - f(x_t,\pi_0)]u_t| &\le \sup_{\theta\in\mathcal{N}_\delta(\pi_0)} h(\|\theta-\pi_0\|)\,\frac{d_n}{n}\sum_{t=1}^n |T(x_t)||u_t| \\ &\le C \sup_{\theta\in\mathcal{N}_\delta(\pi_0)} h(\|\theta-\pi_0\|)\,O_P(1) \to_P 0,\end{aligned}$$

as $\delta \to 0$. Similarly, by (39) in Lemma 8.4 with $g_1(x) = f(x,\pi_0) - f(x,\theta_0)$, we have that

$$\sum_{t=1}^n (f(x_t,\pi_0) - f(x_t,\theta_0))u_t = O_P[(n/d_n)^{1/2}],$$

which implies the required (66).

We next prove the convergence in distribution. As in Theorem 3.2, it suffices to verify the conditions (i)–(iii) and (iv)' of Theorem 3.1, but with

$$D_n = \sqrt{n/d_n}\,\mathbf{I}, \quad Z = \Lambda^{1/2}\,\mathbf{N}L_G^{1/2}(1,0), \quad M \;=\; \Sigma\, L_G(1,0),$$

under Assumptions 4.1.

The proofs for (i) and (ii) are exactly the same as those of Theorem 3.2. Using similar arguments as in the proof of (56), (iv)' follow from Lemma 8.4.

Finally, note that (65) and (66) still hold if we replace $f(x,\theta)$ by $\ddot{f}_{ij}(x,\theta)$ due to Assumption 3.2. By choosing $k_n \to \infty$ in the same way as in the proof of Theorem 3.2, we have, for any $1 \le i, j \le m$,

$$
\begin{aligned}
&\sup_{\theta:\|D_n(\theta-\theta_0)\|\le k_n} \Big|\frac{d_n}{n}\sum_{t=1}^n \ddot{f}_{ij}(x_t,\theta)u_t\Big| \\
&\le \sup_{\theta:\|D_n(\theta-\theta_0)\|\le k_n} \Big|\frac{d_n}{n}\sum_{t=1}^n [\ddot{f}_{ij}(x_t,\theta)-\ddot{f}_{ij}(x_t,\theta_0)]\,u_t\Big| + \Big|\frac{d_n}{n}\sum_{t=1}^n \ddot{f}_{ij}(x_t,\theta_0)u_t\Big| \\
&= o_P(1), \qquad (67)
\end{aligned}
$$

which implies the required (iii). □

**Proof of Theorem 4.2.** The proof goes along the same line as in Theorem 2.1 and Theorem 4.1. Write $v_n = v(\sqrt{n})$, It suffices to show that for any fixed $\pi_0 \in \Theta \cap \mathcal{N}^c$,

$$
\sup_{\theta\in\mathcal{N}_\delta(\pi_0)} \frac{1}{nv_n^2}\sum_{t=1}^n |[f(x_t,\theta)-f(x_t,\pi_0)]u_t| \to_P 0, \qquad (68)
$$

as $\delta \to 0$ uniformly for all large $n$, and

$$
\sum_{t=1}^n (f(x_t,\pi_0)-f(x_t,\theta_0))u_t = o_P(nv_n^2). \qquad (69)
$$

The result (69) follows from (41) of Lemma 8.5. To see (68), by Cauchy-Schwarz inequality then weak law of large number, we have

$$
\begin{aligned}
&\sup_{\theta\in\mathcal{N}_\delta(\pi_0)} \frac{1}{nv_n^2}\sum_{t=1}^n |[f(x_t,\theta)-f(x_t,\pi_0)]u_t| \\
&\le \sup_{\theta\in\mathcal{N}_\delta(\pi_0)} \frac{1}{nv_n^2}\Big(\sum_{t=1}^n [f(x_t,\theta)-f(x_t,\pi_0)]^2\Big)^{1/2}\Big(\sum_{t=1}^n u_t^2\Big)^{1/2} \\
&\le C \sup_{\theta\in\mathcal{N}_\delta(\pi_0)} A_f(\|\theta-\pi_0\|)\Big(\frac{1}{n}\sum_{t=1}^n T_{1f}^2\big(\frac{x_t}{\sqrt{n}}\big)\Big)^{1/2} \to_P 0,
\end{aligned}
$$

as $\delta \to 0$, as required. □

**Proof of Theorem 4.3.** Write $v_n = v(\sqrt{n})$, $\dot{v}_{in} = \dot{v}_i(\sqrt{n})$, $\ddot{v}_{ijn} = \ddot{v}_{ij}(\sqrt{n})$, for $1 \le i, j \le m$. It suffices to verify the conditions (i)–(iii) and (iv)' of Theorem 3.1 with

$$
\begin{aligned}
&D_n = \mathrm{diag}(\sqrt{n}\dot{v}_{1n}, ..., \sqrt{n}\dot{v}_{mn}), \quad M = \int_0^1 \Psi(t)\Psi(t)'dt, \\
&Z = \sigma_{\xi u}\int_0^1 \dot{\Psi}(t)dt + \int_0^1 \Psi(t)dU(t), \qquad (70)
\end{aligned}
$$

under Assumptions 4.1.

The proofs for (i) and (ii) are exactly the same as those in Theorem 3.4. Note that $\sup_{1\le i,j\le m}|\frac{v_n\,\ddot{v}_{ijn}}{\dot{v}_{in}\,\dot{v}_{jn}}| < \infty$. It follows that, by choosing $k_n \to \infty$ in the same way as in the proof of Theorem 3.2, we have, for any $1 \leq i, j \leq m$,

$$
\begin{aligned}
&\sup_{\theta:\|D_n(\theta-\theta_0)\|\leq k_n} \Big|\frac{1}{n\dot{v}_{in}\dot{v}_{jn}} \sum_{t=1}^{n} \ddot{f}_{ij}(x_t,\theta)u_t\Big| \\
&\leq \sup_{\theta:\|D_n(\theta-\theta_0)\|\leq k_n} \Big|\frac{C}{n\ddot{v}_{ijn}} \sum_{t=1}^{n} [\ddot{f}_{ij}(x_t,\theta) - \ddot{f}_{ij}(x_t,\theta_0)]\, u_t\Big| + \Big|\frac{C}{n\ddot{v}_{ijn}} \sum_{t=1}^{n} \ddot{f}_{ij}(x_t,\theta_0)u_t\Big| \\
&= o_P(1),
\end{aligned}
$$

where the convergence of first term follows from (68) with $\pi_0 = \theta_0$ and $f$ replaced by $\ddot{f}$, and that of second term follows from Lemma 8.5. This yields the required (iii).

Finally, (iv)' follows from Lemma 8.5 with

$$g_1(x) = \alpha_3'\dot{h}(x,\theta_0) \quad \text{and} \quad g_2(x) = \alpha_1'\dot{h}(x,\theta_0)\dot{h}(x,\theta_0)'\alpha_2.$$

□

# 11 A Unit Root Test

In this section, we perform a unit root test to check the stationarity of the time series in our example. There exist many works in macro-economics that demonstrate that GDP and $CO_2$ are nonstationary. See, e.g., Wagner (2008). We implement a simple unit root test to verify the nonstationarity of the data we adopt in Section ??. Denote $y_t$ be the time series we want to test for unit root. It can either be the logarithm of GDP or logarithm of $CO_2$. For each country, the series is modeled in the following way:

$$y_t = \alpha y_{t-1} + \varepsilon_t, \quad t = 1, ..., n,$$

where $\varepsilon_t$ is a stationary error process. We are interested in testing the hypothesis: $H_0: \alpha = 1$ versus $H_1: \alpha < 1$. To achieve this, Dickey-Fuller test is performed on the logarithm of GDP and $CO_2$ for each country, by using the `adfTest()` function in the `fUnitRoots` package of the statistical software R. The critical value for a sample size of 58 is -1.947. Table 1 below presents the observed values of the test statistics for the logarithm of GPD and $CO_2$ for all countries. All observed values are greater than the critical value. Therefore, we do not reject the hypothesis that there is a unit root in the model.

# 12 Additional Tables and Figures

Table 1: Dickey-Fuller Test Results

| Countries | DF for log(GDP) | DF for log($CO_2$) |
|---|---|---|
| AUS | 4.9129 | 2.0222 |
| AUT | 3.6022 | 0.7622 |
| BEL | 4.8375 | -0.2559 |
| CAN | 3.8486 | 0.9752 |
| DEN | 5.0166 | 0.2707 |
| FIN | 3.1312 | -0.0394 |
| FRA | 3.0434 | -0.1289 |
| HOL | 4.1681 | 0.8059 |
| IRE | 3.2837 | 1.188 |
| ITA | 2.5805 | -1.1359 |
| JAP | 1.671 | -0.3775 |
| NOR | 4.3584 | 1.3698 |
| SWI | 3.1305 | -0.5665 |
| USA | 4.6961 | 0.2954 |

Table 2: Means and standard errors of $\hat{\theta}_n$ and $\hat{\sigma}_n^2$ with $f(x,\theta) = \exp\{-\theta|x|\}$.

| | | $\rho = 0$ | | $\rho = 0.5$ | | $\rho = 1$ | |
|---|---|---|---|---|---|---|---|
| | $n$ | Mean | Std. error | Mean | Std. error | Mean | Std. error |
| | | | | Scenario **S1** | | | |
| $\hat{\theta}_n$ | 200 | 0.10018 | (0.00121) | 0.10018 | (0.00125) | 0.10011 | (0.00139) |
| | 500 | 0.10014 | (0.00074) | 0.10013 | (0.00077) | 0.10004 | (0.00086) |
| $\hat{\sigma}_n^2$ | 200 | 8.95974 | (0.88064) | 8.96749 | (0.89797) | 8.98783 | (0.89446) |
| | 500 | 8.98116 | (0.57035) | 8.98106 | (0.56659) | 8.99062 | (0.57060) |
| | | | | Scenario **S2** | | | |
| $\hat{\theta}_n$ | 200 | 0.10014 | (0.00131) | 0.10024 | (0.00136) | 0.10014 | (0.00146) |
| | 500 | 0.10009 | (0.00081) | 0.10017 | (0.00083) | 0.10012 | (0.00092) |
| $\hat{\sigma}_n^2$ | 200 | 8.97149 | (0.89930) | 8.96567 | (0.88766) | 9.00270 | (0.89405) |
| | 500 | 8.99296 | (0.57106) | 9.00135 | (0.56876) | 8.99701 | (0.56470) |
| | | | | Scenario **S3** | | | |
| $\hat{\theta}_n$ | 200 | 0.10033 | (0.00192) | 0.10017 | (0.00202) | 0.10026 | (0.00223) |
| | 500 | 0.10022 | (0.00122) | 0.10020 | (0.00124) | 0.10012 | (0.00138) |
| $\hat{\sigma}_n^2$ | 200 | 9.08776 | (0.91758) | 9.11416 | (0.92635) | 9.12292 | (0.92537) |
| | 500 | 9.13221 | (0.58686) | 9.12295 | (0.59715) | 9.14155 | (0.59417) |

Table 3: Means and standard errors of $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_{1n}, \hat{\beta}_{2n})$ and $\hat{\sigma}_n^2$ with $f(x, \alpha, \beta_1, \beta_2) = \alpha + \beta_1 x + \beta_2 x^2$.

| | $n$ | $\rho = 0$ Mean | $\rho = 0$ Std. error | $\rho = 0.5$ Mean | $\rho = 0.5$ Std. error | $\rho = 1$ Mean | $\rho = 1$ Std. error |
|---|---|---|---|---|---|---|---|
| | | | | Scenario **S1** | | | |
| $\hat{\alpha}_n$ | 200 | -19.99444 | (0.52156) | -19.99776 | (0.52417) | -20.00214 | (0.53271) |
| | 500 | -20.00171 | (0.33352) | -20.00164 | (0.33260) | -20.00333 | (0.33106) |
| $\hat{\beta}_{1n}$ | 200 | 9.99995 | (0.04303) | 10.01348 | (0.04427) | 10.02649 | (0.04441) |
| | 500 | 9.99961 | (0.01713) | 10.00537 | (0.01741) | 10.01056 | (0.01790) |
| $\hat{\beta}_{2n}$ | 200 | 0.09999 | (0.00137) | 0.09999 | (0.00133) | 0.10000 | (0.00115) |
| | 500 | 0.10000 | (0.00034) | 0.10001 | (0.00034) | 0.10000 | (0.00029) |
| $\hat{\sigma}_n^2$ | 200 | 8.87340 | (0.89132) | 8.83901 | (0.89370) | 8.79399 | (0.87995) |
| | 500 | 8.94769 | (0.56790) | 8.94198 | (0.57076) | 8.91645 | (0.56403) |
| | | | | Scenario **S2** | | | |
| $\hat{\alpha}_n$ | 200 | -19.99538 | (0.54080) | -19.99210 | (0.54464) | -20.00177 | (0.53812) |
| | 500 | -20.00287 | (0.33868) | -20.00412 | (0.33885) | -20.00142 | (0.34983) |
| $\hat{\beta}_{1n}$ | 200 | 10.00054 | (0.04456) | 10.01464 | (0.04606) | 10.02658 | (0.04718) |
| | 500 | 10.00025 | (0.01747) | 10.00548 | (0.01829) | 10.01142 | (0.01838) |
| $\hat{\beta}_{2n}$ | 200 | 0.10001 | (0.00145) | 0.10000 | (0.00136) | 0.10000 | (0.00118) |
| | 500 | 0.09999 | (0.00037) | 0.10000 | (0.00034) | 0.10000 | (0.00029) |
| $\hat{\sigma}_n^2$ | 200 | 8.85807 | (0.89311) | 8.83928 | (0.89470) | 8.79743 | (0.89541) |
| | 500 | 8.95826 | (0.56801) | 8.94974 | (0.56812) | 8.92162 | (0.56690) |
| | | | | Scenario **S3** | | | |
| $\hat{\alpha}_n$ | 200 | -19.99784 | (0.62713) | -20.00389 | (0.60584) | -19.99447 | (0.62031) |
| | 500 | -19.99644 | (0.39496) | -19.99572 | (0.38897) | -20.00707 | (0.40941) |
| $\hat{\beta}_{1n}$ | 200 | 9.99962 | (0.04949) | 10.01477 | (0.05129) | 10.03178 | (0.05180) |
| | 500 | 10.00011 | (0.02033) | 10.00647 | (0.02073) | 10.01282 | (0.02142) |
| $\hat{\beta}_{2n}$ | 200 | 0.10002 | (0.00165) | 0.09998 | (0.00156) | 0.10000 | (0.00131) |
| | 500 | 0.10000 | (0.00040) | 0.10000 | (0.00039) | 0.10000 | (0.00033) |
| $\hat{\sigma}_n^2$ | 200 | 8.96801 | (0.91677) | 8.93070 | (0.91952) | 8.86505 | (0.93340) |
| | 500 | 9.08728 | (0.58567) | 9.07935 | (0.58298) | 9.03815 | (0.58235) |

Figure 6: Density estimate of $\hat{\theta}_n$.

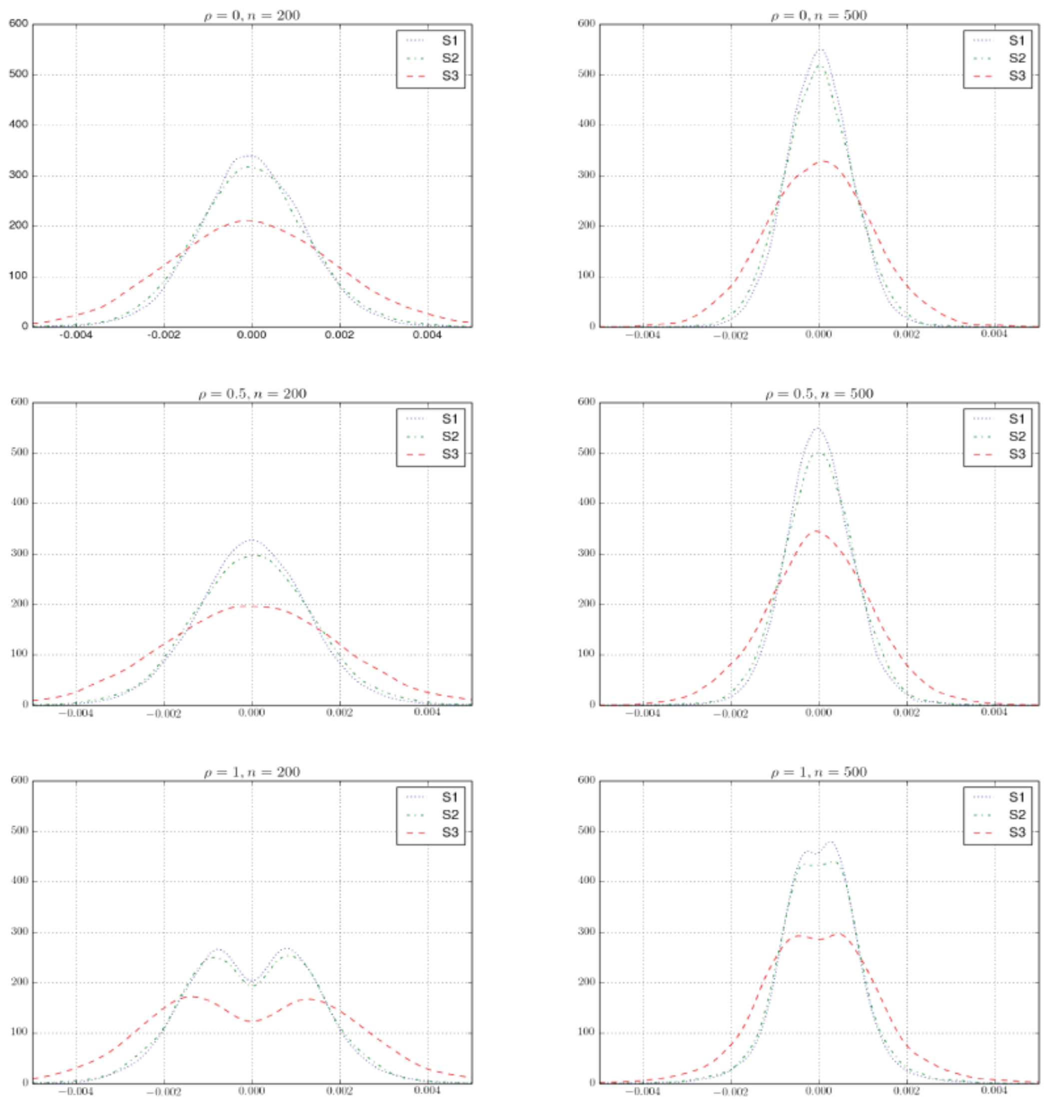

Figure 7: Density estimate of $\hat{\alpha}_n$.

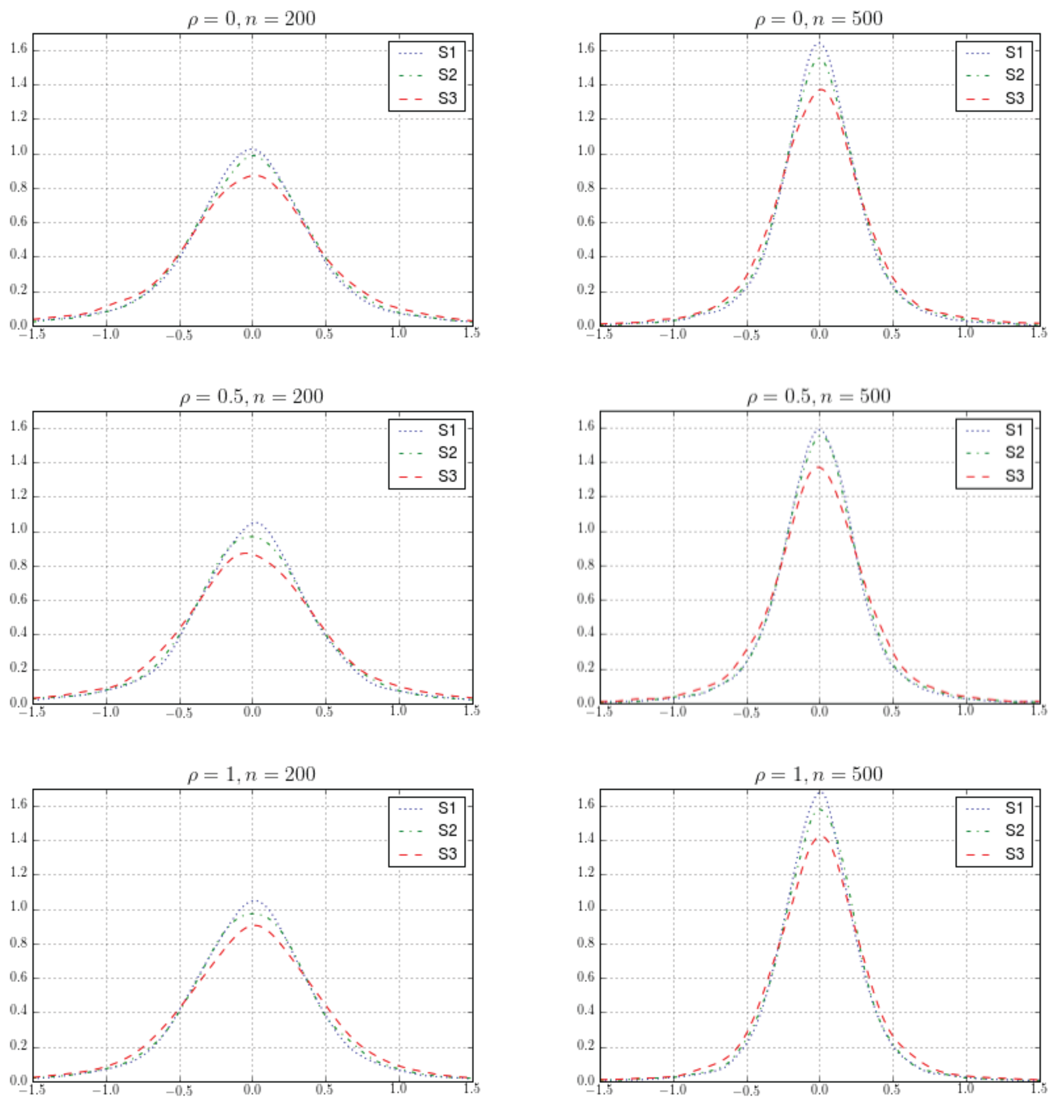

Figure 8: Density estimate of $\hat{\beta}_{1n}$.

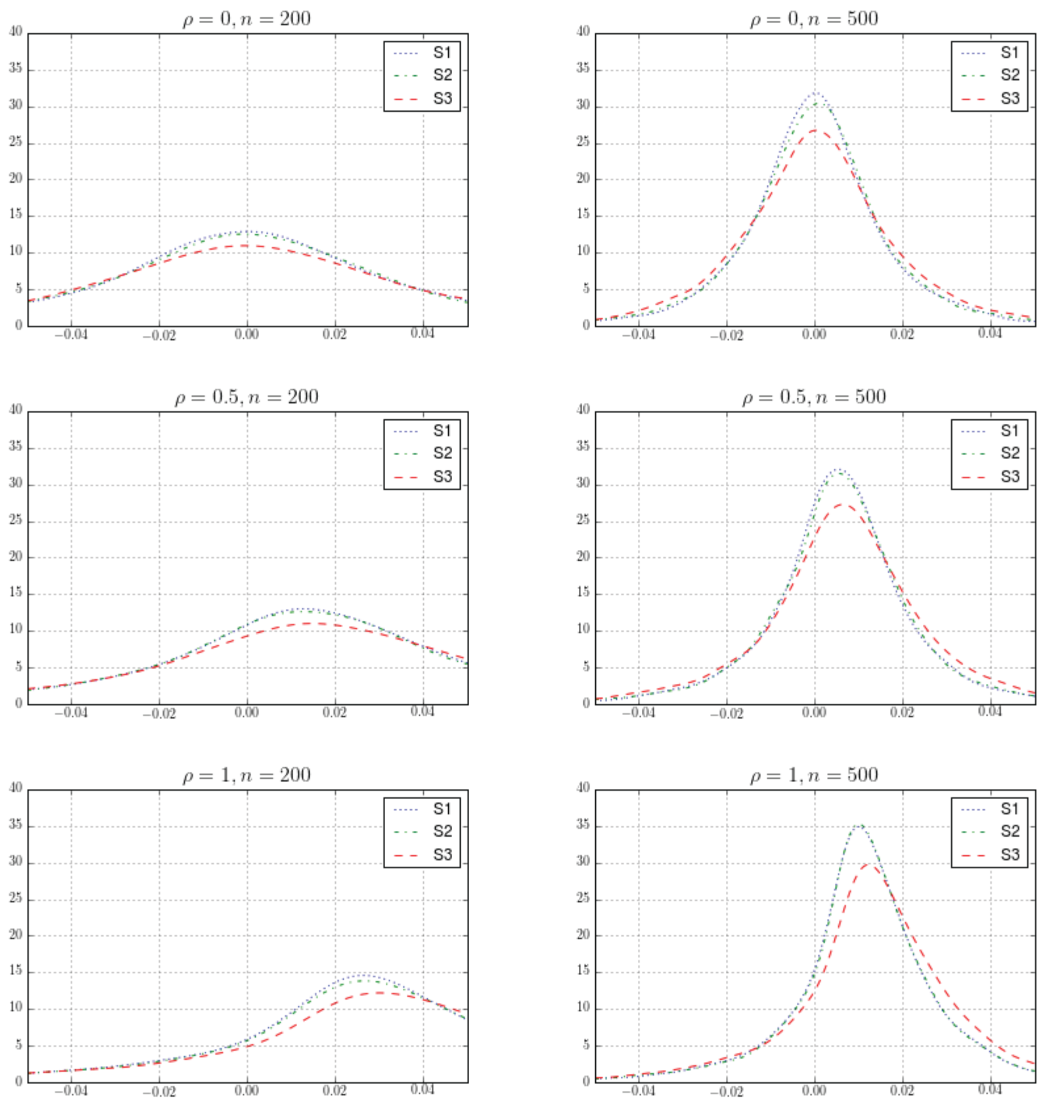

Figure 9: Density estimate of $\hat{\beta}_{2n}$.

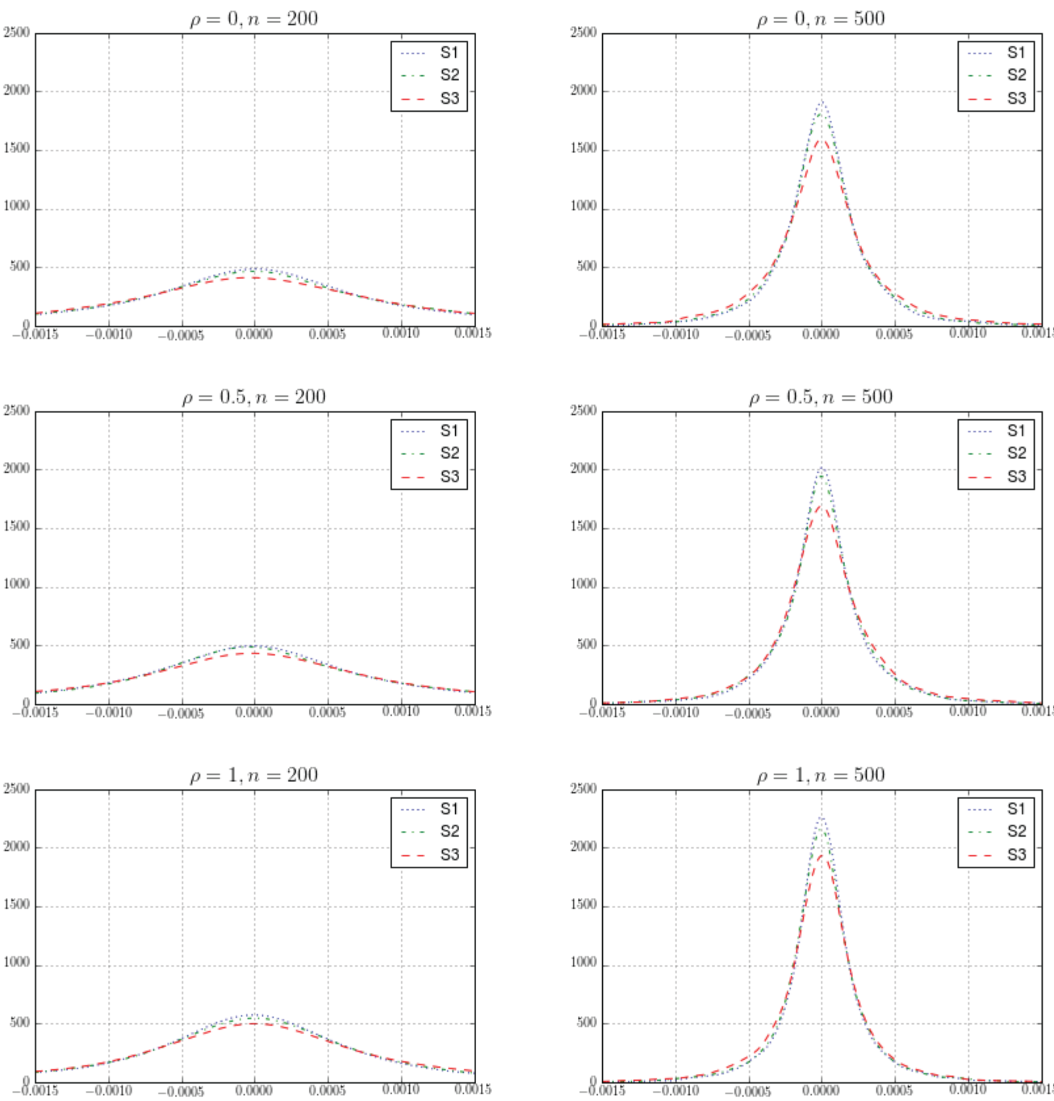

Figure 10: Density estimate of $\hat{\theta}_n$ $t$-ratios.

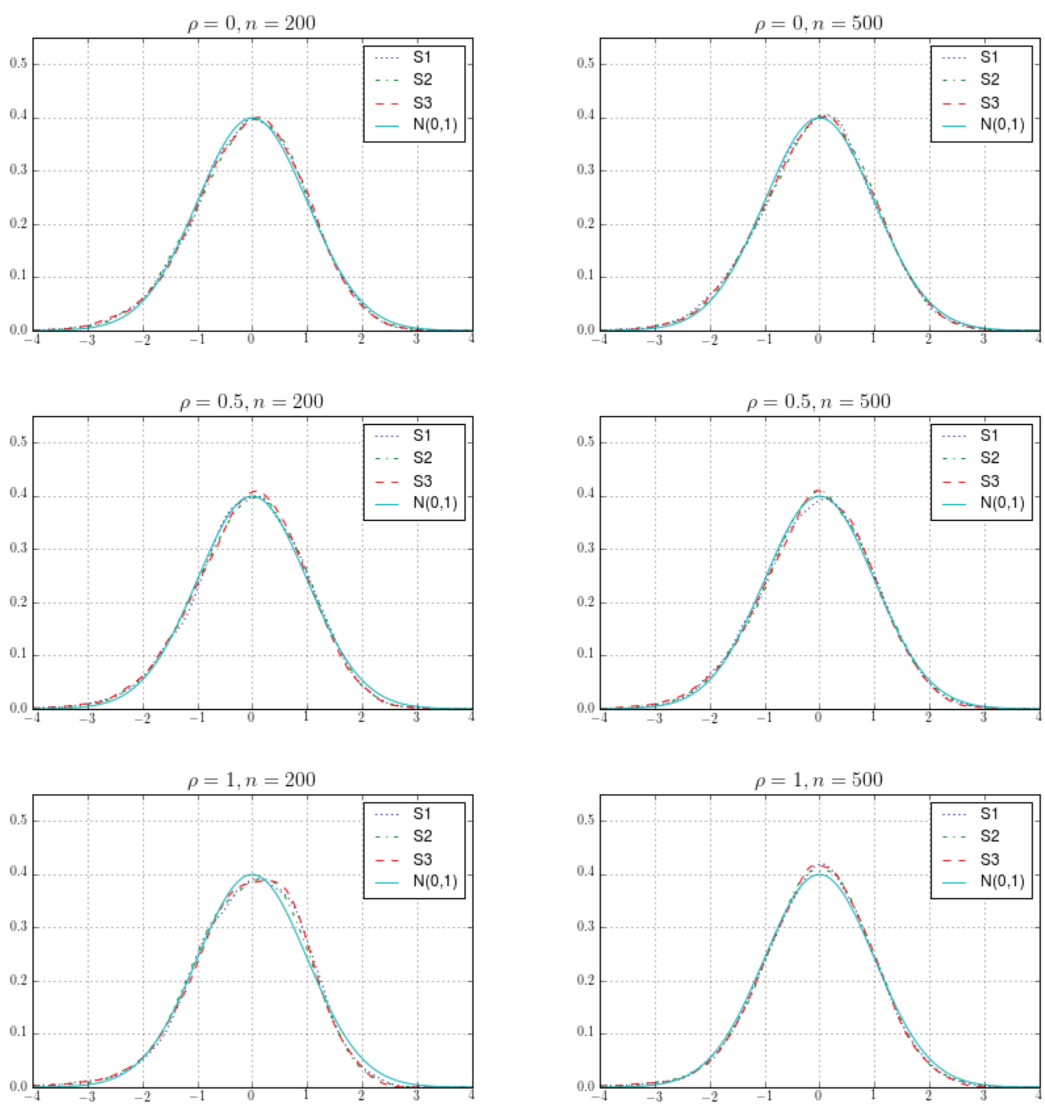

Figure 11: Density estimate of $\hat{\alpha}_n$ $t$-ratios.

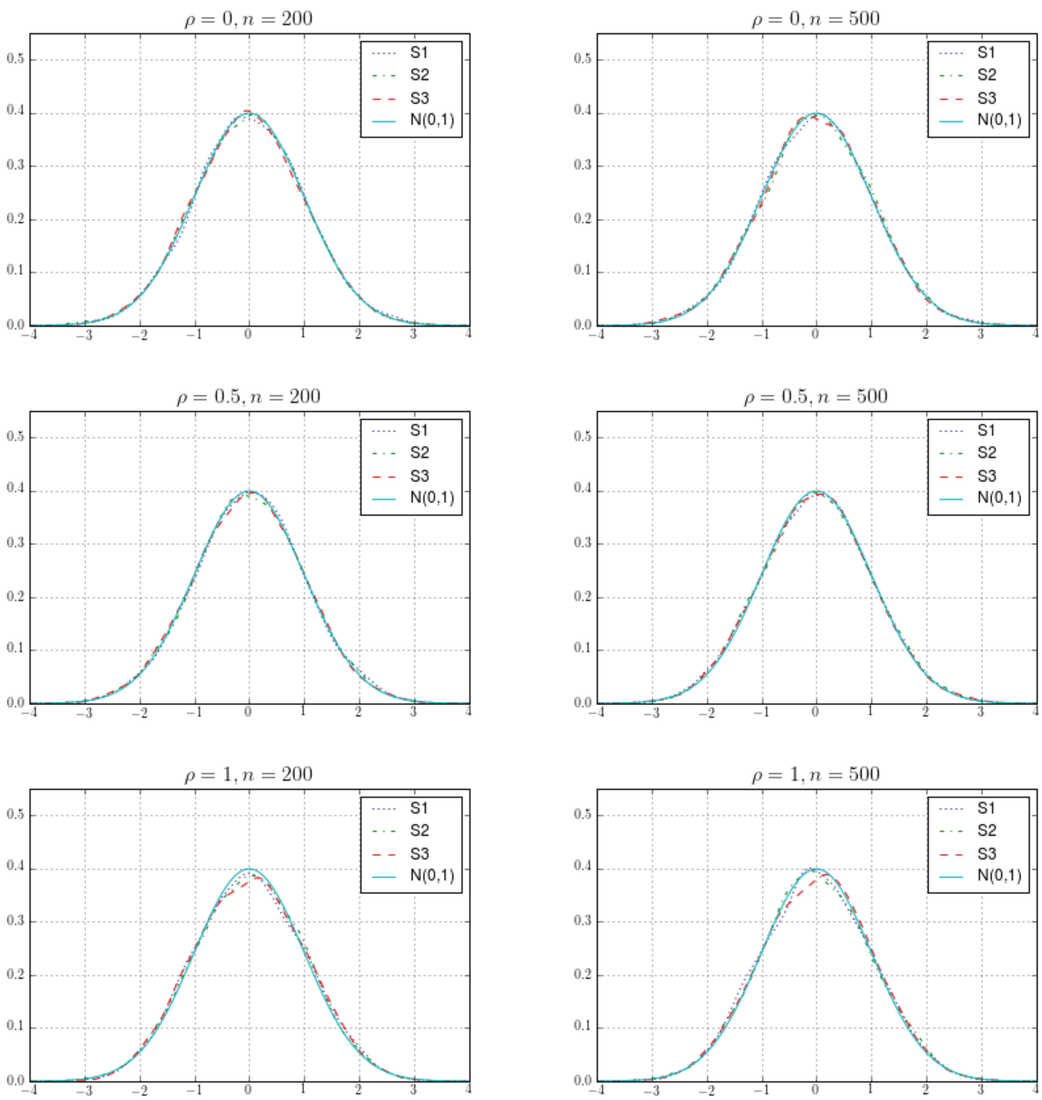

Figure 12: Density estimate of $\hat{\beta}_{1n}$ $t$-ratios.

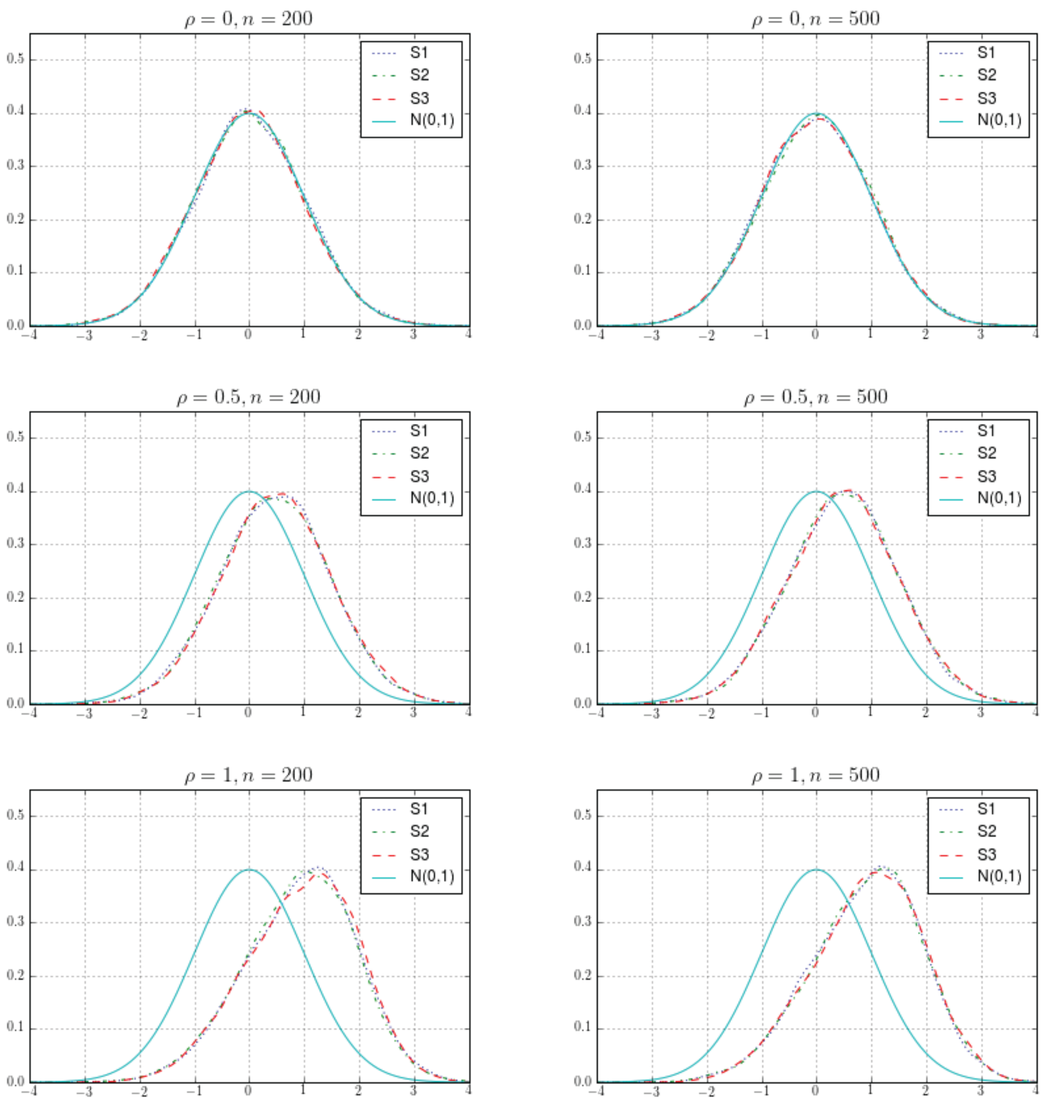

Figure 13: Density estimate of $\hat{\beta}_{2n}$ $t$-ratios.

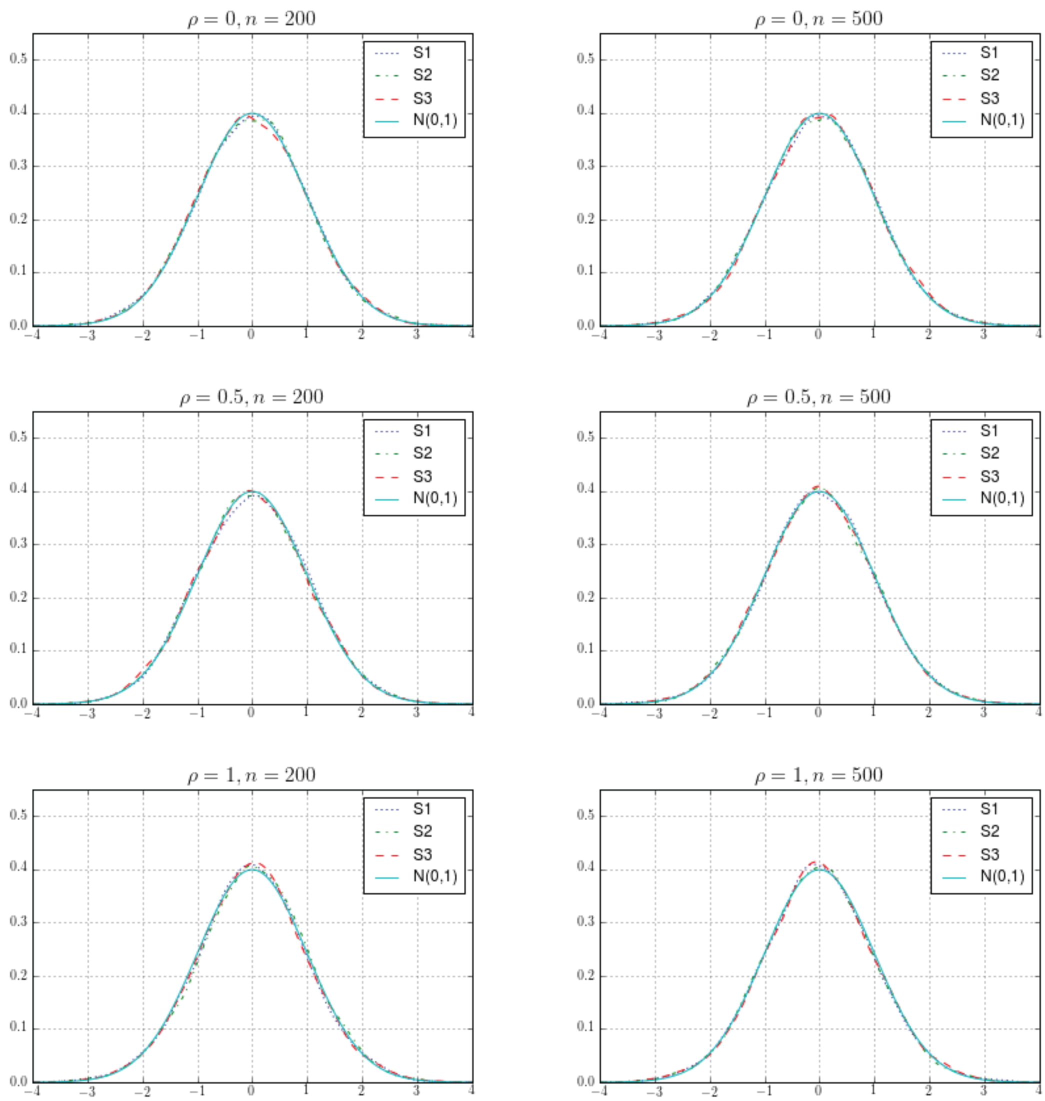

Figure 14: Estimates and 95% Confidence Intervals of $\hat{\alpha}_n$.

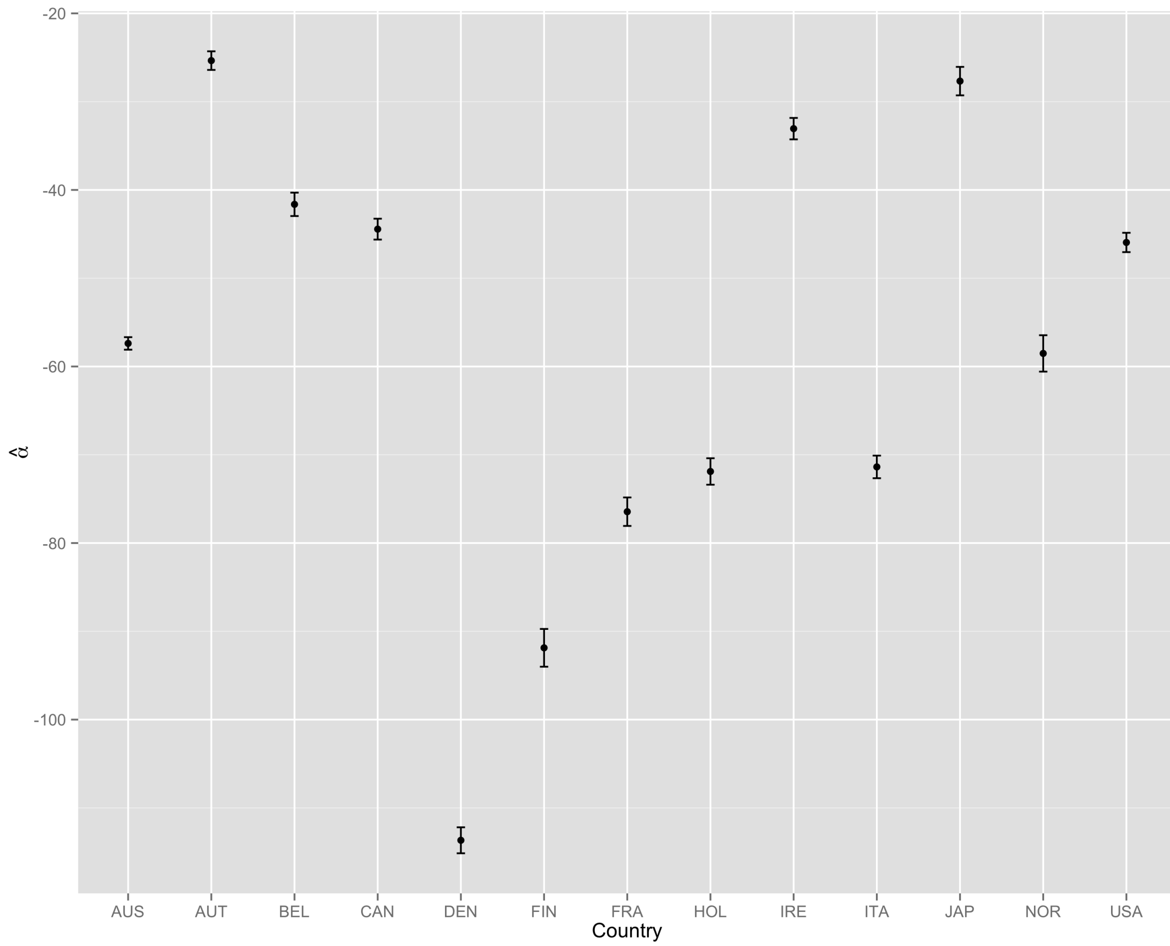


Table 4: Confidence Intervals (95% confidence) of $\hat{\alpha}_n$.

| Country | $\hat{\alpha}_n$ | Lower | Upper |
|---|---|---|---|
| AUS | -57.387 | -58.005 | -56.769 |
| AUT | -25.352 | -26.414 | -24.290 |
| BEL | -41.632 | -43.144 | -40.121 |
| CAN | -44.442 | -45.615 | -43.268 |
| DEN | -113.670 | -115.375 | -111.965 |
| FIN | -91.869 | -94.019 | -89.720 |
| FRA | -76.449 | -78.162 | -74.735 |
| HOL | -71.892 | -73.439 | -70.344 |
| IRE | -33.058 | -34.265 | -31.851 |
| ITA | -71.375 | -72.604 | -70.146 |
| JAP | -27.674 | -29.293 | -26.056 |
| NOR | -58.519 | -60.442 | -56.597 |
| USA | -45.950 | -46.945 | -44.954 |

Table 5: Confidence Intervals (95% confidence) of $\hat{\beta}_{1n}$.

| Country | $\hat{\beta}_{1n}$ | Lower | Upper |
|---|---|---|---|
| AUS | 11.520 | 10.810 | 12.230 |
| AUT | 5.075 | 4.193 | 5.958 |
| BEL | 9.116 | 7.922 | 10.310 |
| CAN | 9.217 | 8.114 | 10.321 |
| DEN | 23.724 | 22.242 | 25.205 |
| FIN | 18.967 | 17.225 | 20.709 |
| FRA | 16.443 | 15.000 | 17.886 |
| HOL | 14.921 | 13.579 | 16.262 |
| IRE | 6.816 | 6.062 | 7.570 |
| ITA | 14.502 | 13.647 | 15.356 |
| JAP | 5.447 | 4.614 | 6.279 |
| NOR | 11.802 | 10.555 | 13.050 |
| USA | 9.536 | 8.449 | 10.623 |

Table 6: Confidence Intervals (95% confidence) of $\hat{\beta}_{2n}$.

| Country | $\hat{\beta}_{2n}$ | Lower | Upper |
|---|---|---|---|
| AUS | -0.562 | -0.730 | -0.394 |
| AUT | -0.246 | -0.389 | -0.102 |
| BEL | -0.485 | -0.721 | -0.248 |
| CAN | -0.462 | -0.732 | -0.192 |
| DEN | -1.226 | -1.586 | -0.865 |
| FIN | -0.967 | -1.256 | -0.678 |
| FRA | -0.875 | -1.190 | -0.559 |
| HOL | -0.762 | -1.093 | -0.432 |
| IRE | -0.341 | -0.443 | -0.238 |
| ITA | -0.729 | -0.882 | -0.576 |
| JAP | -0.258 | -0.365 | -0.151 |
| NOR | -0.586 | -0.822 | -0.351 |
| USA | -0.477 | -0.768 | -0.186 |