Representations of centrally extended Lie superalgebra $\mathfrak{psl}(2|2)$

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Abstract

The symmetries provided by representations of the centrally extended Lie superalgebra $\mathfrak{psl}(2|2)$ are known to play an important role in the spin chain models originated in the planar anti-de Sitter/conformal field theory correspondence and one-dimensional Hubbard model. We give a complete description of finite-dimensional irreducible representations of this superalgebra thus extending the work of Beisert which deals with a generic family of representations. Our description includes a new class of modules with degenerate eigenvalues of the central elements. Moreover, we construct explicit bases in all irreducible representations by applying the techniques of Mickelsson–Zhelobenko algebras.

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1 Introduction

As discovered by Beisert [1, 2, 3], certain spin chain models originated in the planar anti-de Sitter/conformal field theory (AdS/CFT) correspondence admit *hidden symmetries* provided by the action of the Yangian $\mathrm{Y}(\mathfrak{g})$ associated with the centrally extended Lie superalgebra

$$\mathfrak{g} = \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3.$$

This is a semi-direct product of the simple Lie superalgebra $\mathfrak{psl}(2|2)$ of type $A(1,1)$ and the abelian Lie algebra $\mathbb{C}^3$ spanned by elements C, K and P which are central in $\mathfrak{g}$. Due to the results of [5], $\mathfrak{psl}(2|2)$ is distinguished among the basic classical Lie superalgebras by the existence of a three-dimensional central extension. A new R-matrix associated with the extended Lie superalgebra $\mathfrak{g}$ is found by Yamane [14]. Furthermore, $\mathfrak{g}$ can be obtained from the Lie superalgebras of type $D(2,1;\alpha)$ by a particular limit with respect to the parameter α.

The Yangian symmetries of the one-dimensional Hubbard model associated with $\mathrm{Y}(\mathfrak{g})$ were considered in [2]; they extend those provided by the direct sum of two copies of the Yangian for $\mathfrak{sl}(2)$ previously found in [13]. An extensive review of the Yangian symmetries in the spin chain models can be found in [12].

These applications motivate the study of representations of both the Lie superalgebra $\mathfrak{g}$ and its Yangian. In this paper we aim to prove a classification theorem for finite-dimensional irreducible representation of $\mathfrak{g}$. Generic representations of $\mathfrak{g}$ were already described by Beisert [3]. As we demonstrate below, beside these generic modules, the complete classification includes some degenerate representations which were not considered in [3]. In more detail, if L is a finite-dimensional irreducible representation of the Lie superalgebra $\mathfrak{g}$, then each of the central elements C, K and P acts in L as multiplication by a scalar. We will let the lower case letters denote the corresponding scalars,

$$C \mapsto c, \qquad K \mapsto k, \qquad P \mapsto p.$$

The Lie superalgebra $\mathfrak{psl}(2|2)$ is known to admit a family of automorphisms parameterized by elements of the group $\mathrm{SL}(2)$, as described in [4]. As pointed out in [3], by twisting the action of $\mathfrak{g}$ in L by such an automorphism, we obtain another irreducible representation of $\mathfrak{g}$, where the values c, k, p are transformed by

$$\begin{pmatrix} c & -k \\ p & -c \end{pmatrix} \mapsto \begin{pmatrix} u & v \\ w & z \end{pmatrix} \begin{pmatrix} c & -k \\ p & -c \end{pmatrix} \begin{pmatrix} u & v \\ w & z \end{pmatrix}^{-1},$$

and complex numbers u, v, w, z satisfy $uz - vw = 1$. An appropriate transformation of this form brings the 2×2 matrix formed by c, k, p to the Jordan canonical form. In the case, where the canonical form is a diagonal matrix,

$$\begin{pmatrix} c & -k \\ p & -c \end{pmatrix} \mapsto \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix},$$

the values of k and p under the twisted action of $\mathfrak{g}$ are zero, and so the twisted module becomes an irreducible representation of the Lie superalgebra $\mathfrak{sl}(2|2)$. Such representations are well-studied; see e.g. [4], [8], [11] and [15] for an explicit construction of basis vectors and formulas for the action of the generators in this basis. It is essentially this case which was considered in [3] in relation with the symmetries of the S-matrix for the AdS/CFT correspondence. The only remaining possibility is the case where the canonical form is the 2×2 Jordan block,

$$\begin{pmatrix} c & -k \\ p & -c \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This can only happen when $c^2 - pk = 0$ so that both eigenvalues of the matrix are zero. Our goal in this paper is to study the structure of these representations of $\mathfrak{g}$. In what follows we consider the class of finite-dimensional irreducible representations of $\mathfrak{g}$ where both central elements C and K act as the zero operators, while P acts as the identity operator. Our main result is a classification theorem for such representations of $\mathfrak{g}$.

Main Theorem. *A complete list of pairwise non-isomorphic finite-dimensional irreducible representations of $\mathfrak{g}$ where the central elements act by $C \mapsto 0$, $K \mapsto 0$, $P \mapsto 1$, consists of*

1. *the Kac modules $K(m,n)$ with $m, n \in \mathbb{Z}_+$ and $m \neq n$,* $\dim K(m,n) = 16(m+1)(n+1)$,
2. *the modules S_n with $n \in \mathbb{Z}_+$,* $\quad \dim S_n = 8(n+1)(n+2)$.

Here the *Kac modules* $K(m,n)$ over $\mathfrak{g}$ are defined as the induced modules from finite-dimensional irreducible representations of the Lie algebra $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ in the same way as for simple Lie superalgebras, and S_n are certain submodules of $K(n,n)$; see Sec. 3.2 for precise definitions. Comparing this description with the classification theorem for representations of the Lie superalgebra $\mathfrak{psl}(2|2)$ [6], note that nontrivial irreducible quotients of the Kac modules over $\mathfrak{psl}(2|2)$ have dimension $4n(n+2)+2$ (they are also known as *short multiplets*). Hence the family of $\mathfrak{g}$-modules S_n does not have their counterparts within the class of $\mathfrak{psl}(2|2)$-modules.

To give a physical interpretation of the conditions on c, p and k, note that in the original spin chain models [1, 3], the scalar c corresponds to the *energy* of a particle moving on the spin chain, whereas p and k correspond to its *momenta*. Thus, the relation $c^2 - pk = \text{const}$ is the dispersion relation of the particle, and the automorphisms of $\mathfrak{psl}(2|2)$ provided by elements of SL(2) are interpreted as the *Lorentz symmetry* which preserves the dispersion relation. Therefore, the relation $c^2 - pk = 0$ describes a massless particle on the *light-cone*. Due to the Main Theorem, the multiplets of the particle on the light-cone are *shorter* than long multiplets [3] and *longer* than short multiplets. Thus, the particles on the light-cone are described by *middle* multiplets.

Our arguments are based on the theory of Mickelsson–Zhelobenko algebras [16]. We also apply it to construct bases of all finite-dimensional irreducible representations of $\mathfrak{g}$. Formulas for the action of the generators of $\mathfrak{g}$ in such a basis can also be found in an explicit form. Furthermore, this description of representations extends to the case, where the central elements P and K of $\mathfrak{g}$ act as the zero operators, allowing us to essentially reproduce the results of [8], [11] and [15] concerning representations of $\mathfrak{gl}(2|2)$ and $\mathfrak{sl}(2|2)$.

This paper is organized as follows. In Sec. 2 we review the centrally extended Lie superalgebra $\mathfrak{g}$. In Sec. 3 we describe finite-dimensional irreducible representation of $\mathfrak{g}$. After introducing the Mickelsson–Zhelobenko algebra in Sec. 3.1 we construct a basis of the Kac module by the Mickelsson–Zhelobenko generators and establish its irreducibility properties. In Sec. 3.3 the classification theorem is proved. Explicit action of the generators on the Kac modules is described in Sec. 3.4. Appendix A is devoted to relations in the Mickelsson–Zhelobenko algebra. In Appendix B the action of raising operators is produced; it is used to prove irreducibility of the Kac modules. In Appendix C the $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$-highest weight vectors of the Kac modules are listed.

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2 Central extension of Lie superalgebra $\mathfrak{psl}(2|2)$

The general linear Lie superalgebra $\mathfrak{gl}(2|2)$ over $\mathbb{C}$ has the standard basis E_{ij}, $1 \leqslant i, j \leqslant 4$. The $\mathbb{Z}_2$-grading on $\mathfrak{gl}(2|2)$ is defined by setting $\deg E_{ij} = \bar{\imath} + \bar{\jmath}$, where we use the notation $\bar{\imath} = 0$ for $1 \leqslant i \leqslant 2$ and $\bar{\imath} = 1$ for $3 \leqslant i \leqslant 4$. The commutation relations have the form

$$\big[E_{ij}, E_{kl}\big] = \delta_{kj}\, E_{il} - \delta_{il}\, E_{kj}(-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})},$$

where the square brackets denote the super-commutator. Then $\mathfrak{sl}(2|2)$ is the subalgebra of $\mathfrak{gl}(2|2)$ spanned by the elements

$$h_1 = E_{11} - E_{22}, \qquad h_2 = E_{22} + E_{33}, \qquad h_3 = E_{33} - E_{44}$$

and by all elements E_{ij} with $i \neq j$. We have the direct sum decomposition

$$\mathfrak{gl}(2|2) = \mathfrak{sl}(2|2) \oplus \mathbb{C}(E_{11} + E_{22} - E_{33} - E_{44}).$$

Furthermore, the element

$$C = \frac{1}{2}\, h_1 + h_2 - \frac{1}{2}\, h_3 = \frac{1}{2}\,(E_{11} + E_{22} + E_{33} + E_{44})$$

is central in $\mathfrak{sl}(2|2)$, and the simple Lie superalgebra of type $A(1,1)$ is defined as the quotient of $\mathfrak{sl}(2|2)$ by the ideal generated by C. This quotient is denoted by $\mathfrak{psl}(2|2)$. As in [3], we will consider the Lie superalgebra

$$\mathfrak{g} = \mathfrak{sl}(2|2) \ltimes \mathbb{C}^2 = \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3,$$

where $\mathbb{C}^2$ is the abelian Lie algebra with the basis elements K and P, while abelian Lie algebra $\mathbb{C}^3$ is spanned by the elements C, K and P. These elements are central in $\mathfrak{g}$ and the only nontrivial additional relations take the form

$$[E_{13}, E_{24}] = -[E_{23}, E_{14}] = K, \tag{2.1}$$

$$[E_{31}, E_{42}] = -[E_{32}, E_{41}] = P. \tag{2.2}$$

More precisely, the commutations relations in $\mathfrak{g}$ are determined by those for the basis elements

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})} + \bar{\epsilon}_{ik}\, \epsilon_{jl} P + \epsilon_{ik}\, \bar{\epsilon}_{jl} K, \tag{2.3}$$

where the constants ϵ_{ij} and $\bar{\epsilon}_{ij}$ are zero except for the values

$$\epsilon_{12} = -\epsilon_{21} = 1 \qquad \text{and} \qquad \bar{\epsilon}_{34} = -\bar{\epsilon}_{43} = 1.$$

The Lie subalgebra $\mathfrak{g}_0$ of even elements in $\mathfrak{g}$ is the direct sum

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathbb{C}^3, \qquad \mathfrak{k} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \subset \mathfrak{psl}(2|2),$$

where the two copies of $\mathfrak{sl}(2)$ are spanned by the elements E_{12}, E_{21}, h_1 and E_{34}, E_{43}, h_3, respectively.

Given complex numbers u, v, w, z such that $uz - vw = 1$, the corresponding automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ mentioned in the Introduction is determined by the mapping

$$E_{13} \mapsto u\, E_{13} + v\, E_{42}, \qquad E_{42} \mapsto z\, E_{42} + w\, E_{13}, \tag{2.4}$$

and the condition that each element of the subalgebra $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ is stable under ϕ. By [3], the images of the central elements C, K, P are then found from the matrix relation

$$\begin{pmatrix} C & -K \\ P & -C \end{pmatrix} \mapsto \begin{pmatrix} u & v \\ w & z \end{pmatrix} \begin{pmatrix} C & -K \\ P & -C \end{pmatrix} \begin{pmatrix} u & v \\ w & z \end{pmatrix}^{-1}.$$

3 Finite-dimensional irreducible representations

As we pointed out in the Introduction, our main focus will be on finite-dimensional irreducible representations of $\mathfrak{g}$, where the eigenvalues of the central elements are given by

$$C \mapsto 0, \qquad K \mapsto 0, \qquad P \mapsto 1.$$

This means that we will essentially deal with the extended Lie superalgebra $\mathfrak{psl}(2|2) \oplus \mathbb{C}P$, where the only nontrivial additional relations are (2.2).

From the viewpoint of the spin chain model [3], these representations should describe the particle states on the *light-cone* since the *dispersion relations* are given by $c^2 - pk = 0$, where c and p, k correspond to *energy* and *momenta* of the particles, respectively.

3.1 Mickelsson–Zhelobenko algebras

We will use the *Mickelsson–Zhelobenko algebra* $\mathrm{Z}(\mathfrak{g}, \mathfrak{k})$ associated with the pair $\mathfrak{k} \subset \mathfrak{g}$. An extensive theory of such algebras was developed in [16]; see also [9, Ch. 9] and [10] where they were employed for constructions of bases of Gelfand–Tsetlin type in representations of classical Lie algebras and superalgebras. To recall the definitions, denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{k}$ spanned by the basis elements h_1 and h_3. We have the triangular decomposition

$$\mathfrak{k} = \mathfrak{k}^- \oplus \mathfrak{h} \oplus \mathfrak{k}^+,$$

where

$$\mathfrak{k}^- = \text{span of } \{E_{21}, E_{43}\} \qquad \text{and} \qquad \mathfrak{k}^+ = \text{span of } \{E_{12}, E_{34}\}.$$

Let $\mathrm{J} = \mathrm{U}(\mathfrak{g})\,\mathfrak{k}^+$ be the left ideal of $\mathrm{U}(\mathfrak{g})$ generated by $\mathfrak{k}^+$ and consider the quotient

$$\mathrm{M}(\mathfrak{g}, \mathfrak{k}) = \mathrm{U}(\mathfrak{g})/\mathrm{J}.$$

The *Mickelsson algebra* $\mathrm{S}(\mathfrak{g}, \mathfrak{k})$ is defined by

$$\mathrm{S}(\mathfrak{g}, \mathfrak{k}) = \{v \in \mathrm{M}(\mathfrak{g}, \mathfrak{k}) \mid \mathfrak{k}^+ v = 0\}.$$

Given a finite-dimensional $\mathfrak{g}$-module V, its subspace

$$V^+ = \{v \in V \mid \mathfrak{k}^+ v = 0\} \tag{3.1}$$

is a $\mathrm{S}(\mathfrak{g}, \mathfrak{k})$-module whose structure largely determines the structure of V; see [16] for more details. Denote by $\mathrm{R}(\mathfrak{h})$ the field of fractions of the commutative algebra $\mathrm{U}(\mathfrak{h})$. The *Mickelsson–Zhelobenko algebra* $\mathrm{Z}(\mathfrak{g}, \mathfrak{k})$ can be defined as the extension

$$\mathrm{Z}(\mathfrak{g}, \mathfrak{k}) = \mathrm{S}(\mathfrak{g}, \mathfrak{k}) \otimes_{\mathrm{U}(\mathfrak{h})} \mathrm{R}(\mathfrak{h}). \tag{3.2}$$

As was observed by Zhelobenko (see [16]), the algebraic structure of $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ can be described with the use of the *extremal projector* $p = p(\mathfrak{k})$ which is a formal series of elements of $\mathrm{U}(\mathfrak{k})$ with coefficients in $\mathrm{R}(\mathfrak{h})$ given by

$$p = \Bigg(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} E_{21}^k E_{12}^k \frac{1}{(h_1+2)\cdots(h_1+k+1)}\Bigg) \times \Bigg(1 + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} E_{43}^l E_{34}^l \frac{1}{(h_3+2)\cdots(h_3+l+1)}\Bigg).$$

The operator p has the property $p^2 = p$ and satisfies the relations

$$E_{12}\, p = p\, E_{21} = 0 \qquad \text{and} \qquad E_{34}\, p = p\, E_{43} = 0. \tag{3.3}$$

The extremal projector naturally acts on the extension

$$\mathrm{M}'(\mathfrak{g},\mathfrak{k}) = \mathrm{M}(\mathfrak{g},\mathfrak{k}) \otimes_{\mathrm{U}(\mathfrak{h})} \mathrm{R}(\mathfrak{h}).$$

It projects $\mathrm{M}'(\mathfrak{g},\mathfrak{k})$ onto $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ with the kernel $\mathfrak{k}^-\mathrm{M}'(\mathfrak{g},\mathfrak{k})$. In particular, $\mathrm{Z}(\mathfrak{g},\mathfrak{k}) = p\,\mathrm{M}'(\mathfrak{g},\mathfrak{k})$. Moreover, the algebra $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ is generated by the elements

$$z_{ik} = p\, E_{ik}, \qquad z_{ki} = p\, E_{ki}, \qquad i = 1,2 \quad \text{and} \quad k = 3,4,$$

together with C, K and P. We will call the elements z_{ik} and z_{ki} *raising* and *lowering* operators, respectively. They are given by the following explicit formulas.

Lemma 3.1. *The raising operators are found by*

$$\begin{aligned}
z_{14} &= E_{14},\\
z_{13} &= E_{13} + E_{43}E_{14}\frac{1}{h_3+1},\\
z_{24} &= E_{24} - E_{21}E_{14}\frac{1}{h_1+1},\\
z_{23} &= E_{23} - E_{21}E_{13}\frac{1}{h_1+1} + E_{43}E_{24}\frac{1}{h_3+1} - E_{21}E_{43}E_{14}\frac{1}{(h_1+1)(h_3+1)},
\end{aligned}$$

and the lowering operators are

$$\begin{aligned}
z_{41} &= E_{41} + E_{21}E_{42}\frac{1}{h_1+1} - E_{43}E_{31}\frac{1}{h_3+1} - E_{21}E_{43}E_{32}\frac{1}{(h_1+1)(h_3+1)},\\
z_{31} &= E_{31} + E_{21}E_{32}\frac{1}{h_1+1},\\
z_{42} &= E_{42} - E_{43}E_{32}\frac{1}{h_3+1},\\
z_{32} &= E_{32}.
\end{aligned}$$

Proof. These expressions follow by the application of the explicit formula for the extremal projector p. □

We will need expressions for the elements E_{ik} and E_{ki} in terms of the raising and lowering operators provided by the next lemma.

Lemma 3.2. *We have the relations in* $\mathrm{M}'(\mathfrak{g},\mathfrak{k})$*:*

$$
\begin{aligned}
E_{14} &= z_{14},\\
E_{13} &= z_{13} - E_{43} z_{14} \frac{1}{h_3+1},\\
E_{24} &= z_{24} + E_{21} z_{14} \frac{1}{h_1+1},\\
E_{23} &= z_{23} + E_{21} z_{13} \frac{1}{h_1+1} - E_{43} z_{24} \frac{1}{h_3+1} - E_{21} E_{43} z_{14} \frac{1}{(h_1+1)(h_3+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
E_{41} &= z_{41} - E_{21} z_{42} \frac{1}{h_1+1} + E_{43} z_{31} \frac{1}{h_3+1} - E_{21} E_{43} z_{32} \frac{1}{(h_1+1)(h_3+1)},\\
E_{31} &= z_{31} - E_{21} z_{32} \frac{1}{h_1+1},\\
E_{42} &= z_{42} + E_{43} z_{32} \frac{1}{h_3+1},\\
E_{32} &= z_{32}.
\end{aligned}
$$

Proof. The formulas are immediate from Lemma 3.1. □

As follows from [16], the generators of the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ satisfy quadratic relations which can be derived from Lemmas 3.1 and 3.2. In particular, for $i = 1, 2$ and $k = 3, 4$ we have

$$z_{ik}^2 = 0 \qquad \text{and} \qquad z_{ki}^2 = 0.$$

Complete sets of relations in $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ are listed in Appendix A.

3.2 Kac modules

For nonnegative integers m and n we will denote by $L^0(m,n)$ the finite-dimensional irreducible representation of the Lie algebra $\mathfrak{k} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ with the highest weight $[m,n]$. This representation is generated by the highest vector w such that

$$E_{12}\, w = E_{34}\, w = 0, \qquad h_1\, w = m w, \qquad h_3\, w = n w.$$

The vectors

$$E_{21}^k E_{43}^l w, \qquad k = 0, 1, \dots, m, \qquad l = 0, 1, \dots, n, \tag{3.4}$$

form a basis of $L^0(m, n)$. We extend $L^0(m, n)$ to a representation of the subalgebra $\mathfrak{b} \subset \mathfrak{g}$, spanned by $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathbb{C}^3$ and the elements E_{ik} with $i = 1, 2$ and $k = 3, 4$. These additional elements act as the zero operators, while $C \mapsto 0$, $K \mapsto 0$ and $P \mapsto 1$. The corresponding *Kac module* $K(m, n)$ is defined as the induced representation

$$K(m, n) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} L^0(m, n). \tag{3.5}$$

Its basis is formed by the vectors

$$E_{41}^{\theta_1} E_{31}^{\theta_2} E_{42}^{\theta_3} E_{32}^{\theta_4} E_{21}^k E_{43}^l w$$

where each θ_i takes values in $\{0, 1\}$ and k, l are as in (3.4). In particular,

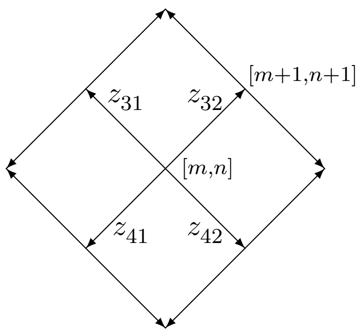
$$\dim K(m, n) = 16(m + 1)(n + 1).$$

As in (3.1) we will write $K^+(m, n)$ for the subspace of $\mathfrak{k}^+$-invariants in $K(m, n)$. Note that the action of the elements z_{ik} and z_{ki} of the Mickelsson–Zhelobenko algebra $\mathrm{Z}(\mathfrak{g}, \mathfrak{k})$ in $K^+(m, n)$ is well-defined. The denominators of these rational functions do not vanish when h_1 and h_3 are replaced by the corresponding eigenvalues of weight vectors. In accordance with [16], a basis of $K^+(m, n)$ can be constructed with the use of ordered products of the lowering operators z_{ki} with $i = 1, 2$ and $k = 3, 4$. Below we use this approach to write down explicit basis elements.

Suppose first that $m, n \geqslant 2$. Consider the elements

$$z_{41}^{\theta_1} z_{31}^{\theta_2} z_{42}^{\theta_3} z_{32}^{\theta_4} w \in K^+(m, n) \tag{3.6}$$

with each θ_i taking values in $\{0, 1\}$. Each element (3.6) can be interpreted as a path in the following labelled oriented graph



where the vertices belong to the lattice $\mathbb{Z}^2$, the middle vertex represents the weight $[m,n]$ of w and the four vectors $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ indicate the action of the respective lowering operators z_{32}, z_{42}, z_{31} and z_{41}. At most one step may be taken in any direction beginning with $(1,1)$, then following with $(1,-1)$, $(-1,1)$ and $(-1,-1)$. The monomial (3.6) is obtained by writing the product of the labells of the vectors used in the path and apply the corresponding operator to w. For instance, the path of length two consisting of $(1,-1)$ then following by $(-1,1)$ corresponds to the monomial $z_{31}\,z_{42}\,w$ of weight $[m,n]$.

We thus obtain the following weight vectors in $K^+(m,n)$:

$$
\begin{array}{rl}
[m,n] & w,\quad z_{41}\,z_{32}\,w,\quad z_{31}\,z_{42}\,w,\quad z_{41}\,z_{31}\,z_{42}\,z_{32}\,w,\\
[m+1,n+1] & z_{32}\,w,\quad z_{31}\,z_{42}\,z_{32}\,w,\\
[m+1,n-1] & z_{42}\,w,\quad z_{41}\,z_{42}\,z_{32}\,w,\\
[m-1,n+1] & z_{31}\,w,\quad z_{41}\,z_{31}\,z_{32}\,w,\\
[m-1,n-1] & z_{41}\,w,\quad z_{41}\,z_{31}\,z_{42}\,w,\\
[m+2,n] & z_{42}\,z_{32}\,w,\\
[m,n+2] & z_{31}\,z_{32}\,w,\\
[m-2,n] & z_{41}\,z_{31}\,w,\\
[m,n-2] & z_{41}\,z_{42}\,w.
\end{array}
$$

In the cases where $m \in \{0,1\}$ or $n \in \{0,1\}$ families of weight vectors in $K^+(m,n)$ are constructed in a way similar to (3.6). We use the interpretation of the elements (3.6) as paths in the same oriented graph with the additional condition that all vertices $[k,l]$ of the path belong to the region $k,l \geqslant 0$. We will call such paths *admissible*. Clearly, if $m,n \geqslant 2$ then all paths are admissible. For instance, in the case $m = n = 0$ this leaves the elements

$$
\begin{array}{rl}
[0,0] & w,\quad z_{41}\,z_{31}\,z_{42}\,z_{32}\,w,\\
[1,1] & z_{32}\,w,\quad z_{31}\,z_{42}\,z_{32}\,w,\\
[2,0] & z_{42}\,z_{32}\,w,\\
[0,2] & z_{31}\,z_{32}\,w,
\end{array}
$$

of the space $K^+(0,0)$. In all remaining cases such vectors are listed in Appendix C.

Proposition 3.3. *A basis of the Kac module $K(m,n)$ is formed by the vectors*

$$
E_{21}^{k}\,E_{43}^{l}\,z_{41}^{\theta_1}\,z_{31}^{\theta_2}\,z_{42}^{\theta_3}\,z_{32}^{\theta_4}\,w, \tag{3.7}
$$

with the condition that the corresponding elements (3.6) *are associated with admissible paths, and where*

$$
k = 0,1,\dots,m-\theta_1-\theta_2+\theta_3+\theta_4 \qquad \text{and} \qquad l = 0,1,\dots,n-\theta_1+\theta_2-\theta_3+\theta_4.
$$

Proof. The restriction of the module $K(m,n)$ to the subalgebra $\mathfrak{k}$ is a direct sum

$$K(m,n)\Big|_{\mathfrak{k}} \cong \bigoplus_{r,s\geqslant 0} c_{r,s}\, L^0(r,s),$$

where the multiplicity $c_{r,s}$ is found by

$$c_{r,s} = \dim K^+(m,n)_{[r,s]},$$

where the subscript $[r,s]$ indicates the corresponding weight subspace. The subspace $K^+(m,n)_{[r,s]}$ coincides with the image of the weight space $K(m,n)_{[r,s]}$ under the action of the extremal projector $p = p(\mathfrak{k})$,

$$K^+(m,n)_{[r,s]} = p\,K(m,n)_{[r,s]}.$$

By the Poincaré–Birkhoff–Witt theorem, the Kac module $K(m,n)$ is spanned by vectors of the form

$$E_{21}^{k}\, E_{43}^{l}\, E_{41}^{\theta_1}\, E_{31}^{\theta_2}\, E_{42}^{\theta_3}\, E_{32}^{\theta_4}\, w$$

where each θ_i takes values in $\{0,1\}$. Due to the properties (3.3) of p, we may conclude that the space $K^+(m,n)_{[r,s]}$ is spanned by the vectors

$$p\,E_{41}^{\theta_1}\, E_{31}^{\theta_2}\, E_{42}^{\theta_3}\, E_{32}^{\theta_4}\, w \tag{3.8}$$

such that $r = m-\theta_1-\theta_2+\theta_3+\theta_4$ and $s = n-\theta_1+\theta_2-\theta_3+\theta_4$. However, each vector (3.8) is a linear combination of admissible elements of the form (3.6). Indeed, this follows by application of the formulas of Lemma 3.2: first replace E_{32} with z_{32}, then use the expression provided by Lemma 3.2 for E_{42} to write the vector as a linear combination of elements $p\,E_{41}^{\theta_1}\, E_{31}^{\theta_2}\, z_{42}^{\theta_3}\, z_{32}^{\theta_4}\, w$ and then use such replacements for E_{31} and E_{41}.

Furthermore, each nonzero element of $K^+(m,n)_{[r,s]}$ generates a $\mathfrak{k}$-submodule of $K(m,n)$ of dimension $(r+1)(s+1)$. Therefore, the module $K(m,n)$ is spanned by all vectors (3.7). On the other hand, the number of these vectors is easily calculated. For $m,n\geqslant 2$ it equals

$$\begin{aligned}
&4(m+1)(n+1)+2(m+2)(n+2)+2(m+2)n+2m(n+2)+2mn\\
&+(m+3)(n+1)+(m+1)(n+3)+(m-1)(n+1)+(m+1)(n-1)\\
&=16(m+1)(n+1)
\end{aligned}$$

which coincides with $\dim K(m,n)$. This proves that the vectors form a basis of $K(m,n)$. The same calculation in the cases where $m\leqslant 1$ or $n\leqslant 1$ confirms that the number of vectors matches $\dim K(m,n)$. $\square$

The proof of Proposition 3.3 essentially contains the decompositions of the Kac modules as $\mathfrak{k}$-modules. In particular, for $m, n \geqslant 2$ we have

$$K(m,n)\Big|_{\mathfrak{k}} \cong 4L^0(m,n) \oplus 2L^0(m+1,n+1) \oplus 2L^0(m+1,n-1) \\ \oplus 2L^0(m-1,n+1) \oplus 2L^0(m-1,n-1) \oplus L^0(m+2,n) \\ \oplus L^0(m,n+2) \oplus L^0(m-2,n) \oplus L^0(m,n-2).$$

Following the terminology used for representations of simple Lie superalgebras [6], we will call the weight $[m,n]$ *typical*, if the Kac module $K(m,n)$ is irreducible. Otherwise, $[m,n]$ will be called *atypical*. We will give necessary and sufficient conditions for $[m,n]$ to be typical. They turn out to coincide with such conditions for representations of $\mathfrak{psl}(2|2)$ (see [6], [7]), but the structure of the atypical Kac modules differs; see also [4]. Our main instrument will be the techniques of Mickelsson–Zhelobenko algebras which will allow us to describe $K(m,n)$ as a module over $\mathfrak{k}$.

Proposition 3.4. *If $m \neq n$ then the Kac module $K(m,n)$ is irreducible.*

Proof. Observe that if R is a nonzero submodule of $K(m,n)$, then the subspace R^+ defined in (3.1) is a nonzero $S(\mathfrak{g},\mathfrak{k})$-submodule of $K^+(m,n)$. Therefore, to describe $\mathfrak{g}$-submodules of $K(m,n)$ it will be sufficient to describe $S(\mathfrak{g},\mathfrak{k})$-submodules of $K^+(m,n)$. Since R^+ is $\mathfrak{h}$-invariant, each weight component of R^+ is contained in R^+. Working case by case for each weight subspace, we verify easily with the use of formulas of Lemmas A.1, A.2, A.3 and Appendix B, that the condition $R^+ \neq \{0\}$ implies that R^+ contains the vector w. For example, suppose that $m, n \geqslant 2$ and that a linear combination

$$c_1 w + c_2 z_{41} z_{32} w + c_3 z_{31} z_{42} w + c_4 z_{41} z_{31} z_{42} z_{32} w, \qquad c_i \in \mathbb{C},$$

belongs to R^+. Applying the operators z_{23} and z_{24} to this element, we obtain the following two relations, respectively,

$$0 = \frac{m-n}{2}\, c_2 + \frac{m+n+2}{2(n+1)}\, c_3 \qquad \text{and} \qquad 0 = -\frac{m-n}{2(n+2)}\, c_2 + \frac{n(m+n+2)}{2(n+1)}\, c_3$$

together with $c_4 = 0$. The relations imply $c_2 = c_3 = 0$ when $m \neq n$. The same argument applied to the remaining weight subspaces implies that $w \in R^+$ and so $R^+ = K^+(m,n)$, which proves that $R = K(m,n)$. Thus, the corresponding Kac module $K(m,n)$ is irreducible. □

Now suppose that $m = n$ and introduce the $\mathrm{S}(\mathfrak{g},\mathfrak{k})$-submodules of $K^+(n,n)$ by

$$S_n^+ = \mathrm{S}(\mathfrak{g},\mathfrak{k})\, z_{32}\, w \qquad \text{and} \qquad T_n^+ = \mathrm{S}(\mathfrak{g},\mathfrak{k})\, z_{41}\, w.$$

The corresponding submodules S_n and T_n of $K(n,n)$ are then defined by

$$S_n = \mathrm{U}(\mathfrak{k})\, S_n^+ \qquad \text{and} \qquad T_n = \mathrm{U}(\mathfrak{k})\, T_n^+. \tag{3.9}$$

Proposition 3.5. *The $\mathfrak{g}$-modules S_n are irreducible for $n \geqslant 0$ and the $\mathfrak{g}$-modules T_n are irreducible for $n \geqslant 1$. Moreover, $T_0 = \{0\}$ and we have a $\mathfrak{g}$-module isomorphism*

$$S_{n-1} \cong T_n, \qquad n \geqslant 1. \tag{3.10}$$

Proof. By using the formulas of Appendices A and B, we can produce explicit bases of S_n^+ and T_n^+. Arranging the basis vectors in accordance with their $\mathfrak{h}$-weights, for S_n^+ with $n \geqslant 1$ we have

$$\begin{array}{rl}
[n,n] & z_{41}\, z_{32}\, w, \quad z_{41}\, z_{31}\, z_{42}\, z_{32}\, w, \\
[n+1,n+1] & z_{32}\, w, \quad z_{31}\, z_{42}\, z_{32}\, w, \\
[n+1,n-1] & z_{41}\, z_{42}\, z_{32}\, w, \\
[n-1,n+1] & z_{41}\, z_{31}\, z_{32}\, w, \\
[n+2,n] & z_{42}\, z_{32}\, w, \\
[n,n+2] & z_{31}\, z_{32}\, w,
\end{array}$$

and for T_n^+ with $n \geqslant 2$ we have

$$\begin{array}{rl}
[n,n] & z_{32}\, z_{41}\, w, \quad z_{31}\, z_{42}\, z_{32}\, z_{41}\, w, \\
[n+1,n-1] & z_{42}\, z_{32}\, z_{41}\, w, \\
[n-1,n+1] & z_{31}\, z_{32}\, z_{41}\, w, \\
[n-1,n-1] & z_{41}\, w, \quad z_{31}\, z_{42}\, z_{41}\, w, \\
[n-2,n] & z_{31}\, z_{41}\, w, \\
[n,n-2] & z_{42}\, z_{41}\, w.
\end{array}$$

Similarly, the basis of S_0^+ is given by

$$\begin{array}{rl}
[0,0] & w, \quad z_{41}\, z_{31}\, z_{42}\, z_{32}\, w, \\
[1,1] & z_{32}\, w, \quad z_{31}\, z_{42}\, z_{32}\, w, \\
[2,0] & z_{42}\, z_{32}\, w, \\
[0,2] & z_{31}\, z_{32}\, w,
\end{array}$$

while the basis of T_1^+ is

$$\begin{array}{rl}
[1,1] & z_{32}\, z_{41}\, w, \quad z_{31}\, z_{42}\, z_{32}\, z_{41}\, w, \\
[2,0] & z_{42}\, z_{32}\, z_{41}\, w, \\
[0,2] & z_{31}\, z_{32}\, z_{41}\, w, \\
[0,0] & z_{41}\, w, \quad z_{41}\, z_{31}\, z_{42}\, w,
\end{array}$$

and $T_0^+ = \{0\}$. Using formulas of Appendices A and B once again, we can see that the nonzero submodules S_n^+ and T_n^+ of the $\mathrm{S}(\mathfrak{g},\mathfrak{k})$-module $K^+(n,n)$ are irreducible. This implies that the corresponding submodules S_n and T_n of $K(n,n)$ are also irreducible.

Finally, to prove the last statement of the proposition, for a given $n \geqslant 1$ denote by w' the highest vector of the $\mathfrak{k}$-module $L^0(n-1,n-1)$. The Mickelsson–Zhelobenko algebra relations imply that for $n \geqslant 2$ we have an $\mathrm{S}(\mathfrak{g},\mathfrak{k})$-module isomorphism

$$\phi : S_{n-1}^+ \to T_n^+, \qquad z_{32}\, w' \mapsto z_{32}\, z_{41}\, w,$$

with the inverse map given by

$$\phi^{-1} : z_{41}\, w \mapsto -\frac{n+1}{n}\, z_{41}\, z_{32}\, w'.$$

For $n = 1$ the statement is equivalent to the existence of an isomorphism $K^+(0,0) \cong T_1^+$. It is provided by the map

$$\phi : K^+(0,0) \to T_1^+, \qquad w' \mapsto z_{41}\, w.$$

This yields the desired isomorphism (3.10). □

Proposition 3.6. *The Kac module $K(n,n)$ over $\mathfrak{g}$ with $n \geqslant 1$ is the direct sum of two irreducible submodules,*

$$K(n,n) = S_n \oplus T_n.$$

The module $K(0,0) = S_0$ is irreducible. Hence, we have an isomorphism

$$K(n,n) \cong S_n \oplus S_{n-1}, \qquad n \geqslant 0,$$

assuming $S_{-1} = \{0\}$.

Proof. This will follow from Proposition 3.5. It suffices to verify that

$$K^+(n,n) = S_n^+ \oplus T_n^+, \qquad n \geqslant 1. \tag{3.11}$$

However, $\dim K^+(n,n) = \dim S_n^+ + \dim T_n^+$ and we have

$$K^+(n,n) = S_n^+ + T_n^+$$

due to the relation

$$w = -\frac{n+2}{n+1}\, z_{41}\, z_{32} w - \frac{n+1}{n}\, z_{32}\, z_{41}\, w;$$

see Lemma A.1. Therefore, the intersection of S_n^+ and T_n^+ is zero and (3.11) follows, thus completing the proof. □

3.3 Classification theorem

We can now prove the classification theorem for representations of the Lie superalgebra $\mathfrak{g} = \mathfrak{psl}(2|2) \ltimes \mathbb{C}^3$, where the central elements act by

$$C \mapsto 0, \qquad K \mapsto 0 \qquad \text{and} \qquad P \mapsto 1. \tag{3.12}$$

Theorem 3.7. *A complete list of pairwise non-isomorphic finite-dimensional irreducible representations of* $\mathfrak{g}$ *with the conditions* (3.12) *consists of*

1. *the Kac modules* $K(m,n)$ *with* $m, n \in \mathbb{Z}_+$ *and* $m \neq n$,
   $\dim K(m,n) = 16(m+1)(n+1)$,
2. *the modules* S_n *with* $n \in \mathbb{Z}_+$, $\quad \dim S_n = 8(n+1)(n+2)$.

Proof. Consider the following triangular decomposition of the Lie superalgebra $\mathfrak{g}$,

$$\mathfrak{g} = \bar{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \bar{\mathfrak{n}}^+,$$

where $\mathfrak{h}$ is spanned by the elements h_1, h_3, C, K and P, whereas the subalgebras $\bar{\mathfrak{n}}^+$ and $\bar{\mathfrak{n}}^-$ are defined by

$$\bar{\mathfrak{n}}^+ = \text{span of } \{E_{12}, E_{34}, E_{31}, E_{32}, E_{14}, E_{24}\},$$
$$\bar{\mathfrak{n}}^- = \text{span of } \{E_{21}, E_{43}, E_{13}, E_{23}, E_{41}, E_{42}\}.$$

Given a pair of complex numbers $\mu = (\mu_1, \mu_3)$, consider the one-dimensional representation $\mathbb{C}_\mu$ of the Lie superalgebra $\mathfrak{h} \oplus \bar{\mathfrak{n}}^+$ defined by

$$\bar{\mathfrak{n}}^+ 1_\mu = 0, \quad h_1 1_\mu = \mu_1 1_\mu, \quad h_3 1_\mu = \mu_3 1_\mu, \quad C 1_\mu = 0, \quad K 1_\mu = 0, \quad P 1_\mu = 1_\mu,$$

where 1_μ denotes the basis vector of $\mathbb{C}_\mu$. The corresponding *Verma module* $\overline{M}(\mu)$ is then defined by

$$\overline{M}(\mu) = \mathrm{U}(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \bar{\mathfrak{n}}^+)} \mathbb{C}_\mu.$$

By a standard argument, $\overline{M}(\mu)$ contains a unique maximal proper submodule V and we set $\overline{L}(\mu) = \overline{M}(\mu)/V$. The module $\overline{L}(\mu)$ is irreducible and any finite-dimensional irreducible representation of $\mathfrak{g}$ is isomorphic to $\overline{L}(\mu)$ for a certain uniquely determined μ. Therefore, to classify finite-dimensional irreducible representations of $\mathfrak{g}$ it suffices to find necessary and sufficient conditions on μ for the module $\overline{L}(\mu)$ to be finite-dimensional.

By considering the $\mathrm{U}(\mathfrak{k})$-cyclic span of the vector 1_μ we conclude that the condition $\dim \overline{L}(\mu) < \infty$ implies that both μ_1 and μ_3 are nonnegative integers. In what follows we will assume that $\mu_1, \mu_3 \in \mathbb{Z}_+$. As a next step, we will demonstrate that $\overline{L}(\mu)$ is infinite-dimensional unless $\mu_3 \geqslant 2$.

Suppose first that $\mu_3 = 0$. The vector

$$v = z_{42} 1_\mu = E_{42} 1_\mu \in \overline{L}(\mu)$$

is nonzero since $E_{31}\, v = P\, 1_\mu = 1_\mu$. On the other hand, the weight of v is $(\mu_1+1,\, -1)$ and $\mathfrak{k}^+\, v = 0$. Therefore, the $\mathfrak{k}$-module $\mathrm{U}(\mathfrak{k})\, v$ is infinite-dimensional and so is $\overline{L}(\mu)$.

Now let $\mu_3 = 1$. If $\mu_1 \geqslant 1$ then the vector

$$u = z_{41}\, z_{42}\, 1_\mu = E_{41}\, E_{42}\, 1_\mu - P\, E_{43}\, 1_\mu = E_{41}\, E_{42}\, 1_\mu - E_{43}\, 1_\mu \in \overline{L}(\mu)$$

is nonzero since

$$E_{24}\, E_{14}\, u = -\frac{\mu_1(\mu_1+1)}{2}\, 1_\mu$$

while the weight of u is $(\mu_1, -1)$. If $\mu_1 = 0$ then the vector $y = E_{41}\, 1_\mu$ is nonzero since $E_{32}\, y = -P\, 1_\mu = -1_\mu$ and the weight of y is $(-1, 0)$.

As a final step, we will show that each representation $\overline{L}(\mu)$ with $\mu_3 \geqslant 2$ is finite-dimensional and isomorphic to exactly one module in the list given in the formulation of the theorem. By the construction of the Kac modules $K(m,n)$, for $\mu = (m, n+2)$ we have the homomorphism

$$\overline{M}(m, n+2) \to K(m,n), \qquad 1_\mu \mapsto z_{31}\, z_{32}\, w.$$

By Propositions 3.4 and 3.5, this yields isomorphisms

$$\overline{L}(m, n+2) \cong K(m,n), \qquad m, n \in \mathbb{Z}_+, \quad m \neq n,$$

and

$$\overline{L}(n, n+2) \cong S_n, \qquad n \in \mathbb{Z}_+.$$

The theorem is proved. □

The techniques of Mickelsson–Zhelobenko algebras which we used to prove Theorem 3.7 can also be applied to reproduce the well-known descriptions of finite-dimensional irreducible representations of the Lie superalgebras $\mathfrak{psl}(2|2)$ and $\mathfrak{sl}(2|2)$ [7]; see also [8], [11] and [15]. Namely, the above arguments can be easily modified for the case where the central element P of $\mathfrak{g}$ acts as the zero operator. The corresponding Kac modules $K^\circ(m,n)$ over $\mathfrak{psl}(2|2)$ are irreducible for $m \neq n$. However, the structure of $K^\circ(n,n)$ differs from that of the $\mathfrak{g}$-module $K(n,n)$; the corresponding submodules S°_n and T°_n of $K^\circ(n,n)$, which are defined exactly as in (3.9), are not irreducible for $n \geqslant 1$. The intersection $U_n = S^\circ_n \cap T^\circ_n$ is nonzero for $n \geqslant 1$ and we have

$$U_n^+ = \text{span of } \{z_{41}\, z_{32}\, w, \quad z_{41}\, z_{31}\, z_{42}\, z_{32}\, w, \quad z_{41}\, z_{42}\, z_{32}\, w, \quad z_{41}\, z_{31}\, z_{32}\, w\}.$$

The sum $R_n = S^\circ_n + T^\circ_n$ is a proper submodule of $K^\circ(n,n)$ (cf. Proposition 3.5) and the quotient $L^\circ(n,n) = K^\circ(n,n)/R_n$ is irreducible. The vectors w, $z_{31}\, w$, $z_{42}\, w$, $z_{31}\, z_{42}\, w$ form a basis of $L^{\circ+}(n,n)$ for $n \geqslant 1$.

To summarize, we get the following description of $\mathfrak{psl}(2|2)$-modules; see [6] and [7].

Corollary 3.8. *A complete list of pairwise non-isomorphic finite-dimensional irreducible representations of* $\mathfrak{psl}(2|2)$ *consists of*

1. *the Kac modules* $K^\circ(m,n)$ *with* $m,n \in \mathbb{Z}_+$ *and* $m \neq n$,
   $\dim K^\circ(m,n) = 16(m+1)(n+1)$,
2. *the modules* $L^\circ(n,n)$ *with* $n \geqslant 1$, $\quad \dim L^\circ(n,n) = 4n(n+2)+2$,
3. *the trivial one-dimensional module* $L^\circ(0,0)$. □

To state the corresponding results for the Lie superalgebra $\mathfrak{g}' = \mathfrak{sl}(2|2) = \mathfrak{psl}(2|2) \oplus \mathbb{C}C$, consider the Kac modules $K(m,n;2c)$ over $\mathfrak{g}'$ (with $m,n \in \mathbb{Z}_+$), which are defined as in (3.5), except that the central elements now act by $C \mapsto c$, $K \mapsto 0$ and $P \mapsto 0$, and we assume that the complex number c is nonzero. The $\mathfrak{g}'$-module $K(m,n;2c)$ is irreducible if and only if

$$m - n \neq \pm 2c \qquad \text{and} \qquad m + n + 2 \neq \pm 2c. \tag{3.13}$$

These conditions define the class of *typical* representations of $\mathfrak{g}'$. The remaining *atypical* representations are nontrivial quotients of $K(m,n;2c)$ in the cases where (3.13) does not hold. To describe the corresponding submodules, consider the Mickelsson algebra $\mathrm{S}(\mathfrak{g}',\mathfrak{k})$ and introduce submodules

$$\begin{aligned} S &\subset K(m,n;m-n), & T &\subset K(m,n;n-m), \\ X &\subset K(m,n;-m-n-2), & Y &\subset K(m,n;m+n+2), \end{aligned}$$

by setting $V = \mathrm{U}(\mathfrak{k})V^+$, where V denotes one of the four submodules, and V^+ is the $\mathrm{S}(\mathfrak{g}',\mathfrak{k})$-submodule of the respective Kac module,

$$\begin{aligned} S^+ &= \mathrm{S}(\mathfrak{g}',\mathfrak{k})\, z_{32}\, w, & T^+ &= \mathrm{S}(\mathfrak{g}',\mathfrak{k})\, z_{41}\, w \\ X^+ &= \mathrm{S}(\mathfrak{g}',\mathfrak{k})\, z_{31}\, w, & Y^+ &= \mathrm{S}(\mathfrak{g}',\mathfrak{k})\, z_{42}\, w. \end{aligned}$$

Corollary 3.9. *A complete list of pairwise non-isomorphic finite-dimensional irreducible representations of* $\mathfrak{sl}(2|2)$ *with a non-zero eigenvalue* c *of the central element* C *consists of*

1. *the Kac modules* $K(m,n;2c)$ *with the conditions* (3.13),
   $\dim K(m,n;2c) = 16(m+1)(n+1)$,
2. *the modules* $K(m,n;m-n)/S$,
   $\dim K(m,n;m-n)/S = 4\big(m(n+1) + (m+1)n\big)$,
3. *the modules* $K(m,n;-m+n)/T$,
   $\dim K(m,n;-m+n)/T = 4\big((m+1)(n+2) + (m+2)(n+1)\big)$,

4. the modules $K(m,n;-m-n-2)/X$,
$\dim K(m,n;-m-n-2)/X = 4\big((m+2)(n+1)+(m+1)n\big)$,

5. the modules $K(m,n;m+n+2)/Y$,
$\dim K(m,n;-m-n-2)/Y = 4\big((m+1)(n+2)+m(n+1)\big)$. □

3.4 Explicit construction of representations

Our proof of the classification theorem (Theorem 3.7) was based on explicit bases of irreducible representations V. They all have the form

$$E_{21}^{k}\,E_{43}^{l}\,z_{41}^{\theta_1}\,z_{31}^{\theta_2}\,z_{42}^{\theta_3}\,z_{32}^{\theta_4}\,w, \tag{3.14}$$

with some conditions on the parameters, where w is the highest vector of the $\mathfrak{k}$-module $L^0(m,n)$. The matrix elements for the action of the generators of $\mathfrak{g}$ in this basis can be found from the Mickelsson–Zhelobenko algebra relations in a standard way; cf.[9, Ch. 9] and [10]. First observe that if $k > m-\theta_1-\theta_2+\theta_3+\theta_4$ or $l > n-\theta_1+\theta_2-\theta_3+\theta_4$ then the corresponding vector (3.14) is zero. This is easily verified by considering all possible values of the parameters θ_i. For example, if $\theta_1=\theta_3=\theta_4=0$ and $\theta_2=1$ then for $m \geqslant 1$ we have

$$\begin{aligned} E_{21}^{m}\,z_{31}\,w &= E_{21}^{m}\Big(E_{31}+E_{21}\,E_{32}\,\frac{1}{m+1}\Big)\,w \\ &= E_{31}\,E_{21}^{m}w + \frac{1}{m+1}\,E_{32}\,E_{21}^{m+1}w - (m+1)\,E_{31}\,E_{21}^{m}\,\frac{1}{m+1}\,w = 0 \end{aligned}$$

since $E_{21}^{m+1}w = 0$ in $K(m,n)$. Therefore, $E_{21}^{k}\,z_{31}\,w = 0$ for all $k > m$. This determines the action of the subalgebra $\mathfrak{k}$ on the basis vectors.

Furthermore, using the commutation relations of $\mathfrak{g}$ we can reduce the calculation to the case where one of the generators of the form E_{ik} or E_{ki} with $i = 1,2$ and $k = 3,4$ acts on the vector $v \in V^{+}$. Then we write this generator in the form provided by Lemma 3.2 and apply the formulas for the action of the elements z_{ik} and z_{ki} on the basis of V^{+}. To illustrate, consider the action of E_{13} on the basis vector (3.14),

$$E_{13}\,E_{21}^{k}\,E_{43}^{l}\,z_{41}^{\theta_1}\,z_{31}^{\theta_2}\,z_{42}^{\theta_3}\,z_{32}^{\theta_4}\,w = E_{21}^{k}\,E_{43}^{l}\,E_{13}\,z_{41}^{\theta_1}\,z_{31}^{\theta_2}\,z_{42}^{\theta_3}\,z_{32}^{\theta_4}\,w - k\,E_{21}^{k-1}E_{43}^{l}\,E_{23}\,z_{41}^{\theta_1}\,z_{31}^{\theta_2}\,z_{42}^{\theta_3}\,z_{32}^{\theta_4}\,w.$$

Next, replace E_{13} and E_{23} by their expression provided by Lemma 3.2. In particular, the first vector on the right hand side becomes

$$E_{21}^{k}\,E_{43}^{l}\,\Big(z_{13} - E_{43}\,z_{14}\frac{1}{h_3+1}\Big)\,z_{41}^{\theta_1}\,z_{31}^{\theta_2}\,z_{42}^{\theta_3}\,z_{32}^{\theta_4}\,w$$

so that the calculation is completed by applying the formulas of Appendix B for the action of z_{13} and z_{14} on the vector $z_{41}^{\theta_1}\,z_{31}^{\theta_2}\,z_{42}^{\theta_3}\,z_{32}^{\theta_4}\,w$.

We will omit explicit matrix element formulas to avoid significant extension of the paper for the reason that their reproduction is straightforward from the formulas of Appendices A and B.

Note that the same techniques of Mickelsson–Zhelobenko algebras can also be used to reproduce explicit basis constructions for representations of the Lie superalgebras $\mathfrak{sl}(2|2)$ and $\mathfrak{gl}(2|2)$ given in [8], [11] and [15].

A Relations in the Mickelsson–Zhelobenko algebra

The following relations in $\mathrm{Z}(\mathfrak{g},\mathfrak{k})$ are written without a specialization of the values of the central elements C, K and P.

Lemma A.1. *We have the relations for the lowering operators:*

$$\begin{aligned}
z_{31}z_{41} &= -z_{41}z_{31}\frac{h_3}{h_3+1},\\
z_{42}z_{41} &= -z_{41}z_{42}\frac{h_1}{h_1+1},\\
z_{32}z_{41} &= -P\frac{h_3}{h_3+1} - z_{31}z_{42}\frac{h_1-h_3}{(h_1+1)(h_3+1)} - z_{41}z_{32}\frac{h_3(h_3+2)}{(h_3+1)^2},\\
z_{42}z_{31} &= P - z_{31}z_{42} + z_{41}z_{32}\frac{h_1+h_3+2}{(h_1+1)(h_3+1)},\\
z_{32}z_{31} &= -z_{31}z_{32}\frac{h_1}{h_1+1},\\
z_{32}z_{42} &= -z_{42}z_{32}\frac{h_3}{h_3+1}.
\end{aligned}$$

Lemma A.2. *We have the relations for the raising operators:*

$$\begin{aligned}
z_{14}z_{13} &= -z_{13}z_{14}\frac{h_3}{h_3+1},\\
z_{14}z_{24} &= -z_{24}z_{14}\frac{h_1}{h_1+1},\\
z_{14}z_{23} &= -K\frac{h_3}{h_3+1} - z_{24}z_{13}\frac{h_1-h_3}{(h_1+1)(h_3+1)} - z_{23}z_{14}\frac{h_3(h_3+2)}{(h_3+1)^2},\\
z_{13}z_{24} &= K - z_{24}z_{13} + z_{23}z_{14}\frac{h_1+h_3+2}{(h_1+1)(h_3+1)},\\
z_{13}z_{23} &= -z_{23}z_{13}\frac{h_1}{h_1+1},\\
z_{24}z_{23} &= -z_{23}z_{24}\frac{h_3}{h_3+1}.
\end{aligned}$$

Lemma A.3. *We have the relations for the raising and lowering operators;*

with z_{14}*:*

$$\begin{aligned}
z_{14}z_{41} &= \frac{h_1h_3(h_1-h_3+2C)}{2(h_1+1)(h_3+1)} + z_{31}z_{13}\frac{h_1(h_1+2)}{(h_1+1)^2(h_3+1)} - z_{32}z_{23}\frac{1}{(h_1+1)(h_3+1)} \\
&\quad - z_{41}z_{14}\frac{h_1h_3(h_1+2)(h_3+2)}{(h_1+1)^2(h_3+1)^2} + z_{42}z_{24}\frac{h_3(h_3+2)}{(h_1+1)(h_3+1)^2}, \\
z_{14}z_{31} &= -z_{31}z_{14}\frac{h_1(h_1+2)}{(h_1+1)^2} + z_{32}z_{24}\frac{1}{h_1+1}, \\
z_{14}z_{42} &= -z_{42}z_{14}\frac{h_3(h_3+2)}{(h_3+1)^2} + z_{32}z_{13}\frac{1}{h_3+1}, \\
z_{14}z_{32} &= -z_{32}z_{14},
\end{aligned}$$

with z_{13}*:*

$$\begin{aligned}
z_{13}z_{41} &= -z_{41}z_{13}\frac{h_1(h_1+2)}{(h_1+1)^2} + z_{42}z_{23}\frac{1}{h_1+1}, \\
z_{13}z_{31} &= \frac{h_1(h_1+h_3+2+2C)}{2(h_1+1)} - z_{31}z_{13}\frac{h_1(h_1+2)}{(h_1+1)^2} + z_{32}z_{23}\frac{1}{h_1+1} \\
&\quad - z_{41}z_{14}\frac{h_1(h_1+2)}{(h_1+1)^2(h_3+1)} + z_{42}z_{24}\frac{1}{(h_1+1)(h_3+1)}, \\
z_{13}z_{42} &= -z_{42}z_{13}, \\
z_{13}z_{32} &= -z_{32}z_{13} - z_{42}z_{14}\frac{1}{h_3+1},
\end{aligned}$$

with z_{24}*:*

$$\begin{aligned}
z_{24}z_{41} &= -z_{41}z_{24}\frac{h_3(h_3+2)}{(h_3+1)^2} + z_{31}z_{23}\frac{1}{h_3+1}, \\
z_{24}z_{31} &= -z_{31}z_{24}, \\
z_{24}z_{42} &= -\frac{h_3(h_1+h_3+2-2C)}{2(h_3+1)} + z_{31}z_{13}\frac{1}{(h_1+1)(h_3+1)} + z_{32}z_{23}\frac{1}{h_3+1} \\
&\quad - z_{41}z_{14}\frac{h_3(h_3+2)}{(h_1+1)(h_3+1)^2} - z_{42}z_{24}\frac{h_3(h_3+2)}{(h_3+1)^2}, \\
z_{24}z_{32} &= -z_{32}z_{24} - z_{31}z_{14}\frac{1}{h_1+1},
\end{aligned}$$

with z_{23}*:*

$$\begin{aligned}
z_{23}z_{41} &= -z_{41}z_{23},\\
z_{23}z_{31} &= -z_{31}z_{23} - z_{41}z_{24}\frac{1}{h_3+1},\\
z_{23}z_{42} &= -z_{42}z_{23} - z_{41}z_{13}\frac{1}{h_1+1},\\
z_{23}z_{32} &= -\frac{h_1-h_3-2C}{2} - z_{31}z_{13}\frac{1}{h_1+1} - z_{32}z_{23} - z_{41}z_{14}\frac{1}{(h_1+1)(h_3+1)} - z_{42}z_{24}\frac{1}{h_3+1}.
\end{aligned}$$

B Action of raising operators

Lemmas A.1, A.2 and A.3 imply the following relations for the action of the raising operators on the vectors of the space $K^+(m,n)$ of $\mathfrak{k}^+$-invariants of the Kac module $K(m,n)$.

Action on w:

$$z_{14}\cdot w = z_{24}\cdot w = z_{13}\cdot w = z_{23}\cdot w = 0.$$

Action on $z_{41}z_{32}w$:

$$\begin{aligned}
z_{14}\cdot z_{41}z_{32}w &= \frac{(m-n)[(m+1)(n+1)+1]}{2(m+2)(n+2)}z_{32}w,\\
z_{24}\cdot z_{41}z_{32}w &= -\frac{m-n}{2(n+2)}z_{31}w,\\
z_{13}\cdot z_{41}z_{32}w &= -\frac{m-n}{2(m+2)}z_{42}w,\\
z_{23}\cdot z_{41}z_{32}w &= \frac{m-n}{2}z_{41}w.
\end{aligned}$$

Action on $z_{31}z_{42}w$:

$$\begin{aligned}
z_{14}\cdot z_{31}z_{42}w &= -\frac{n(m+n+2)}{2(m+2)(n+1)}z_{32}w,\\
z_{24}\cdot z_{31}z_{42}w &= \frac{n(m+n+2)}{2(n+1)}z_{31}w,\\
z_{13}\cdot z_{31}z_{42}w &= \frac{(mn+m+n)(m+n+2)}{2(m+2)(n+1)}z_{42}w,\\
z_{23}\cdot z_{31}z_{42}w &= \frac{m+n+2}{2(n+1)}z_{41}w.
\end{aligned}$$

Action on $z_{41}z_{31}z_{42}z_{32}w$:

$$z_{14} \cdot z_{41}z_{31}z_{42}z_{32}w = \frac{m+n+2}{2(m+2)} P z_{32}w + \frac{(m-n)(m+1)(mn+m+2n+1)}{2(m+2)^2(n+2)} \times \left[1 + \frac{(m+n+2)(mn+m+2n+3)}{(m+1)(n+1)(mn+m+2n+1)}\right] z_{31}z_{42}z_{32}w,$$

$$z_{24} \cdot z_{41}z_{31}z_{42}z_{32}w = -\frac{(m+n+2)(n+1)(n^2+5n+7)}{2(n+2)^3} z_{41}z_{31}z_{32}w,$$

$$z_{13} \cdot z_{41}z_{31}z_{42}z_{32}w = -\frac{(m+n+2)(mn+2m+n+3)}{2(m+2)^2} z_{41}z_{42}z_{32}w - \frac{m-n}{2(m+2)} P z_{42}w,$$

$$z_{23} \cdot z_{41}z_{31}z_{42}z_{32}w = \frac{m-n}{2} z_{41}z_{31}z_{42}w.$$

Action on $z_{32}w$:

$$z_{23} \cdot z_{32}w = -\frac{m-n}{2} w, \qquad z_{14} \cdot z_{32}w = z_{24} \cdot z_{32}w = z_{13} \cdot z_{32}w = 0.$$

Action on $z_{31}z_{42}z_{32}w$:

$$\begin{aligned} z_{14} \cdot z_{31}z_{42}z_{32}w &= 0, \\ z_{24} \cdot z_{31}z_{42}z_{32}w &= \frac{m+n+2}{2} z_{31}z_{32}w, \\ z_{13} \cdot z_{31}z_{42}z_{32}w &= \frac{m+n+2}{2} z_{42}z_{32}w, \\ z_{23} \cdot z_{31}z_{42}z_{32}w &= -\frac{m-n}{2} z_{31}z_{42}w + \frac{m+n+2}{2(n+1)} z_{41}z_{32}w. \end{aligned}$$

Action on $z_{42}w$:

$$z_{24} \cdot z_{42}w = -\frac{n(m+n+2)}{2(n+1)} w, \qquad z_{14} \cdot z_{42}w = z_{13} \cdot z_{42}w = z_{23} \cdot z_{42}w = 0.$$

Action on $z_{41}z_{42}z_{32}w$:

$$\begin{aligned} z_{14} \cdot z_{41}z_{42}z_{32}w &= \frac{n(m-n)}{2(n+1)} z_{42}z_{32}w, \\ z_{24} \cdot z_{41}z_{42}z_{32}w &= \frac{n(n+2)(m+n+2)}{2(n+1)^2} z_{41}z_{32}w + \frac{m-n}{2(n+1)} z_{31}z_{42}w, \\ z_{13} \cdot z_{41}z_{42}z_{32}w &= 0, \\ z_{23} \cdot z_{41}z_{42}z_{32}w &= -\frac{m-n}{2} z_{41}z_{42}w. \end{aligned}$$

Action on $z_{31}w$:

$$z_{13}\cdot z_{31}w=\frac{m(m+n+2)}{2(m+1)}w,\qquad z_{14}\cdot z_{31}w=z_{24}\cdot z_{31}w=z_{23}\cdot z_{31}w=0.$$

Action on $z_{41}z_{31}z_{32}w$:

$$\begin{aligned}
&z_{14}\cdot z_{41}z_{31}z_{32}w=\frac{m(m-n)}{2(m+1)}z_{31}z_{32}w,\\
&z_{24}\cdot z_{41}z_{31}z_{32}w=0,\\
&z_{13}\cdot z_{41}z_{31}z_{32}w=-\frac{m(m+n+2)}{2(m+1)}z_{41}z_{32}w-\frac{m-n}{2(m+1)}(z_{31}z_{42}w-Pw),\\
&z_{23}\cdot z_{41}z_{31}z_{32}w=-\frac{m-n}{2}z_{41}z_{31}w.
\end{aligned}$$

Action on $z_{41}w$:

$$z_{14}\cdot z_{41}w=\frac{mn(m-n)}{2(m+1)(n+1)}w,\qquad z_{24}\cdot z_{41}w=z_{13}\cdot z_{41}w=z_{23}\cdot z_{41}w=0.$$

Action on $z_{41}z_{31}z_{42}w$:

$$\begin{aligned}
z_{14}\cdot z_{41}z_{31}z_{42}w&=\frac{(m-n)(mn+m+n+2)}{2(m+1)(n+1)}z_{31}z_{42}w+\frac{(m+n+2)n(n+2)}{2(m+1)(n+1)^2}z_{41}z_{32}w\\
&\quad+\frac{(m+n+2)n}{2(m+1)}Pw,\\
z_{24}\cdot z_{41}z_{31}z_{42}w&=-\frac{n(m+n+2)}{2(n+1)}z_{41}z_{31}w,\\
z_{13}\cdot z_{41}z_{31}z_{42}w&=-\frac{m(m+n+2)}{2(m+1)}z_{41}z_{42}w,\\
z_{23}\cdot z_{41}z_{31}z_{42}w&=0.
\end{aligned}$$

Action on $z_{42}z_{32}w$:

$$\begin{aligned}
&z_{14}\cdot z_{42}z_{32}w=0, &&z_{24}\cdot z_{42}z_{32}w=\frac{m+n+2}{2}z_{32}w,\\
&z_{13}\cdot z_{42}z_{32}w=0, &&z_{23}\cdot z_{42}z_{32}w=\frac{m-n}{2}z_{42}w.
\end{aligned}$$

Action on $z_{31}z_{32}w$:

$$\begin{aligned}
&z_{14}\cdot z_{31}z_{32}w=0, &&z_{24}\cdot z_{31}z_{32}w=0,\\
&z_{13}\cdot z_{31}z_{32}w=\frac{m+n+2}{2}z_{32}w, &&z_{23}\cdot z_{31}z_{32}w=\frac{m-n}{2}z_{31}w.
\end{aligned}$$

Action on $z_{41}z_{31}w$:

$$z_{14}\cdot z_{41}z_{31}w = \frac{(m-1)(m-n)}{2m}z_{31}w, \qquad z_{24}\cdot z_{41}z_{31}w = 0,$$
$$z_{13}\cdot z_{41}z_{31}w = -\frac{(m-1)(m+n+2)}{2m}z_{41}w, \qquad z_{23}\cdot z_{41}z_{31}w = 0.$$

Action on $z_{41}z_{42}w$:

$$z_{14}\cdot z_{41}z_{42}w = \frac{(n-1)(m-n)}{2n}z_{42}w, \qquad z_{24}\cdot z_{41}z_{42}w = \frac{(n-1)(m+n+2)}{2n}z_{41}w,$$
$$z_{13}\cdot z_{41}z_{42}w = 0, \qquad z_{23}\cdot z_{41}z_{42}w = 0.$$

C Bases of $\mathfrak{k}^+$-invariants in Kac modules

$K^+(0,0)$:

$$\begin{array}{ll} [0,0] & w,\quad z_{41}z_{31}z_{42}z_{32}w, \\ [1,1] & z_{32}w,\quad z_{31}z_{42}z_{32}w, \\ [2,0] & z_{42}z_{32}w, \\ [0,2] & z_{31}z_{32}w. \end{array}$$

$K^+(1,0)$:

$$\begin{array}{ll} [1,0] & w,\quad z_{41}z_{32}w,\quad z_{41}z_{31}z_{42}z_{32}w, \\ [2,1] & z_{32}w,\quad z_{31}z_{42}z_{32}w, \\ [0,1] & z_{31}w,\quad z_{41}z_{31}z_{32}w, \\ [3,0] & z_{42}z_{32}w, \\ [1,2] & z_{31}z_{32}w. \end{array}$$

$K^+(0,1)$:

$$\begin{array}{ll} [0,1] & w,\quad z_{41}z_{32}w,\quad z_{41}z_{31}z_{42}z_{32}w, \\ [1,2] & z_{32}w,\quad z_{31}z_{42}z_{32}w, \\ [1,0] & z_{42}w,\quad z_{41}z_{42}z_{32}w, \\ [2,1] & z_{42}z_{32}w, \\ [0,3] & z_{31}z_{32}w. \end{array}$$

$K^+(1,1)$:

$$\begin{array}{ll}
[1,1] & w, \quad z_{41}z_{31}z_{42}z_{32}w, \quad z_{31}z_{42}w, \quad z_{41}z_{32}w, \\
[2,2] & z_{32}w, \quad z_{31}z_{42}z_{32}w, \\
[2,0] & z_{42}w, \quad z_{41}z_{42}z_{32}w, \\
[0,2] & z_{31}w, \quad z_{41}z_{31}z_{32}w, \\
[0,0] & z_{41}w, \quad z_{41}z_{31}z_{42}w, \\
[3,1] & z_{42}z_{32}w, \\
[1,3] & z_{31}z_{32}w.
\end{array}$$

$K^+(m,0)$ with $m \geqslant 2$:

$$\begin{array}{rl}
[m,0] & w, \quad z_{41}z_{32}w, \quad z_{41}z_{31}z_{42}z_{32}w, \\
[m+1,1] & z_{32}w, \quad z_{31}z_{42}z_{32}w, \\
[m-1,1] & z_{31}w, \quad z_{41}z_{31}z_{32}w, \\
[m+2,0] & z_{42}z_{32}w, \\
[m,2] & z_{31}z_{32}w, \\
[m-2,0] & z_{41}z_{31}w.
\end{array}$$

$K^+(0,n)$ with $n \geqslant 2$:

$$\begin{array}{rl}
[0,n] & w, \quad z_{41}z_{32}w, \quad z_{41}z_{31}z_{42}z_{32}w, \\
[1,n+1] & z_{32}w, \quad z_{31}z_{42}z_{32}w, \\
[1,n-1] & z_{42}w, \quad z_{41}z_{42}z_{32}w, \\
[2,n] & z_{42}z_{32}w, \\
[0,n+2] & z_{31}z_{32}w, \\
[0,n-2] & z_{41}z_{42}w.
\end{array}$$

$K^+(m,1)$ with $m \geqslant 2$:

$$\begin{array}{rl}
[m,1] & w, \quad z_{41}z_{32}w, \quad z_{31}z_{42}w, \quad z_{41}z_{31}z_{42}z_{32}w, \\
[m+1,2] & z_{32}w, \quad z_{31}z_{42}z_{32}w, \\
[m+1,0] & z_{42}w, \quad z_{41}z_{42}z_{32}w, \\
[m-1,2] & z_{31}w, \quad z_{41}z_{31}z_{32}w, \\
[m-1,0] & z_{41}w, \quad z_{41}z_{31}z_{42}w, \\
[m+2,1] & z_{42}z_{32}w, \\
[m,3] & z_{31}z_{32}w, \\
[m-2,1] & z_{41}z_{31}w.
\end{array}$$

$K^+(1,n)$ with $n \geqslant 2$:

$$\begin{array}{rl} [1,n] & w, \quad z_{41}z_{32}w, \quad z_{31}z_{42}w, \quad z_{41}z_{31}z_{42}z_{32}w, \\ [2,n+1] & z_{32}w, \quad z_{31}z_{42}z_{32}w, \\ [2,n-1] & z_{42}w, \quad z_{41}z_{42}z_{32}w, \\ [0,n+1] & z_{31}w, \quad z_{41}z_{31}z_{32}w, \\ [0,n-1] & z_{41}w, \quad z_{41}z_{31}z_{42}w, \\ [3,n] & z_{42}z_{32}w, \\ [1,n+2] & z_{31}z_{32}w, \\ [1,n-2] & z_{41}z_{42}w. \end{array}$$

$K^+(m,n)$ with $m,n \geqslant 2$:

$$\begin{array}{rl} [m,n] & w, \quad z_{41}z_{32}w, \quad z_{31}z_{42}w, \quad z_{41}z_{31}z_{42}z_{32}w, \\ [m+1,n+1] & z_{32}w, \quad z_{31}z_{42}z_{32}w, \\ [m+1,n-1] & z_{42}w, \quad z_{41}z_{42}z_{32}w, \\ [m-1,n+1] & z_{31}w, \quad z_{41}z_{31}z_{32}w, \\ [m-1,n-1] & z_{41}w, \quad z_{41}z_{31}z_{42}w, \\ [m+2,n] & z_{42}z_{32}w, \\ [m,n+2] & z_{31}z_{32}w, \\ [m-2,n] & z_{41}z_{31}w, \\ [m,n-2] & z_{41}z_{42}w. \end{array}$$

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