Invariants of the vacuum module associated with the Lie superalgebra $\mathfrak{gl}(1|1)$

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Abstract

We describe the algebra of invariants of the vacuum module associated with the affinization of the Lie superalgebra $\mathfrak{gl}(1|1)$. We give a formula for its Hilbert–Poincaré series in a fermionic (cancellation-free) form which turns out to coincide with the generating function of the plane partitions over the $(1,1)$-hook. Our arguments are based on a super version of the Beilinson–Drinfeld–Raïs–Tauvel theorem which we prove by producing an explicit basis of invariants of the symmetric algebra of polynomial currents associated with $\mathfrak{gl}(1|1)$. We identify the invariants with affine supersymmetric polynomials via a version of the Chevalley theorem.

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1 Introduction

Suppose that $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ and $\kappa \in \mathbb{C}$. The vacuum module $V_\kappa(\mathfrak{g})$ at the level κ over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ has a vertex algebra structure. The center of this vertex algebra is trivial unless the level is *critical*; this is a unique value of κ depending on the normalization of the invariant symmetric bilinear form on $\mathfrak{g}$. In this case the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is an algebra of polynomials in infinitely many variables as described by a theorem of Feigin and Frenkel [5]; see also [10]. One may regard the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ as a commutative subalgebra of the universal enveloping algebra $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$. This leads to connections with the Gaudin model and to constructions of commutative subalgebras of $\mathrm{U}(\mathfrak{g})$ and its tensor powers; see [6], [7], [8] and [19]. Explicit constructions of generators of the Feigin–Frenkel center for the classical Lie algebras $\mathfrak{g}$ were given in [2], [3] and [13]. The Harish–Chandra images of the generators as elements of the corresponding classical $\mathcal{W}$-algebras were calculated in [14] with the use of the Yangian characters.

As explained in Kac [11, Sec. 4.7], the construction of the vertex algebra $V_\kappa(\mathfrak{g})$ can be extended to any Lie superalgebra $\mathfrak{g}$ equipped with an invariant supersymmetric bilinear form. In the case of general linear Lie superalgebras $\mathfrak{g} = \mathfrak{gl}(m|n)$, constructions of several families of elements of the center $\mathfrak{z}\big(\widehat{\mathfrak{gl}}(m|n)\big)$ at the critical level were given in [15]. It was conjectured there (see Remark 3.4(ii)) that each of the families generates the center. The main result of this paper is a proof of the conjecture in the case $m = n = 1$. We believe this result will be a key ingredient for the proof of the conjecture for arbitrary m and n.

In more detail, let $\mathfrak{g}$ denote $\mathfrak{gl}(1|1)$, the four-dimensional Lie superalgebra with the even basis elements E_{11}, E_{22} and odd elements E_{21}, E_{12}. Introduce the invariant supersymmetric bilinear form $(.|.)$ on $\mathfrak{g}$ by

$$(E_{ij}|E_{kl}) = \delta_{ij}\,\delta_{kl}(-1)^{\bar{\imath}+\bar{k}},$$

where $\bar{1} = 0$ and $\bar{2} = 1$. The *affinization* $\widehat{\mathfrak{g}}$ is then defined as the centrally extended Lie superalgebra of Laurent polynomials,

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K, \tag{1.1}$$

where the element K is even and central in $\widehat{\mathfrak{g}}$, the remaining commutation relations are given by

$$\big[E_{ij}[r], E_{kl}[s]\big] = \delta_{kj}\, E_{il}[r+s] - \delta_{il}\, E_{kj}[r+s](-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})} + r\,\delta_{r,-s}\delta_{ij}\,\delta_{kl}\,K\,(-1)^{\bar{\imath}+\bar{k}} \tag{1.2}$$

with $E_{ij}[r] = E_{ij}\,t^r$ for $r \in \mathbb{Z}$.[1] The vacuum module $V_{\mathrm{cri}}(\mathfrak{g})$ at the critical level over $\widehat{\mathfrak{g}}$ is defined as the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K - 1$. The center of the vertex algebra $V_{\mathrm{cri}}(\mathfrak{g})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{S \in V_{\mathrm{cri}}(\mathfrak{g}) \mid \mathfrak{g}[t]\, S = 0\}.$$

[1] The element K corresponds to K' in the corrected version of [15] in `arXiv:0911.3447v4`.

Elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ are called *Segal–Sugawara vectors*. Three families of such vectors were constructed in [15, Sec. 3.1]. To recall the construction, introduce the extended Lie superalgebra $\widehat{\mathfrak{g}} \oplus \mathbb{C}\tau$ with $\tau = -d/dt$, defined by the commutation relations

$$\big[\tau, X[r]\big] = -r\, X[r-1], \qquad \big[\tau, K\big] = 0. \tag{1.3}$$

Given an invertible 2×2 matrix Z over a (not necessarily super-commutative) superalgebra,

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \qquad \text{with} \qquad Z^{-1} = \begin{bmatrix} z'_{11} & z'_{12} \\ z'_{21} & z'_{22} \end{bmatrix},$$

where z_{11}, z_{22} are even and z_{12}, z_{21} are odd, the *Berezinian* of Z is $\operatorname{Ber} Z = z_{11} z'_{22}$. The *supertrace* of an arbitrary matrix of this form is $\operatorname{str} Z = z_{11} - z_{22}$. By [15, Theorem 3.2], all coefficients s_{kl} in the expansion of the supertrace

$$\operatorname{str} \begin{bmatrix} \tau + E_{11}[-1] & E_{12}[-1] \\ -E_{21}[-1] & \tau - E_{22}[-1] \end{bmatrix}^k = s_{k0}\, \tau^k + s_{k1}\, \tau^{k-1} + \cdots + s_{kk}$$

are Segal–Sugawara vectors. Here the entries of the matrices belong to the universal enveloping algebra of the Lie superalgebra $\widehat{\mathfrak{g}}_- \oplus \mathbb{C}\tau$ with $\widehat{\mathfrak{g}}_- = t^{-1}\mathfrak{g}[t^{-1}]$ so that the coefficients s_{kl} are understood as elements of the vacuum module $V_{\text{cri}}(\mathfrak{g})$. Furthermore, use a formal variable u to define the elements b_{kl} of $V_{\text{cri}}(\mathfrak{g})$ by the expansion

$$\operatorname{Ber} \begin{bmatrix} 1 + u\big(\tau + E_{11}[-1]\big) & u\, E_{12}[-1] \\ -u\, E_{21}[-1]) & 1 + u\big(\tau - E_{22}[-1]\big) \end{bmatrix} = \sum_{k=0}^{\infty} \sum_{l=0}^{k} b_{kl}\, u^k \tau^{k-l}. \tag{1.4}$$

By [15, Corollary 3.3], all elements b_{kl} are Segal–Sugawara vectors and they are related to the s_{kl} by means of the Newton identities. Yet another family of Segal–Sugawara vectors is related to the b_{kl} by the MacMahon Master Theorem; see [15, Theorem 2.2 and Corollary 3.3]. Namely, denoting the series (1.4) by $b(u)$, the Segal–Sugawara vectors h_{kl} are found as the coefficients in the expansion

$$b(-u)^{-1} = \sum_{k=0}^{\infty} \sum_{l=0}^{k} h_{kl}\, u^k \tau^{k-l}.$$

Both b_{kl} and h_{kl} are also given as certain weighted traces involving the antisymmetrizer and symmetrizer, respectively, in the group algebra $\mathbb{C}[\mathfrak{S}_k]$; we recall these formulas in Sec. 2 below.

The center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of the vertex algebra $V_{\text{cri}}(\mathfrak{g})$ is invariant under the *translation operator* $T : V_{\text{cri}}(\mathfrak{g}) \to V_{\text{cri}}(\mathfrak{g})$ which is the derivation $T = -d/dt$ of the algebra $\mathrm{U}(\widehat{\mathfrak{g}}_-)$ determined by the properties

$$\big[T, E_{ij}[r]\big] = -r\, E_{ij}[r-1], \tag{1.5}$$

where $E_{ij}[r]$ is understood as the operator of left multiplication by $E_{ij}[r]$.

The following is our first main result which can be regarded as an analogue of the Feigin–Frenkel theorem for the Lie superalgebra $\mathfrak{gl}(1|1)$.

Theorem A. *Each of the families* $\{T^r s_{kk}\}$, $\{T^r b_{kk}\}$ *and* $\{T^r h_{kk}\}$ *with* $r \geqslant 0$ *and* $k \geqslant 1$ *generates the center* $\mathfrak{z}(\widehat{\mathfrak{g}})$ *of the vertex algebra* $V_{\text{cri}}(\mathfrak{g})$.

As explained in [10, Chap. 3] by the example $\mathfrak{g} = \mathfrak{sl}(2)$, the proof can be reduced to verifying the corresponding property of the *classical limit* of the vacuum module $V_{\text{cri}}(\mathfrak{g})$; that is, to describing the invariants of the $\mathfrak{g}[t]$-module $\mathrm{S}\big(\mathfrak{g}[t,t^{-1}]/\mathfrak{g}[t]\big)$. We regard $\mathfrak{g}[t,t^{-1}]/\mathfrak{g}[t] \cong \widehat{\mathfrak{g}}_-$ as a $\mathfrak{g}[t]$-module with the adjoint action and extend it to the symmetric algebra $\mathrm{S}(\widehat{\mathfrak{g}}_-)$. The classical limit of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of invariants

$$\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]} = \{P \in \mathrm{S}(\widehat{\mathfrak{g}}_-) \mid \mathfrak{g}[t]\, P = 0\}. \tag{1.6}$$

In the case where $\mathfrak{g}$ is a simple Lie algebra, the algebra (1.6) is described by the Beilinson–Drinfeld theorem (see [10, Theorem 3.4.2]), and this description is also implied by an earlier work of Raïs and Tauvel [18]. Namely, recall that the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants in the symmetric algebra $\mathrm{S}(\mathfrak{g})$ admits an algebraically independent family of generators,

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[P_1,\dots,P_n], \qquad n = \operatorname{rank}\mathfrak{g}; \tag{1.7}$$

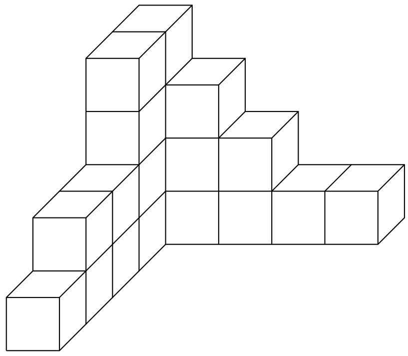
see e.g. [4, Sec. 7.3]. Identify the generators P_i with their images under the embedding $\mathrm{S}(\mathfrak{g}) \hookrightarrow \mathrm{S}(\widehat{\mathfrak{g}}_-)$ taking $X \in \mathfrak{g}$ to $X[-1]$. Then the family $\{T^r P_k\}$ is algebraically independent and

$$\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]} = \mathbb{C}[T^r P_1,\dots,T^r P_n \mid r \geqslant 0],$$

where $T = -d/dt$ now denotes the derivation of the algebra $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ determined by the same properties (1.5) applied to the symmetric algebra.

If $\mathfrak{g}$ is a simple Lie superalgebra, then the structure of the algebra of invariants $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is more complicated; see e.g. [20], [21]. In particular, it does not admit an algebraically independent family of generators. One could still expect that a natural super-analogue of the Beilinson–Drinfeld–Raïs–Tauvel theorem holds: if $\{P_k \mid k \geqslant 1\}$ is a family of generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then the derivatives $\{T^r P_k \mid k \geqslant 1,\ r \geqslant 0\}$ generate the algebra $\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$. We prove this analogue for $\mathfrak{g} = \mathfrak{gl}(1|1)$ by producing a basis of the algebra of invariants and showing that each basis element is expressed in terms of the generators.

The Chevalley images of the basis elements turn out to form a basis of the algebra $\Lambda^{\text{aff}}(1|1)$ of *affine supersymmetric polynomials*. We establish this property by employing a lemma of Sergeev [20, 21]. As an application, we calculate the Hilbert–Poincaré series of this algebra which thus coincides with that of the algebra of $\mathfrak{g}[t]$-invariants of $\mathrm{S}(\widehat{\mathfrak{g}}_-)$. Moreover, this series turns out to coincide with the generating function of the plane partitions over the $(1,1)$-hook. In more detail, such a plane partition can be regarded as a finite sequence of Young diagrams $\lambda^{(1)} \supset \cdots \supset \lambda^{(r)}$, where each term of the sequence is a hook diagram $(a, 1^b)$. Equivalently, a plane partition can be viewed as a corner "brick wall" formed by unit cubes or "bricks", the i-th level of the wall has the shape of the hook $\lambda^{(i)}$, as illustrated:



The corresponding sequence of hooks in this example is $(5, 1^4) \supset (3, 1^3) \supset (2, 1) \supset (1^2)$. The generating function of such plane partitions is known and given by the expression

$$\frac{1}{(q)_\infty^2} \sum_{k=0}^{\infty} (-1)^k \, q^{\frac{k^2+k}{2}} = 1 + q + 3q^2 + 6q^3 + 12q^4 + 21q^5 + 38q^6 + 63q^7 + 106q^8 + \dots, \quad (1.8)$$

where

$$(q)_\infty = \prod_{i=1}^{\infty} (1 - q^i)$$

and the coefficient of q^N is the number of plane partitions with N cubes. This formula was proved in [9] and its extension for plane partitions over the (m, n)-hook was conjectured. The conjecture was proved in [16]. Our second main result is the following.

Theorem B. *The algebra of* $\mathfrak{g}[t]$*-invariants of* $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ *is isomorphic to the algebra* $\Lambda^{\mathrm{aff}}(1|1)$ *of affine supersymmetric polynomials. The Hilbert–Poincaré series of both algebras are given by*

$$\frac{1}{(q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k^2}, \qquad \textit{with} \quad (q)_k = \prod_{i=1}^{k} (1 - q^i). \quad (1.9)$$

Moreover, this series coincides with the generating function of the plane partitions over the $(1, 1)$*-hook and so equals* (1.8).

We also give a conjectural characterization property of the affine supersymmetric polynomials analogous to that of the supersymmetric polynomials; see Section 3.3. It is implied by the invariance property of elements of the symmetric algebra in the same way as in the finite-dimensional case; cf. [20].

In the Appendix we prove a simple formula for the Hilbert–Poincaré series of the algebra $\Lambda(m|n)$ of supersymmetric polynomials. In different forms this series was previously calculated in [17] and [22].

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2 Invariants of the symmetric algebra

To reproduce the formulas for the elements $\{b_{kl}\}$ and $\{h_{kl}\}$ from [15, Corollary 3.3], we will denote by $Z = [Z_{ij}]$ the matrix used in the Introduction:

$$Z = \begin{bmatrix} \tau + E_{11}[-1] & E_{12}[-1] \\ -E_{21}[-1] & \tau - E_{22}[-1] \end{bmatrix}.$$

We will identify Z with an element of the tensor product superalgebra $\operatorname{End} \mathbb{C}^{1|1} \otimes \mathrm{U}$ by

$$Z = \sum_{i,j=1}^{2} e_{ij} \otimes Z_{ij} (-1)^{\bar{\imath}\bar{\jmath} + \bar{\jmath}},$$

where the e_{ij} denote the standard matrix units and U stands for the universal enveloping algebra of $\widehat{\mathfrak{g}}_{-} \oplus \mathbb{C}\tau$. Consider tensor products

$$\operatorname{End} \mathbb{C}^{1|1} \otimes \ldots \otimes \operatorname{End} \mathbb{C}^{1|1} \otimes \mathrm{U} \tag{2.1}$$

with k copies of $\operatorname{End} \mathbb{C}^{1|1}$ and for $a = 1, \ldots, k$ write Z_a for the matrix Z corresponding to the a-th copy of the endomorphism superalgebra so that the components in all remaining copies are the identity matrices. The symmetric group $\mathfrak{S}_k$ acts naturally on the tensor product space $(\mathbb{C}^{1|1})^{\otimes k}$. We let H_k and A_k denote the respective images of the normalized symmetrizer and antisymmetrizer

$$\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma \in \mathbb{C}[\mathfrak{S}_k], \qquad \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot \sigma \in \mathbb{C}[\mathfrak{S}_k] \tag{2.2}$$

in (2.1) with the identity component in U. The elements b_{kl} are h_{kl} are then found by the expansions

$$\operatorname{str}_{1,\ldots,k} A_k Z_1 \ldots Z_k = b_{k0} \tau^k + b_{k1} \tau^{k-1} + \cdots + b_{kk}, \tag{2.3}$$

$$\operatorname{str}_{1,\ldots,k} H_k Z_1 \ldots Z_k = h_{k0} \tau^k + h_{k1} \tau^{k-1} + \cdots + h_{kk}, \tag{2.4}$$

where str denotes the supertrace taken with respect to all k copies of $\operatorname{End} \mathbb{C}^{1|1}$.

Due to the relationships between the elements of the three families $\{s_{kl}\}$, $\{b_{kl}\}$ and $\{h_{kl}\}$, it will be sufficient to prove Theorem A for one of them. We will work with the elements $h_{kl} \in \mathrm{U}(\widehat{\mathfrak{g}}_-)$ most of the time, whose explicit form implied by (2.4) is provided by [15, Proposition 2.3]. Denote by $\bar{h}_{kl}$ their symbols in the associated graded algebra $\mathrm{gr}\, \mathrm{U}(\widehat{\mathfrak{g}}_-) \cong \mathrm{S}(\widehat{\mathfrak{g}}_-)$. They are easily calculated and we have, in particular,

$$\bar{h}_{kk} = E_{11}[-1]^{k-1}\big(E_{11}[-1] + E_{22}[-1]\big) + (k-1)\, E_{11}[-1]^{k-2} E_{21}[-1] E_{12}[-1],$$

where we keep the same notation $E_{ij}[r]$ for the generators of the symmetric algebra. Observe that these elements are recovered from the invariants

$$E_{11}^{k-1}\big(E_{11} + E_{22}\big) + (k-1)\, E_{11}^{k-2} E_{21} E_{12} \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$$

through the embedding $\mathrm{S}(\mathfrak{g}) \hookrightarrow \mathrm{S}(\widehat{\mathfrak{g}}_-)$ sending $X \in \mathfrak{g}$ to $X[-1]$. In the same way, the symbols $\bar{b}_{kk}$ and $\bar{s}_{kk}$ are obtained as the images in $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ of the respective $\mathfrak{g}$-invariants

$$E_{22}^{k-1}\big(E_{11} + E_{22}\big) - (k-1)\, E_{22}^{k-2} E_{21} E_{12} \qquad \text{and} \qquad \mathrm{str} \begin{bmatrix} E_{11} & E_{12} \\ -E_{21} & -E_{22} \end{bmatrix}^k \tag{2.5}$$

in $\mathrm{S}(\mathfrak{g})$. The translation operator T gives rise to a derivation on $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ which we will denote by the same symbol; it is determined by the properties implied by (1.5). We have an easily verified generating function identity

$$\sum_{r=0}^{\infty} \frac{T^r\, \bar{h}_{kk}}{r!}\, z^r = E_{11}(z)^{k-1}\, \big(E_{11}(z) + E_{22}(z)\big) + (k-1)\, E_{11}(z)^{k-2}\, E_{21}(z) E_{12}(z), \tag{2.6}$$

where

$$E_{ij}(z) = \sum_{r=0}^{\infty} E_{ij}[-r-1]\, z^r.$$

It is a consequence of the commutative vertex algebra structure on $\mathrm{S}(\widehat{\mathfrak{g}}_-)$; first we note that

$$\sum_{r=0}^{\infty} \frac{T^r\, E_{ij}[-1]}{r!}\, z^r = E_{ij}(z)$$

and then apply the property

$$\sum_{r=0}^{\infty} \frac{T^r\, \big(E_{ij}[-1]\, E_{kl}[-1]\big)}{r!}\, z^r = E_{ij}(z)\, E_{kl}(z).$$

Similarly, the invariants (2.5) give rise to the power series

$$E_{22}(z)^{k-1}\, \big(E_{11}(z) + E_{22}(z)\big) - (k-1)\, E_{22}(z)^{k-2}\, E_{21}(z) E_{12}(z) \tag{2.7}$$

and

$$\operatorname{str}\begin{bmatrix} E_{11}(z) & E_{12}(z) \\ -E_{21}(z) & -E_{22}(z) \end{bmatrix}^k, \tag{2.8}$$

respectively.

The next theorem is an analogue of the Beilinson–Drinfeld–Raïs–Tauvel theorem; see [18] (and [1] for a similar argument) and [10, Theorem 3.4.2].

Theorem 2.1. *The coefficients of each of the series* (2.6), (2.7) *and* (2.8) *generate the algebra* $\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$.

Proof. The rest of Section 2 is devoted to the proof of Theorem 2.1. Our strategy will be to construct a basis of the algebra of $\mathfrak{g}[t]$-invariants of $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ and then to show that every basis element can be expressed as a polynomial in the coefficients of the series (2.6); i.e., in the elements $T^r\bar{h}_{kk}$. It will be convenient to use the following basis elements of the Lie superalgebra $\widehat{\mathfrak{g}}_-$: for $i \geqslant 0$ set

$$a_i = E_{11}[-i-1], \qquad c_i = E_{11}[-i-1] + E_{22}[-i-1], \tag{2.9}$$

and

$$\varphi_i = E_{21}[-i-1], \qquad \psi_i = E_{12}[-i-1],$$

so that the c_i are central in $\widehat{\mathfrak{g}}_-$. Moreover, for $i \geqslant 0$ introduce elements of the symmetric algebra $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ by

$$y_i = \sum_{a+b=i} \varphi_a \psi_b.$$

Lemma 2.2. *Every element of the algebra of* $\mathfrak{g}[t]$*-invariants of* $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ *can be written as a polynomial in the elements* a_i, c_i *and* y_i *with* $i \geqslant 0$.

Proof. Any element $P \in \mathrm{S}(\widehat{\mathfrak{g}}_-)$ can be written in the form

$$P = \sum_{I,J} P_{IJ}\, \varphi_{i_1} \dots \varphi_{i_n} \psi_{j_1} \dots \psi_{j_m} \tag{2.10}$$

with the conditions $0 \leqslant i_1 < \dots < i_n$ and $j_1 > \dots > j_m \geqslant 0$ for uniquely determined polynomials P_{IJ} in the a_i and c_i, where $I = \{i_1, \dots, i_n\}$ and $J = \{j_1, \dots, j_m\}$. Suppose now that P is invariant. It will be sufficient to use only the conditions that $E_{22}[r]\, P = 0$ for $r = 0, 1$. They are clearly satisfied by any polynomial in a_i, c_i and y_i. For the action of $E_{22}[r]$ we have

$$E_{22}[r]\,\varphi_i = \begin{cases} \varphi_{i-r} & \text{if } \ i \geqslant r, \\ 0 & \text{if } \ i < r, \end{cases} \qquad\qquad E_{22}[r]\,\psi_i = \begin{cases} -\psi_{i-r} & \text{if } \ i \geqslant r, \\ \ \ 0 & \text{if } \ i < r. \end{cases}$$

Hence, $P_{IJ} = 0$ unless I and J have the same cardinality, as implied by the relation $E_{22}[0]\,P = 0$. Given such an invariant P, we can find an element

$$Q = \sum_K Q_K\, y_{k_1} \dots y_{k_l} \qquad \text{with} \qquad k_1 \geqslant \dots \geqslant k_l \geqslant 0,$$

where $K = \{k_1, \dots, k_l\}$ and each Q_K is a polynomial in the a_i and c_i, such that the expansion (2.10) for $P + Q$ does not contain monomials of the form $\varphi_0 \dots \varphi_{n-1}\, \psi_{j_1} \dots \psi_{j_n}$ for any $n \geqslant 0$. This follows easily by induction on the n-tuples $(j_1, \dots, j_n)$ with the lexicographic order; we assume that if $m < n$ then any m-tuple $(h_1, \dots, h_m)$ precedes $(j_1, \dots, j_n)$. Indeed, the largest monomial is eliminated by taking the sum

$$\varphi_0 \dots \varphi_{n-1}\, \psi_{j_1} \dots \psi_{j_n} + \text{const}\, y_{j_1}\, y_{j_2+1} \dots y_{j_n+n-1}$$

for an appropriate value of the constant.

Furthermore, assuming that none of the monomials $\varphi_0 \dots \varphi_{n-1}\, \psi_{j_1} \dots \psi_{j_n}$ occurs in P, we will show that $P = 0$. Suppose for the contrary that $P \neq 0$ and take the minimum n-tuple $(i_1, \dots, i_n)$ in the lexicographic order such that $\varphi_{i_1} \dots \varphi_{i_n}$ occurs in the expansion of P; its coefficient is a nonzero linear combination of the products $\psi_{j_1} \dots \psi_{j_n}$. By our assumption, $(i_1, \dots, i_n) = (0, 1, \dots, s-1, i_{s+1}, \dots, i_n)$ for some $0 \leqslant s \leqslant n-1$ and $i_{s+1} > s$. The condition $E_{22}[1]\,P = 0$ then brings a contradiction since the coefficient of the monomial $\varphi_0 \varphi_1 \dots \varphi_{s-1} \varphi_{i_{s+1}-1} \dots \varphi_{i_n}$ in the expansion of $E_{22}[1]\,P$ will be nonzero. $\square$

Introduce formal power series

$$c(z) = \sum_{i=0}^{\infty} c_i z^i, \qquad \varphi(z) = \sum_{i=0}^{\infty} \varphi_i z^i, \qquad \psi(z) = \sum_{i=0}^{\infty} \psi_i z^i, \qquad y(z) = \sum_{i=0}^{\infty} y_i z^i. \tag{2.11}$$

Note that $y(z) = \varphi(z)\psi(z)$ and we have the relations

$$\psi(z)^2 = 0, \qquad y(z)\psi(z) = 0, \qquad y(z)^2 = 0. \tag{2.12}$$

Suppose that $z = (z_1, \dots, z_n)$ is a family of independent variables and $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition. Its parts are nonnegative integers satisfying the condition $\lambda_1 \geqslant \dots \geqslant \lambda_n \geqslant 0$. The length $\ell(\lambda)$ is the number of nonzero parts. The corresponding Schur polynomial $s_\lambda(z)$ is defined as the ratio of two alternants,

$$s_\lambda(z) = \frac{\begin{vmatrix} z_1^{\lambda_1+n-1} & \dots & z_n^{\lambda_1+n-1} \\ \vdots & \vdots & \vdots \\ z_1^{\lambda_n} & \dots & z_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} z_1^{n-1} & \dots & z_n^{n-1} \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \end{vmatrix}},$$

where the denominator is the Vandermonde determinant

$$\Delta = \prod_{i<j} (z_i - z_j);$$

see e.g. [12, Ch. 1] for other presentations and properties of the Schur polynomials.

By the last relation in (2.12), the power series $y(z_1) \dots y(z_n)$ is divisible by Δ. The ratio is skew-symmetric with respect to permutations of the z_i which implies that $y(z_1) \dots y(z_n)$ is divisible by the square of Δ. Since the Schur polynomials form a basis of the algebra of symmetric polynomials in $z_1, \dots, z_n$, we can define elements $Y_\lambda^{(n)} \in \mathrm{S}(\widehat{\mathfrak{g}}_-)$ by the expansion

$$\frac{y(z_1) \dots y(z_n)}{\prod_{i \neq j} (z_i - z_j)} = \sum_{\lambda,\, \ell(\lambda) \leqslant n} Y_\lambda^{(n)} s_\lambda(z). \tag{2.13}$$

By Lemma 2.2, the algebra of invariants $\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$ is contained in the subalgebra S° of $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ generated by the elements a_i, c_i and y_i. The subalgebra of S° generated by the a_i and c_i with $i \geqslant 0$ can be regarded as the algebra of polynomials in these variables, which we denote by H. We regard S° as an H-module; elements of H act by multiplication.

Lemma 2.3. *The family*

$$\{Y_\lambda^{(n)} \mid n = 0, 1, \dots, \quad \ell(\lambda) \leqslant n\}$$

forms a basis of the H*-module* S°*.*

Proof. The expansion (2.13) implies that the family spans the H-module S°. To prove the linear independence over H, express the elements $Y_\lambda^{(n)}$ in terms of the generators φ_i and ψ_i. Since these generators are odd, we have the expansions

$$\frac{\varphi(z_1) \dots \varphi(z_n)}{\Delta} = \sum_{\mu,\, \ell(\mu) \leqslant n} \varphi_{\mu_1 + n - 1} \dots \varphi_{\mu_n} \, s_\mu(z)$$

and

$$\frac{\psi(z_1) \dots \psi(z_n)}{\Delta} = \sum_{\nu,\, \ell(\nu) \leqslant n} \psi_{\nu_1 + n - 1} \dots \psi_{\nu_n} \, s_\nu(z),$$

summed over partitions μ and ν. Furthermore,

$$\frac{y(z_1) \dots y(z_n)}{\prod_{i \neq j} (z_i - z_j)} = \frac{\varphi(z_1) \dots \varphi(z_n) \, \psi(z_1) \dots \psi(z_n)}{\Delta^2},$$

and so, taking into account (2.13), we conclude that

$$Y_\lambda^{(n)} = \sum_{\mu,\, \nu} c_{\mu\nu}^\lambda \, \varphi_{\mu_1 + n - 1} \dots \varphi_{\mu_n} \, \psi_{\nu_1 + n - 1} \dots \psi_{\nu_n}, \tag{2.14}$$

where the sum is taken over partitions μ and ν of lengths not exceeding n, and the $c^{\lambda}_{\mu\nu}$ are the Littlewood–Richardson coefficients defined by the relation

$$s_\mu(z)s_\nu(z) = \sum_{\lambda} c^{\lambda}_{\mu\nu} s_\lambda(z),$$

see e.g. [12, Ch. 1]. Note that $c^{\lambda}_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$, where $|\lambda| = \lambda_1 + \dots + \lambda_n$ denotes the weight of λ. Since $c^{\lambda}_{\lambda\varnothing} = 1$, and the monomials $\varphi_{\lambda_1+n-1} \dots \varphi_{\lambda_n} \psi_{n-1} \dots \psi_0$ are linearly independent over H, then so are the elements $Y^{(n)}_{\lambda}$. $\square$

Remark 2.4. Two more bases of the H-module S° (which we will not use) are formed by the monomials $y_{l_1} \dots y_{l_n}$ with $n \geqslant 0$ and the conditions $l_i - l_{i+1} \geqslant 2$ for $i = 1, \dots, n-1$ and $l_n \geqslant 0$ (cf. [24]) and by the monomials $y_{k_1} \dots y_{k_n}$ with $k_1 \geqslant \dots \geqslant k_n \geqslant n-1$. $\square$

Now suppose that $P \in \mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$. By Lemmas 2.2 and 2.3, there is a unique presentation

$$P = \sum_{m \geqslant 0} \sum_{\mu,\, \ell(\mu) \leqslant m} P^{(m)}_{\mu} Y^{(m)}_{\mu}, \tag{2.15}$$

where the coefficients $P^{(m)}_{\mu}$ are certain polynomials in a_i and c_i.

Lemma 2.5. *Given a decomposition* (2.15) *for an invariant* $P \in \mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$, *let* $n \geqslant 0$ *have the property that* $P^{(m)}_{\mu} = 0$ *for all* $m > n$ *and let a partition* $\lambda = (\lambda_1, \dots, \lambda_n)$ *be such that* $P^{(n)}_{\lambda} \neq 0$ *but* $P^{(n)}_{\mu} = 0$ *for all* μ *with* $|\mu| > |\lambda|$. *Then* $P^{(n)}_{\lambda}$ *does not depend on the variables* a_i *with* $i \geqslant n$.

Proof. We will use the condition $E_{12}[0]\, P = 0$. The operator $E_{12}[0]$ can be written in the form

$$E_{12}[0] = -\sum_{j \geqslant 0} \psi_j \partial_j + \sum_{r \geqslant 0} c_r \partial_{\varphi_r}, \tag{2.16}$$

where we denote $\partial_j = \partial/\partial a_j$ and ∂_{φ_r} is the left derivative over φ_r. The condition on n implies

$$\sum_{j \geqslant 0} \psi_j \sum_{\mu,\, \ell(\mu) \leqslant n} \partial_j(P^{(n)}_{\mu})\, Y^{(n)}_{\mu} = 0.$$

Take $i \geqslant n$ and consider the coefficient of the monomial $\varphi_{\lambda_1+n-1} \dots \varphi_{\lambda_n} \psi_i \psi_{n-1} \dots \psi_0$ on the left hand side. By the condition on λ, this monomial can only occur for $j = i$ and $\mu = \lambda$ thus implying $\partial_i(P^{(n)}_{\lambda}) = 0$, as required. $\square$

In what follows we will call by a *leading component* any product of the form $R\, Y^{(n)}_{\lambda}$, where R is a polynomial in the a_i and c_i which does not depend on the variables a_i with $i \geqslant n$. By Lemma 2.5, every invariant has a leading component. Our next goal is to show that there exists an invariant $P \in \mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$ containing any given leading component and no other leading components in the expansion (2.15). It suffices to do this for monomials of the form

$$Y_\lambda^{(n)}\, \partial_0^{-k_0} \dots \partial_{n-1}^{-k_{n-1}}\, 1, \tag{2.17}$$

where we regard ∂_i^{-1} as a partial integration operator with respect to a_i so that

$$\partial_0^{-k_0} \dots \partial_{n-1}^{-k_{n-1}}\, 1 = \frac{a_0^{k_0} \dots a_{n-1}^{k_{n-1}}}{k_0! \dots k_{n-1}!}. \tag{2.18}$$

Given a value of n, introduce another family of independent variables $t_0, \dots, t_{n-1}$ and consider the formal power series in the variables z_i and t_i whose coefficients are polynomials in the a_i,

$$F(z_1, \dots, z_n; t_0, \dots, t_{n-1}) = \prod_{i=0}^{n-1} \Big(1 - \partial_i^{-1} t_i\Big)^{-1}$$
$$\times \prod_{j=0}^{\infty} \left(1 - \partial_{n+j}^{-1} \Big(t_{n-1}\, s_{(j+1)}(z) - t_{n-2}\, s_{(j+1,1)}(z) + \dots + (-1)^{n-1} t_0\, s_{(j+1,1^{n-1})}(z)\Big)\right)^{-1} 1,$$

where $s_{(j+1,1^k)}(z)$ is the Schur polynomial in the variables $z_1, \dots, z_n$ associated with the hook partition $(j+1, 1^k)$. In particular, the series $F(z_1, \dots, z_n; t_0, \dots, t_{n-1})$ is symmetric in $z_1, \dots, z_n$.

Lemma 2.6. *We have the identity*

$$F(z_1, \dots, z_n; t_0, \dots, t_{n-1}) = \prod_{i=0}^{\infty} \left(1 - \partial_i^{-1} \Big(z_1^i\, T_n^{(1)} + \dots + z_n^i\, T_n^{(n)}\Big)\right)^{-1} 1,$$

where

$$T_n^{(k)} = \frac{t_{n-1} - t_{n-2}\, e_1(z_1, \dots, \widehat{z}_k, \dots, z_n) + \dots + (-1)^{n-1} t_0\, e_{n-1}(z_1, \dots, \widehat{z}_k, \dots, z_n)}{(z_k - z_1) \dots \wedge \dots (z_k - z_n)}$$

and $e_1, \dots, e_{n-1}$ *denote the elementary symmetric polynomials; the hats and wedges indicate symbols or zero factors to be skipped.*

Proof. The rational function $T_n^{(k)}$ is written as the ratio

$$T_n^{(k)} = \frac{\begin{vmatrix} z_1^{n-1} & \dots & t_{n-1} & \dots & z_n^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1 & \dots & t_1 & \dots & z_n \\ 1 & \dots & t_0 & \dots & 1 \end{vmatrix}}{\Delta},$$

where the t_i occupy the k-th column in the numerator. Hence, the $T_n^{(k)}$ are the solutions of the system of equations

$$z_1^i T_n^{(1)} + \dots + z_n^i T_n^{(n)} = t_i, \qquad i = 0, 1, \dots, n-1.$$

Furthermore, if $i \geqslant n$ and $1 \leqslant m \leqslant n$ then the coefficient of t_{n-m} in the expression $z_1^i T_n^{(1)} + \dots + z_n^i T_n^{(n)}$ equals the ratio

$$\frac{\begin{vmatrix} z_1^{n-1} & \dots & z_n^{n-1} \\ \dots & \dots & \dots \\ z_1^i & \dots & z_n^i \\ \dots & \dots & \dots \\ 1 & \dots & 1 \end{vmatrix}}{\Delta},$$

where $z_1^i, \dots, z_n^i$ replace row m of the Vandermonde determinant in the numerator. This ratio coincides with $(-1)^{m-1} s_{(i-n+1,1^{m-1})}$, as required. $\square$

We are now in a position to prove a key lemma providing explicit $\mathfrak{g}[t]$-invariants in $\mathrm{S}(\widehat{\mathfrak{g}}_-)$. Recall the formal power series (2.11) and set

$$A(z_1, \dots, z_n; t_0, \dots, t_{n-1}) = \prod_{k=1}^{n} \Big(c(z_k) + y(z_k) T_n^{(k)} \Big) F(z_1, \dots, z_n; t_0, \dots, t_{n-1}). \qquad (2.19)$$

This is a formal power series in the z_i and t_i, symmetric in $z_1, \dots, z_n$, whose coefficients are elements of the subalgebra S° of $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ generated by the a_i, c_i and y_i.

Lemma 2.7. *All coefficients of the series* $A(z_1, \dots, z_n; t_0, \dots, t_{n-1})$ *belong to* $\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$.

Proof. It is enough to show that $E_{12}[0]\, A(z_1, \dots, z_n; t_0, \dots, t_{n-1}) = 0$. Recall that the action of $E_{12}[0]$ is given by the operator (2.16). Lemma 2.6 implies

$$\partial_i F(z_1, \dots, z_n; t_0, \dots, t_{n-1}) = \Big(z_1^i T_n^{(1)} + \dots + z_n^i T_n^{(n)} \Big) F(z_1, \dots, z_n; t_0, \dots, t_{n-1})$$

for all $i \geqslant 0$. Hence,

$$E_{12}[0]\, F(z_1, \dots, z_n; t_0, \dots, t_{n-1}) = -\Big(\psi(z_1) T_n^{(1)} + \dots + \psi(z_n) T_n^{(n)} \Big) F(z_1, \dots, z_n; t_0, \dots, t_{n-1}).$$

On the other hand, $E_{12}[0]\, y(z_k) = c(z_k)\, \psi(z_k)$ and so

$$E_{12}[0] \prod_{k=1}^{n} \Big(c(z_k) + y(z_k) T_n^{(k)} \Big) = \sum_{i=1}^{n} c(z_i)\, \psi(z_i)\, T_n^{(i)} \prod_{k \neq i} \Big(c(z_k) + y(z_k) T_n^{(k)} \Big).$$

Since $y(z_k)\psi(z_k) = 0$ by (2.12), we have $E_{12}[0]\, A(z_1, \dots, z_n; t_0, \dots, t_{n-1}) = 0$. $\square$

Now expand (2.19) along the basis formed by the products of monomials in the t_i and Schur polynomials in $z_1, \dots, z_n$. Take a partition λ of length not exceeding n and consider the coefficient of the basis element

$$t_0^{k_0} \dots t_{n-2}^{k_{n-2}} t_{n-1}^{k_{n-1}+n} s_\lambda(z), \qquad k_i \geqslant 0, \tag{2.20}$$

in the expansion. Furthermore, use Lemma 2.3 to write this coefficient as a linear combination of the basis elements $Y_\mu^{(m)}$. By (2.13), this linear combination contains a leading component in the form (2.17). All other elements $Y_\mu^{(m)}$ occurring in the linear combination will have the property $m \leqslant n$; moreover, if $m = n$ then $|\mu| < |\lambda|$. Therefore, eliminating all other leading components with the use of an easy induction, we get an invariant containing a unique leading component. Thus, taking into account Lemma 2.5, we may conclude that the coefficients of the basis elements (2.20) in the expansion of (2.19) with n running over nonnegative integers form a basis of $\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$ as a module over the algebra of polynomials in the c_i with $i \geqslant 0$.

Take $n = 1$ in (2.19) and observe that the coefficient of t_0^{k-1} in $A(z_1; t_0)$ equals

$$\frac{1}{(k-1)!}\Big(a(z)^{k-1} c(z) + (k-1)\, a(z)^{k-2} y(z)\Big), \tag{2.21}$$

where

$$a(z) = \sum_{i=0}^{\infty} a_i z^i.$$

This is immediate from the identity

$$\sum_{0 \leqslant i_1 \leqslant \dots \leqslant i_p} z^{i_1 + \dots + i_p}\, \partial_{i_1}^{-1} \dots \partial_{i_p}^{-1}\, 1 = \frac{a(z)^p}{p!}$$

which holds for any $p \geqslant 0$. Since the series (2.6) equals $(k-1)!$ times (2.21), the proof of Theorem 2.1 will be completed if we show that all coefficients of the series (2.19) for all values of n are expressed as polynomials in the coefficients of $A(z_1; t_0)$. This is the statement of the next lemma.

Lemma 2.8. *We have the identity*

$$A(z_1, \dots, z_n; t_0, \dots, t_{n-1}) = A(z_1; T_n^{(1)}) \dots A(z_n; T_n^{(n)}).$$

Proof. We have

$$A(z_1; t_0) \dots A(z_n; t_{n-1}) = \prod_{k=1}^{n} \Big(c(z_k) + y(z_k)\, t_{k-1}\Big)\, F(z_1; t_0) \dots F(z_n; t_{n-1}). \tag{2.22}$$

Write

$$F(z_1;t_0)\dots F(z_n;t_{n-1}) = \prod_{i=0}^{\infty}\Big(1-\partial_i^{-1}z_1^i t_0\Big)^{-1}1\dots\Big(1-\partial_i^{-1}z_n^i t_{n-1}\Big)^{-1}1.$$

Expanding the series and using the identity

$$\partial_i^{-k_1}1\dots\partial_i^{-k_n}1 = \binom{k_1+\dots+k_n}{k_1,\dots,k_n}\partial_i^{-k_1-\dots-k_n}1$$

we find that

$$F(z_1;t_0)\dots F(z_n;t_{n-1}) = \prod_{i=0}^{\infty}\bigg(1-\partial_i^{-1}\Big(z_1^i t_0+\dots+z_n^i t_{n-1}\Big)\bigg)^{-1}1.$$

Hence, replacing $t_i \mapsto T_n^{(i+1)}$ for $i=0,\dots,n-1$ in (2.22) we recover the formal power series $A(z_1,\dots,z_n;t_0,\dots,t_{n-1})$, as required. □

This completes the proof of Theorem 2.1. □

3 Affine supersymmetric polynomials

Recall that a polynomial $P(u,v) = P(u_1,\dots,u_m,v_1,\dots,v_n)$ in two sets of independent variables $u=(u_1,\dots,u_m)$ and $v=(v_1,\dots,v_n)$ is called *supersymmetric*, if it is symmetric in each of the sets separately and the following cancellation property holds: the result of the substitution $u_m=-v_n=t$ into $P(u,v)$ is independent of t. We denote the algebra of supersymmetric polynomials by $\Lambda(m|n)$. The supersymmetric Schur polynomials parameterized by all Young diagrams not containing the box $(m+1,n+1)$ form a basis of this algebra. Moreover, each of the families of elementary, complete and power sums supersymmetric functions generates $\Lambda(m|n)$; see e.g. [12, Ch. 1].

Given a supersymmetric polynomial $P(u,v)$, replace each variable u_i and v_j by the respective formal power series

$$u_i(z) = \sum_{r=0}^{\infty} u_{i\,r}z^r, \qquad v_j(z) = \sum_{r=0}^{\infty} v_{j\,r}z^r,$$

and write

$$P\big(u_1(z),\dots,u_m(z),v_1(z),\dots,v_n(z)\big) = \sum_{r=0}^{\infty} P_r\,z^r,$$

where the coefficients P_r are polynomials in the variables $u_{1r},\dots,u_{mr},v_{1r},\dots,v_{nr}$ with r running over the set of nonnegative integers. Equivalently, P_r is found as the derivative

$$P_r = \frac{T^rP}{r!},$$

where $P = P(u, v)$ is regarded as a polynomial in the variables $u_{i\,0} = u_i$ and $v_{j\,0} = v_j$, and the derivation T acts on the variables by the rule (cf. Sec. 2):

$$T : u_{i\,r} \mapsto (r+1)\,u_{i\,r+1}, \qquad v_{j\,r} \mapsto (r+1)\,v_{j\,r+1}.$$

Definition 3.1. We denote by $\Lambda^{\mathrm{aff}}(m|n)$ the subalgebra of the algebra of polynomials in the variables $u_{i\,r}$ and $v_{j\,r}$ generated by all coefficients P_r associated with all supersymmetric polynomials $P(u, v)$. Any element of $\Lambda^{\mathrm{aff}}(m|n)$ will be called an *affine supersymmetric polynomial.* $\square$

It is clear that the algebra $\Lambda^{\mathrm{aff}}(m|n)$ is generated by the coefficients P_r associated to any family $\{P\}$ of generators of the algebra $\Lambda(m|n)$. For instance, considering the supersymmetric power sums

$$u_1^k + \dots + u_m^k - (-1)^k (v_1^k + \dots + v_n^k)$$

we get the following explicit formulas for generators of $\Lambda^{\mathrm{aff}}(m|n)$:

$$\sum_{i=1}^{m} \sum_{r_1+\dots+r_k=r} u_{i\,r_1} \dots u_{i\,r_k} - (-1)^k \sum_{j=1}^{n} \sum_{r_1+\dots+r_k=r} v_{j\,r_1} \dots v_{j\,r_k}, \qquad k \geqslant 1, \quad r \geqslant 0, \tag{3.1}$$

where the second sums are taken over the k-tuples $(r_1, \dots, r_k)$ of nonnegative integers.

Setting

$$\deg u_{i\,r} = r+1 \qquad \text{and} \qquad \deg v_{j\,r} = r+1$$

defines a grading on the algebra of polynomials in the $u_{i\,r}$ and $v_{j\,r}$. In particular, the degree of the generator in (3.1) equals $k + r$. The subalgebra $\Lambda^{\mathrm{aff}}(m|n)$ inherits the grading so that we have the direct sum decomposition

$$\Lambda^{\mathrm{aff}}(m|n) = \bigoplus_{N \geqslant 0} \Lambda^{\mathrm{aff}}(m|n)^N,$$

where $\Lambda^{\mathrm{aff}}(m|n)^N$ denotes the subspace of $\Lambda^{\mathrm{aff}}(m|n)$ spanned by homogeneous elements of degree N and we set $\Lambda^{\mathrm{aff}}(m|n)^0 := \mathbb{C}$. We let $H_{m,n}(q)$ denote the corresponding Hilbert–Poincaré series

$$H_{m,n}(q) = \sum_{N=0}^{\infty} \dim \Lambda^{\mathrm{aff}}(m|n)^N\, q^N.$$

As in the Introduction, by a plane partition over the (m, n)-hook we mean a finite sequence of Young diagrams (or partitions) $\lambda^{(1)} \supset \dots \supset \lambda^{(r)}$ such that $\lambda^{(1)}$ does not contain the box $(m+1, n+1)$. Such a plane partition can be viewed as an array formed by unit cubes, the i-th level of the array has the shape $\lambda^{(i)}$. An explicit formula for the generating function of the plane partitions was conjectured in [9] and proved in [16]. For $n \geqslant m \geqslant 1$ it has the form

$$f_{m,n}(q) = \frac{1}{(q)_\infty^{m+n}} \sum_{k_1 \geqslant \dots \geqslant k_m \geqslant 0} \Bigg((-1)^{k_1+\dots+k_m} \, q^{\frac{1}{2}\sum_{i=1}^m (k_i^2+(2i-1)k_i)} \\ \times \prod_{1 \leqslant i<j \leqslant m} (1-q^{k_i-k_j+j-i}) \prod_{1 \leqslant i<j \leqslant n} (1-q^{k_i-k_j+j-i}) \Bigg),$$

where $k_j := 0$ for $j > m$ and the coefficient of q^N in the series is the number of plane partitions over the (m, n)-hook containing exactly N unit cubes.

Conjecture 3.2. *The dimension* $\dim \Lambda^{\mathrm{aff}}(m|n)^N$ *equals the number of plane partitions over the* (m, n)*-hook containing exactly* N *unit cubes. Equivalently, if* $n \geqslant m \geqslant 1$ *then the Hilbert–Poincaré series* $H_{m,n}(q)$ *coincides with* $f_{m,n}(q)$. □

The conjecture holds for $n = 0$ (or $m = 0$); that is, for the algebra of *affine symmetric polynomials* $\Lambda^{\mathrm{aff}}(m)$. This algebra admits a family of algebraically independent generators which can be obtained, for instance, by taking $n = 0$ in (3.1):

$$\sum_{i=1}^m \sum_{r_1+\dots+r_k=r} u_{ir_1} \dots u_{ir_k}, \qquad k = 1, \dots, m, \qquad r \geqslant 0.$$

The Hilbert–Poincaré series is then found by

$$\prod_{k=1}^m \prod_{r \geqslant k} (1-q^r)^{-1} = \frac{1}{(q)_\infty^m} \prod_{i=1}^{m-1} (1-q^i)^{m-i}$$

which coincides with $f_{0,m}(q)$; cf. [10, Sec. 4.3].

Below we prove Conjecture 3.2 for $m = n = 1$; see Sec. 3.2. First we give an alternative expression for the generating function $f_{m,n}(q)$ in this case.

Proposition 3.3. *We have*

$$f_{1,1}(q) = \frac{1}{(q)_\infty} \sum_{k=0}^\infty \frac{q^{k^2+k}}{(q)_k^2}.$$

Proof. By definition,

$$f_{1,1}(q) = \frac{1}{(q)_\infty^2} \sum_{k=0}^\infty (-1)^k \, q^{\frac{k^2+k}{2}}.$$

The desired identity follows from a more general relation which holds for $s \geqslant 0$:

$$\sum_{k=0}^\infty \frac{q^{k^2+k}}{(q)_k^2} - \frac{1}{(q)_\infty} \sum_{k=0}^{s-1} (-1)^k \, q^{\frac{k^2+k}{2}} = (-1)^s \sum_{k=s}^\infty \frac{q^{k^2-(s-1)k+\frac{s^2-s}{2}}}{(q)_k \, (q)_{k-s}}. \tag{3.2}$$

We prove (3.2) by induction on s. It holds trivially for $s=0$ so suppose that $s \geqslant 1$. To complete the induction step we need to show that

$$(-1)^s \sum_{k=s}^{\infty} \frac{q^{k^2-(s-1)k+\frac{s^2-s}{2}}}{(q)_k\,(q)_{k-s}} - (-1)^s\, \frac{q^{\frac{s^2+s}{2}}}{(q)_\infty} = (-1)^{s+1} \sum_{k=s+1}^{\infty} \frac{q^{k^2-sk+\frac{s^2+s}{2}}}{(q)_k\,(q)_{k-s-1}}.$$

This is immediate from the identity

$$\frac{1}{(q)_\infty} = \sum_{k=s}^{\infty} \frac{q^{k(k-s)}}{(q)_k\,(q)_{k-s}}$$

which holds for $s \geqslant 0$ and is easily verified as follows. Both sides are generating functions for all partitions. This is clear for the left hand side, while the expression on the right hand side is obtained by first assigning the maximum size rectangle of the form $(k-s)\times k$ contained in a Young diagram. Then the generating function of the Young diagrams with a fixed value of s is given by

$$\frac{q^{k(k-s)}}{(q)_k\,(q)_{k-s}}$$

as required. $\square$

3.1 Chevalley homomorphism

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g} = \mathfrak{gl}(m|n)$, where the subalgebras $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$ are spanned by the basis elements E_{ij} with $i<j$, $i=j$ and $i>j$, respectively. The Chevalley homomorphism

$$\varsigma : \mathrm{S}(\mathfrak{g}) \to \mathrm{S}(\mathfrak{h})$$

is the projection modulo the ideal $\mathrm{S}(\mathfrak{g})(\mathfrak{n}_- \cup \mathfrak{n}_+)$. The restriction of ς to the subalgebra of invariants yields an isomorphism between $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and the algebra of supersymmetric polynomials in two sets of variables; see e.g. [20].

Consider an affine analogue of ς defined as the projection

$$\widehat{\varsigma} : \mathrm{S}(\widehat{\mathfrak{g}}_-) \to \mathrm{S}(\widehat{\mathfrak{h}}_-) \tag{3.3}$$

modulo the ideal $\mathrm{S}(\widehat{\mathfrak{g}}_-)\big(t^{-1}\mathfrak{n}_-[t^{-1}] \cup t^{-1}\mathfrak{n}_+[t^{-1}]\big)$, where we set $\widehat{\mathfrak{h}}_- = t^{-1}\mathfrak{h}[t^{-1}]$. We identify $\mathrm{S}(\widehat{\mathfrak{h}}_-)$ with the algebra of polynomials in the variables $u_{1r}, \dots, u_{mr}, v_{1r}, \dots, v_{nr}$ with $r \geqslant 0$ by setting

$$u_{ir} = E_{ii}[-r-1] \qquad \text{and} \qquad v_{jr} = E_{j+m\,j+m}[-r-1].$$

Proposition 3.4. *The restriction of the homomorphism* (3.3) *to the subalgebra* $\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$ *is injective.*

Proof. Suppose that $Q \in \mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$ and $\widehat{\varsigma}(Q) = 0$. Take a positive integer p such that Q does not depend on the generators $E_{ij}[r]$ with $r < -p$. By the definition of the $\mathfrak{g}[t]$-action on $\mathrm{S}(\widehat{\mathfrak{g}}_-)$, we have $t^p\,\mathfrak{g}[t]\,Q = 0$. Denote by $\mathfrak{g}_p$ the quotient of $\mathfrak{g}[t]$ by the ideal $t^p\,\mathfrak{g}[t]$ and denote by $\mathfrak{g}_{p,-}$ the quotient of $\widehat{\mathfrak{g}}_- = t^{-1}\mathfrak{g}[t^{-1}]$ by the ideal $t^{-p-1}\mathfrak{g}[t^{-1}]$. The proposition will follow if we show that the restriction map

$$\mathrm{S}(\mathfrak{g}_{p,-})^{\mathfrak{g}_p} \to \mathrm{S}(\mathfrak{h}_{p,-})$$

is injective for any positive integer p, where $\mathfrak{h}_{p,-}$ denotes the quotient of $\widehat{\mathfrak{h}}_-$ by the ideal $t^{-p-1}\mathfrak{h}[t^{-1}]$. We will derive this claim from the following general result of Sergeev [20, Proposition 1.1]; see also [21, Lemma 4.3] for a shorter and more direct proof.

Lemma 3.5. *Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra and V a finite-dimensional $\mathfrak{g}$-module. Given a subspace W of V, suppose that there exists an even element $w_0 \in W$ such that the map*

$$\mathfrak{g} \times W \to V, \qquad (x, w) \mapsto x\,w_0 + w$$

is surjective. Then the restriction map $\mathrm{S}(V^*)^{\mathfrak{g}} \to \mathrm{S}(W^*)$ is injective. □

To apply the lemma we take $\mathfrak{g} = \mathfrak{g}_p$ and let $V = \mathfrak{g}_p$ be the adjoint $\mathfrak{g}_p$-module. The dual module V^* is isomorphic to $\mathfrak{g}_{p,-}$; the isomorphism takes the element $E_{kl}[s]^*$ dual to the basis vector $E_{kl}[s]$ of $\mathfrak{g}_p$ to the element $E_{lk}[-s-1](-1)^{\bar{l}}$, where $\bar{l} = 0$ or 1 depending on whether $l \leqslant m$ or $l > m$. The subspace W is the quotient $\mathfrak{h}_p$ of $\mathfrak{h}[t]$ by the ideal $t^p\,\mathfrak{h}[t]$. The dual space W^* is identified with $\mathfrak{h}_{p,-}$. The assumptions of Lemma 3.5 will hold for any element

$$w_0 = \sum_{k=1}^{m+n} \gamma_k\, E_{kk}[0]$$

with $\gamma_i \neq \gamma_j$ for $i \neq j$. Indeed, this is clear from the relations

$$[E_{ij}[r], w_0] = (\gamma_j - \gamma_i)\, E_{ij}[r].$$

The proposition is proved. □

3.2 Hilbert–Poincaré series

We are now in a position to prove Theorem B. In particular, this will prove Conjecture 3.2 in the case $m = n = 1$ which we will be considering here. So $\mathfrak{g} = \mathfrak{gl}(1|1)$ and the grading on the symmetric algebra $\mathrm{S}(\widehat{\mathfrak{g}}_-)$ is defined by

$$\deg E_{ij}[-r-1] = r + 1. \tag{3.4}$$

We have $\deg \varphi_i = \deg \psi_i = i + 1$, and by (2.14)

$$\deg Y^{(k)}_\lambda = |\lambda| + k\,(k+1), \qquad \ell(\lambda) \leqslant k. \tag{3.5}$$

For a given k, let d_N be the number of basis elements $Y^{(k)}_\lambda$ of degree N. By (3.5), the generating function is given by

$$\sum_{N\geqslant 0} d_N\, q^N = \frac{q^{k^2+k}}{(q)_k}.$$

Since $\deg a_i = i+1$ the generating function of the monomials (2.18) (with $n := k$) is $(q)_k^{-1}$. Similarly, the generating function of the algebra of polynomials in the c_i is $(q)_\infty^{-1}$ and so the Hilbert–Poincaré series of $\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]}$ is given by

$$\frac{1}{(q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q)_k^2}.$$

Furthermore, the image of the series (2.6) under the projection (3.3) equals

$$E_{11}(z)^{k-1}\big(E_{11}(z)+E_{22}(z)\big). \tag{3.6}$$

Since the elements $u_1^{k-1}(u_1+v_1)$ with $k\geqslant 1$ generate the algebra of supersymmetric polynomials $\Lambda(1|1)$, the coefficients of the series (3.6) generate the algebra $\Lambda^{\mathrm{aff}}(1|1)$ of affine supersymmetric polynomials. Here, as before, we identify the variables by $u_{1r} = E_{11}[-r-1]$ and $v_{1r} = E_{22}[-r-1]$. Therefore, by Proposition 3.4, we have a Chevalley-type isomorphism of graded algebras

$$\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]} \cong \Lambda^{\mathrm{aff}}(1|1).$$

Together with Proposition 3.3 this completes the proof of Theorem B.

We believe this isomorphism extends to the case of $\mathfrak{g}=\mathfrak{gl}(m|n)$ for arbitrary m and n.

Conjecture 3.6. *The restriction of the map* (3.3) *to the subalgebra of* $\mathfrak{g}[t]$*-invariants yields an isomorphism of graded algebras*

$$\mathrm{S}(\widehat{\mathfrak{g}}_-)^{\mathfrak{g}[t]} \cong \Lambda^{\mathrm{aff}}(m|n).$$

In particular, their Hilbert–Poincaré series coincide. □

Conjecture 3.6 holds for $n=0$ or $m=0$ as implied by the Beilinson–Drinfeld–Raïs–Tauvel theorem; see [10, Sec. 4.3].

3.3 Affine cancellation property

Using notation (2.9), we will regard $\Lambda^{\mathrm{aff}}(1|1)$ as the subalgebra of the algebra of polynomials in the variables $a_r = u_{1r}$ and $c_r = u_{1r}+v_{1r}$ with $r\geqslant 0$. Working over Laurent polynomials in c_0 define elements d_r by the relation

$$d(z) := \sum_{r=0}^{\infty} d_r z^r = c(z)^{-1}, \qquad c(z)=\sum_{r=0}^{\infty} c_r z^r.$$

Explicitly,

$$d_r = c_0^{-1} \sum_{\alpha_1+2\,\alpha_2+\dots+r\,\alpha_r=r} \frac{(\alpha_1+\dots+\alpha_r)!}{\alpha_1!\,\alpha_2!\dots\alpha_r!} \Big(-\frac{c_1}{c_0}\Big)^{\alpha_1} \dots \Big(-\frac{c_r}{c_0}\Big)^{\alpha_r},$$

summed over nonnegative integers α_i. Consider the operator

$$\mathcal{D} = \sum_{r=0}^{\infty} d_r\,\partial_r, \qquad \partial_r = \partial/\partial a_r.$$

Proposition 3.7. *If $P \in \Lambda^{\mathrm{aff}}(1|1)$ then $\mathcal{D}P$ does not contain negative powers of c_0.*

Proof. The algebra $\Lambda^{\mathrm{aff}}(1|1)$ is generated by the coefficients of the series $a(z)^k c(z)$ with $k \geqslant 0$. We have

$$\mathcal{D}\,a(z)^k c(z) = k\,a(z)^{k-1} d(z) c(z) = k\,a(z)^{k-1}.$$

Thus, the required property holds for generators of the algebra $\Lambda^{\mathrm{aff}}(1|1)$. Since $\mathcal{D}$ is a derivation, it will hold for all its elements. $\square$

We conjecture that the property given by Proposition 3.7 is characteristic for the affine supersymmetric polynomials.

Conjecture 3.8. *A polynomial P in the variables a_r and c_r belongs to $\Lambda^{\mathrm{aff}}(1|1)$ if and only if $\mathcal{D}P$ does not contain negative powers of c_0.*

A Generating function for the supersymmetric polynomials in $m+n$ variables

Two different forms of the Hilbert–Poincaré series for the algebra $\Lambda(m|n)$ were given in [17] and [22]. Both proofs rely on the parametrization of basis elements of $\Lambda(m|n)$ by Young diagrams contained in the (m,n)-hook. We give yet another formula for the series and derive it from the characterization of the supersymmetric polynomials via the cancellation property; cf. [23].

Proposition A.1. *The Hilbert–Poincaré series of the algebra $\Lambda(m|n)$ is found by*

$$\chi_{m,n}(q) = \sum_{k=0}^{\min\{m,n\}} \frac{q^{(m-k)(n-k)}}{(q)_{m-k}\,(q)_{n-k}}.$$

Proof. As before, we consider two sets of variables $u = (u_1,\dots,u_m)$ and $v = (v_1,\dots,v_n)$. For $m,n \geqslant 1$ we have a surjective homomorphism

$$\Lambda(m,n) \to \Lambda(m-1,n-1), \qquad u_m \mapsto 0, \quad v_n \mapsto 0.$$

Its kernel coincides with the space

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(u_i+v_j)\,\mathbb{C}\,[u,v]^{\mathfrak{S}_m\times\mathfrak{S}_n},$$

where $\mathbb{C}\,[u,v]^{\mathfrak{S}_m\times\mathfrak{S}_n}$ is the algebra of bisymmetric polynomials. Hence we have a recurrence relation

$$\chi_{m,n}(q)=\chi_{m-1,n-1}(q)+\frac{q^{mn}}{(q)_m\,(q)_n}$$

which leads to the desired formula. □

The recurrence relation can also be easily seen from the parametrization of basis elements by Young diagrams. The term $\chi_{m-1,n-1}(q)$ accounts for the diagrams contained in the $(m-1,n-1)$-hook, whereas the generating function of the diagrams in the (m,n)-hook containing the box (m,n) is $q^{mn}/(q)_m\,(q)_n$; cf. the proof of Proposition 3.3.

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