Multiperiod Mean-Standard-Deviation Time Consistent Portfolio Selection *

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Abstract

We study a multiperiod portfolio selection problem in which a single period mean-standard-deviation criterion is used to construct a separable multiperiod selection criterion. Using this criterion, we obtain a closed form optimal strategy which depends on selection schemes of investor's risk preference. As a consequence, we develop a multiperiod portfolio selection scheme. In doing so, we adapt a pseudo dynamic programming principle from other existing results. The analysis is performed in the market of risky assets only, however, we allow both market transitions and intermediate cash injections and offtakes.

Keywords: Discrete time; Dynamic programming; Time-consistency; Mean-standard-deviation; Non-self-financing.

1 Introduction

Portfolio selection problem has been of a great interest by both academics and practitioners. There are various selection criteria available. Some examples include the classical mean-variance (MV) criterion introduced by Markowitz (1952), the safety-first criterion proposed by Roy (1952), and the criterion which targets a particular wealth level used by Skaf and Boyd (2009). In this paper, we choose a mean-standard-deviation (MSD) criterion which (in the single period

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case) has the form:

$$J_x(\boldsymbol{u}) \quad = \quad \mathbb{E}_x(W^{\boldsymbol{u}}) - \kappa\sqrt{Var_x(W^{\boldsymbol{u}})},$$

where $W^{\boldsymbol{u}}$ denotes investor's wealth at the end of the investment horizon, which depends on investor's initial wealth x, his investment strategy $\boldsymbol{u}$, and a parameter $\kappa > 0$ which characterizes investor's risk tolerance. All terms will be defined in a more precise way later. There are several reasons to choose this criterion. The most significant one is the fact that it provides a partial understanding on how to choose a dynamic portfolio for the class of translation-invariant and positive-homogeneous (TIPH) risk measures. The TIPH risk measure class contains many interesting examples such as the well-known Value at Risk (VaR), and the Conditional Value at Risk (CVaR). In a single period portfolio selection model, it has been shown (see for example Landsman and Makov (2011)) that if the asset returns follow a (joint) elliptical distribution, optimizing a risk measure from the TIPH class is equivalent to optimizing the MSD criterion.

There has been extensive research in the past regarding single period portfolio selection by using MSD criterion. For example, Landsman (2008) found a closed form solution by using matrix partitions. Owadally (2012) proposed two alternative ways in which the obtained solutions are more efficient computationally. The first approach is based on the relationship between optimizing the MSD criterion and optimizing the MV criterion which is close to a precommitment approach. For portfolio selection by precommitment approach, we refer to Li and Ng (2000); Çakmak and Özekici (2006). The second approach utilizes the standard Lagrange argument together with some facts from linear algebra. One may note that both Landsman (2008) and Owadally (2012) consider a market of risky assets. Later on, a risk free asset is added to the model in Landsman and Makov (2012), however only a trivial solution is obtained (when a budget constraint only is imposed).

Just like in the second method given by Owadally (2012) we follow a standard Lagrange method to solve the single period problem. However, the main interest of this paper, is to extend the single period framework to a multiperiod model. In doing so, we note that the MSD and the MV criterion face the same difficulty due to the presence of the variance term in their formulation. The difficulty is that we can not apply the standard dynamic programming principle (DPP). In recent years, it is quite popular to use the time consistency concept to establish a pseudo DPP. This concept has been widely applied in the multiperiod portfolio selection problem with the MV criterion. We mention a few references here: Björk and Murgoci (2010); Wu (2013); Chen et al. (2013); Bensoussan et al. (2014) for discrete time, and Björk et al. (2014); Bensoussan et al. (2014) for continuous time setting. There are different definitions of time consistency. Here, we concentrate on the time consistency of optimal strategy with respect to a multiperiod selection criterion. To formulate a pseudo DPP, it has been argued that a rational investor should choose his strategy consistently through time. In other words, the investors only choose among strategies which they are going to follow in the future (see

Strotz (1955-1956)). Thus, in discrete time, by utilizing this time consistency approach one can select an optimal strategy through a period-wise optimization and backward recursion. A meaningful explanation is given through a game theory point of view, and such a strategy has been called an equilibrium control (henceforth referred to as a weakly time consistent optimal strategy). It inherits the equilibrium concept that arises in game theory. We refer to Björk and Murgoci (2010); Wu (2013); Bensoussan et al. (2014) and the references therein for more details. With a rather strong form of time consistency as proposed, for example by Kang and Filar (2006), an extra property of a time consistent optimal strategy is required. This property states that any sub-strategy of a weakly time consistent optimal strategy is also optimal for the corresponding subsequent periods. This is essentially satisfied for an optimal strategy that can be obtained through the standard DPP. Inspired by the work of Kovacevic and Pflug (2009), Chen et al. (2013) constructed a multiperiod separable selection criterion. With respect to this criterion, they proved that the optimal strategy obtained through the pseudo DPP satisfies the extra property of strong time consistency. They obtained a closed form optimal strategy with a multiperiod separable selection criterion of MV type. Later on, their work has been extended by Chen et al. (2014) to allow market transitions.

To the authors' best knowledge, the multiperiod portfolio selection problem in which a MSD type criterion is used as a selection criterion is only briefly mentioned in Kronborg and Steffensen (2015). However, the authors consider a model with two assets only, where one of the assets is supposed to be risk free. Within their setting, a trivial result (a special case of Landsman and Makov (2012)) only is obtained. In essence, the outcome is that is the reward is large enough, it would be advisable to invest as much as possible in the risky asset, whereas when the reward is too little in comparison to the investor's risk tolerance, the strategy is to invest in the risk free asset only. A similar result is obtained in the corresponding continuous time problem (see Kryger and Steffensen (2010); Kronborg and Steffensen (2015)). Our contribution in this paper is to extend the single period model to a multiperiod selection scheme. We take the single period MSD criterion and formulate a separable multiperiod selection criteria of MSD type (similar to Chen et al. (2013) for the MV case). By applying the aforementioned pseudo DPP, we obtain a closed form optimal strategy. The analysis is performed in a market of risky assets only. However, we allow for market transitions, and also for intermediate cash injections and offtakes. Thus, the wealth process of the investor is no longer self-financing in our setting. As far as we are aware, for multiperiod portfolio selection problem, the only work in which intermediate cash injections and offtakes are allowed and closed form solution is obtained, is by Wu and Li (2012). However unlike our work, the authors consider the multiperiod MV criterion, and follow a precommitment approach.

The paper is organized in the following way. In Section 2, we set up our model. In Section 3, we derive the optimal strategy, obtain the multiperiod portfolio selection scheme, and compute the optimal conditional expectation and conditional variance of the terminal wealth. Numerical illustrations and comparisons are performed in Section 4. Finally, we conclude the paper in

Section 5. To prepare for our work, we introduce some notations and conventions.

We use a bold letter to distinguish a vector $\boldsymbol{v} \in \mathbb{R}^d$ from a scalar $v \in \mathbb{R}$. All vectors are column vectors. Moreover,

- for any matrix $\boldsymbol{B}$, $\boldsymbol{B}^T$ denotes its transpose, $\bar{\boldsymbol{B}}(i)$ denotes the sum of the elements of its ith row; if $\boldsymbol{B}$ is square, $\boldsymbol{B}^m$ denotes its mth power, where $m \geq 0$, and $\boldsymbol{B}^0 = \boldsymbol{I}$ (identity matrix);
- for any vector $\boldsymbol{v}$, we denote by $\boldsymbol{v}^i$ its ith component, and by $diag(\boldsymbol{v})$ a diagonal matrix whose diagonal elements $diag(\boldsymbol{v})_{ii} = \boldsymbol{v}^i$ for all i;
- for any matrix $\boldsymbol{B}$, and a vector $\boldsymbol{v}$, we define $\boldsymbol{B}_{\boldsymbol{v}}$ as the matrix product of $\boldsymbol{B}$ and $diag(\boldsymbol{v})$;
- for any sequence of matrices $(\boldsymbol{B}_n)_{n>0}$, and $m < l$, we put $\sum_{n=\ell}^{m} \boldsymbol{B}_n = 0$, and $\prod_{n=\ell}^{m} \boldsymbol{B}_n = I$.

2 Problem Formulation

2.1 The Market and the Investor

Consider a market which has a finite number of different states such as "Normal", "Bull" and "Bear". From time to time the market may shift from one state to another. The transitions of the market are captured by a discrete time homogeneous Markov Chain $\{\theta_n, n \geq 0\}$, with a state space $S = \{1, ..., k\}$, and a transition matrix $\boldsymbol{Q} = (q_{ij})_{k\times k}$. There are $d > 1$ risky assets in the market with random return rates $\boldsymbol{r}_n^1, \ldots, \boldsymbol{r}_n^d$ evolving over time interval $[0, N]$. The vector process of return rates $(\boldsymbol{r}_n^1, \ldots, \boldsymbol{r}_n^d)^T$ will be denoted by $\boldsymbol{r}_n$ whose dynamics is given by an equation

$$\boldsymbol{r}_{n+1}(\theta_n) = \boldsymbol{m}_n(\theta_n) + \boldsymbol{s}_n(\theta_n)\boldsymbol{\epsilon}_{n+1} \in \mathbb{R}^d, \tag{1}$$

(see for example, Costa and Araujo (2008), for this commonly used model). The process $(\boldsymbol{\epsilon}_n)_{n>0}$ is a sequence of independent identically distributed (i.i.d) d-dimensional zero mean random vectors, with covariance matrix I. The functions $\boldsymbol{m}_n : S \to \mathbb{R}^d$ and $\boldsymbol{s}_n : S \to \mathbb{R}^{d\times d}$ are deterministic for each $n = 0, ..., N-1$. In what follows it will be sometimes convenient to use the notation $\boldsymbol{r}_n(\theta_n)$ for $\boldsymbol{r}_n$. Then, for a given market state $\theta_n = j$, the ith component $\boldsymbol{r}_{n+1}^i(j)$ of $\boldsymbol{r}_{n+1}(j)$ represents the return of the ith risky asset over time period $[n, n+1]$. Thus, for every one dollar, we obtain

$$\boldsymbol{R}_{n+1}(\theta_n) = \mathbf{1} + \boldsymbol{r}_{n+1}(\theta_n), \tag{2}$$

where $\mathbf{1} \in \mathbb{R}^d$ is a vector of ones. An investor who has a finite investment horizon $[0, N]$, chooses a strategy at time 0, and adjusts his strategy at times $n = 1, ..., N-1$. We denote the strategy of the investor as

$$\boldsymbol{u} = (\boldsymbol{u}_0(\theta_0), ..., \boldsymbol{u}_{N-1}(\theta_{N-1}))^T, \tag{3}$$

where each $\boldsymbol{u}_n : S \to U$ is a deterministic function and

$$U = \{\boldsymbol{u} \in \mathbb{R}^d : \mathbf{1}^T \boldsymbol{u} = 1\}. \tag{4}$$

For any given market state $\theta_n = j$, the ith component $\boldsymbol{u}_n^i(j) \in \mathbb{R}$ represents the proportions of wealth allocated by the investor to the ith asset. The set $\mathcal{U}^0$ of all such strategies will be interpreted as a set of strategies admissible at time 0. For all $m > 0$, we call $\boldsymbol{u}^m = (\boldsymbol{u}_n)_{n \geq m}$ a sub-strategy of $\boldsymbol{u}$, and use $\mathcal{U}^m$ to denote the set of such admissible sub-strategies.

In this work we assume that at every stage the investor can make cash injections and offtakes. More precisely, at time $n \leq N-1$, the investor can manually change his account by amount of $C_n := C_n(\theta_n)$, where $C_n : S \to \mathbb{R}$ is a deterministic function. If $C_n \geq 0$, this represents a net cash injection, and if $C_n < 0$, this represents a net offtake. Thus, if the market is in a good state, the investor may choose to add money to his portfolio and if the market is in a bad state he may wish to take some cash out. When injections and offtakes are deterministic, this can be interpreted as those investments and/or consumptions which the investor has already planned to add and/or withdraw at the beginning of his investment horizon.

The wealth process $(W_n)_{n \geq 0}$ of the investor is modeled by an $\mathbb{R}$-valued discrete time stochastic process with the dynamics

$$W_{n+1} = W_n \boldsymbol{R}_{n+1}^T(\theta_n)\boldsymbol{u}_n(\theta_n) + C_n(\theta_n). \tag{5}$$

Moreover, for $n = 0, ..., N-1$, and all $i \in S$, we write

$$\begin{aligned} \boldsymbol{M}_n(i) &:= \mathbb{E}(\boldsymbol{R}_{n+1}(\theta_n) | \theta_n = i) = \mathbf{1} + \boldsymbol{m}_n(i), \\ \boldsymbol{\Sigma}_n(i) &:= Var(\boldsymbol{R}_{n+1}(\theta_n) | \theta_n = i) = \boldsymbol{s}_n(i)\boldsymbol{s}_n^T(i). \end{aligned}$$

2.2 Assumptions

Now, in the rest of this work, we make the following assumptions.

(A1). Suppose θ_0 is deterministic. Assume that the Markov chain $\{\theta_n, n \geq 0\}$ is generated by

$$\theta_{n+1} = F(\theta_n, \xi_{n+1}),$$

where $F : S \times \mathbb{R}^d \to S$ is a mapping , and $(\xi_n)_{n>0}$ is a sequence of independent $\mathbb{R}$-valued random variables (see Theorem 58.1 in Levine (2010) for this way of generating Markov chain).

(A2). The sequences $(\xi_n)_{n>0}$ and $(\boldsymbol{\epsilon}_n)_{n>0}$ are independent.

(A3). All random quantities are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a discrete time filtration $\{\mathcal{F}_n\}_{n\geq 0}$. We assume $\mathcal{F}_n = \sigma\Big(\mathcal{G}_n \vee \mathcal{Y}_n\Big)$ for $n \geq 1$, where $\mathcal{G}_n = \sigma(\xi_m, 1 \leq m \leq n)$, and $\mathcal{Y}_n = \sigma(\boldsymbol{\epsilon}_m, 1 \leq m \leq n)$, and $\mathcal{F}_0$ is trivial.

(A4). Without loss of generality, we assume W_0 is deterministic.

(A5). The matrices $\boldsymbol{\Sigma}_n(i)$ are positive definite for $n = 0, ..., N-1$, and all $i \in S$.

(A6). Short selling is allowed, and there are no tax and transaction costs.

We note that, under **(A1)**-**(A3)**, W_n and θ_{n+1} are conditionally independent (given θ_n). Under **(A5)**, we know that $\boldsymbol{\Sigma}_n(i)$ is invertible and its inverse is positive definite for $n = 0, ..., N-1$, and for all $i \in S$. To simplify our presentation, for $n = 0, ..., N-1$, and all $i \in S$, we further define

$$
\begin{aligned}
a_n(i) &= \mathbf{1}^T \boldsymbol{\Sigma}_n^{-1}(i)\mathbf{1}, \qquad b_n(i) = \mathbf{1}^T \boldsymbol{\Sigma}_n^{-1}(i)\boldsymbol{M}_n(i), \\
h_n(i) &= \boldsymbol{M}_n(i)^T \boldsymbol{\Sigma}_n^{-1}(i)\boldsymbol{M}_n(i), \quad g_n(i) = h_n(i) - \frac{b_n^2(i)}{a_n(i)}.
\end{aligned}
$$

Since $\boldsymbol{\Sigma}_n^{-1}(i)$ is positive definite, it is clear that $a_n(i) > 0$.

2.3 The Control Problem

Next, we formulate the problem of interest as a discrete time optimal control problem. We will introduce a definition of a selection criterion for single period and formulate a separable multiperiod selection criterion of MSD type. We adapt the definition of probability functional and separable expected conditional mapping proposed by Kovacevic and Pflug (2009), and extended by Chen et al. (2013).

Definition 2.1. *A single period selection criterion over a given time period* $[n, n+1]$*, where* $n \geq 0$*, is an* $\mathcal{F}_n$*-measurable functional* $\mathcal{J}_n(\cdot) : L^2(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$.

For a fixed $x \in \mathbb{R}$ and $i \in S$, a single period MSD criterion over $[n, n+1]$ takes the form:

$$
\mathcal{J}_{n,x,i}(W_{n+1}) \quad = \quad \mathbb{E}_{n,x,i}(W_{n+1}) - \kappa_n(i)\sqrt{Var_{n,x,i}(W_{n+1})}, \tag{6}
$$

where

$$
\begin{aligned}
\mathbb{E}_{n,x,i}(W_{n+1}) &= \mathbb{E}(W_{n+1}|W_n = x, \theta_n = i), \\
Var_{n,x,i}(W_{n+1}) &= Var(W_{n+1}|W_n = x, \theta_n = i).
\end{aligned}
$$

We may add a superscript $\boldsymbol{u}$ to W_n when we wish to emphasize the dependence of the wealth process on a (corresponding) strategy.

Next, by using the single period MSD criteria, for any starting time $n = 0, ..., N-1$, we can formulate a separable multiperiod selection criterion of MSD type,

$$\begin{aligned} J_{n,x,i}(\boldsymbol{u}^n) &= \mathbb{E}\Big(\sum_{m=n}^{N-2} \mathcal{J}_{m,W_m,\theta_m}(W_{m+1}) \\ &\quad + \mathcal{J}_{N-1,W_{N-1},\theta_{N-1}}(W_N) | W_n = x, \theta_n = i \Big). \end{aligned} \tag{7}$$

Here and after, we use superscripts to emphasis the dependence of J on n, x, and i.

Next, we borrow the definition of time consistency from Kang and Filar (2006).

Definition 2.2. *Given any starting time $n = 0, ..., N-1$, a strategy $\boldsymbol{u}^{n,*} = (\boldsymbol{u}^*_n(\theta_n), ..., \boldsymbol{u}^*_{N-1}(\theta_{N-1}))$ is said to be a strongly time consistent optimal strategy with respect to $J_{n,x,i}(\boldsymbol{u}^n)$ if it satisfies the following two conditions.*

- ***Condition 1**: Let $\mathcal{A}^n \subset \mathcal{U}^n$ be a set of strategies of the form $\boldsymbol{u}^n = (\boldsymbol{v}(i), \boldsymbol{u}^*_{n+1}(\theta_{n+1}), ..., \boldsymbol{u}^*_{N-1}(\theta_{N-1}))$, where $\boldsymbol{v}(i) \in \mathbb{R}^d$ is arbitrary. Then, we have*

$$\sup_{\boldsymbol{u}^n \in \mathcal{A}^n} J_{n,x,i}(\boldsymbol{u}^n) = J_{n,x,i}(\boldsymbol{u}^{n,*}). \tag{8}$$

- ***Condition 2**: For $m = n+1, .., N-1$,*

$$\sup_{\boldsymbol{u}^m \in \mathcal{U}^m} J_{m,x,i}(\boldsymbol{u}^m) = J_{m,x,i}(\boldsymbol{u}^{m,*}), \tag{9}$$

where $\boldsymbol{u}^{m,*} = (\boldsymbol{u}^*_m(i), ..., \boldsymbol{u}^*_{N-1}(\theta_{N-1}))$.

*If **Condition 1** is satisfied, then we say that the strategy is a weakly time consistent optimal strategy with respect to $J_{n,x,i}(\cdot)$.*

Condition 1 states that a weakly time-consistent optimal strategy is obtained through a period-wise optimization and backward recursion. This formulates a pseudo DPP and provides a way of selecting an optimal strategy. In contrast, a strongly time-consistent optimal strategy has an extra property which guarantees that any sub-strategy of a weakly time-consistent optimal strategy is also optimal for the corresponding subsequent period. This extra property makes a strongly time-consistent optimal strategy similar to an optimal strategy obtained by standard DPP.

Let

$$V(n, x, i) = \sup_{\boldsymbol{u}^n \in \mathcal{A}^n} J_{n,x,i}(\boldsymbol{u}^n).$$

Based on the arguments in Chen et al. (2013), if there exists a weakly time consistent optimal strategy for the above problem, then such strategy satisfies the strong time consistency conditions presented in Definition 2.2. Our mission is to solve this optimal control problem and find such optimal strategy.

3 Optimal Portfolio Selection under Market Transitions and Intermediate Cash Injections

3.1 Properties of Single Period Mean-Standard-Deviation Selection Criterion

The name single period MSD criterion is actually a special term taken from Actuarial science, and its negative opposite in some sense is called the standard deviation premium (see **?** and the reference therein). Before we move to the derivation of the optimal strategy, let us briefly outline some properties of the single period MSD criterion. Without loss of generality, we consider the first period. Since the cash injections (and offtakes) are no longer relevant, we have

$$\mathcal{J}_{0,x,i}(W_1^{\boldsymbol{u}}) = \mathbb{E}_{0,x,i}(W_1^{\boldsymbol{u}}) - \kappa_0\sqrt{Var_{0,x,i}(W_1^{\boldsymbol{u}})},$$

where

$$\mathbb{E}_{0,x,i}(W_1^{\boldsymbol{u}}) = x\boldsymbol{u}_0^T(i)\boldsymbol{M}_0(i), \text{ and } Var_{0,x,i}(W_1^{\boldsymbol{u}}) = x^2\boldsymbol{u}_0^T(i)\boldsymbol{\Sigma}_0(i)\boldsymbol{u}_0(i).$$

For a single period problem, the market transitions and time dependence are also irrelevant. To simplify notations, we drop the subscripts in κ, $\boldsymbol{M}$, $\boldsymbol{\Sigma}$, $\boldsymbol{u}$, W, and $\mathcal{J}$, and also drop the market state argument i in $\boldsymbol{M}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{u}$. Then, we have the following lemma.

Lemma 3.1. *The single period MSD criterion satisfies the following properties.*

- ***(Translation Invariance)**: For all $\boldsymbol{u} \in \mathbb{R}^d$, and $y \in \mathbb{R}$ we have*

$$\mathcal{J}(W^{\boldsymbol{u}} + y) = \mathcal{J}(W^{\boldsymbol{u}}) + y.$$

- ***(Positive Homogeneity)**: For all $\boldsymbol{u} \in \mathbb{R}^d$, and $\alpha \geq 0$ we have*

$$\mathcal{J}(\alpha W^{\boldsymbol{u}}) = \alpha\mathcal{J}(W^{\boldsymbol{u}}).$$

- ***(Scaling Property)**: There exists some functional $\hat{J}$ such that for all $x \in \mathbb{R}$,*

$$\mathcal{J}(W^{\boldsymbol{u}}) = x\hat{J}(\boldsymbol{u}).$$

- ***(Concavity)**: Assume that $x \in (0,\infty)$. Then, $\mathcal{J} : \boldsymbol{u} \to x\big(\boldsymbol{u}^T\boldsymbol{M} - \sqrt{\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}}\big)$ is a strictly concave function of $\boldsymbol{u}$, i.e., for all $\boldsymbol{u}, \hat{\boldsymbol{u}} \in \mathbb{R}^d$, $\boldsymbol{u} \neq \hat{\boldsymbol{u}}$ and $\xi \in (0,1)$, we have*

$$\xi\mathcal{J}\big(\boldsymbol{u}\big) + (1-\xi)\mathcal{J}\big(\hat{\boldsymbol{u}}\big) < \mathcal{J}\big(\xi\boldsymbol{u} + (1-\xi)\hat{\boldsymbol{u}}\big). \tag{10}$$

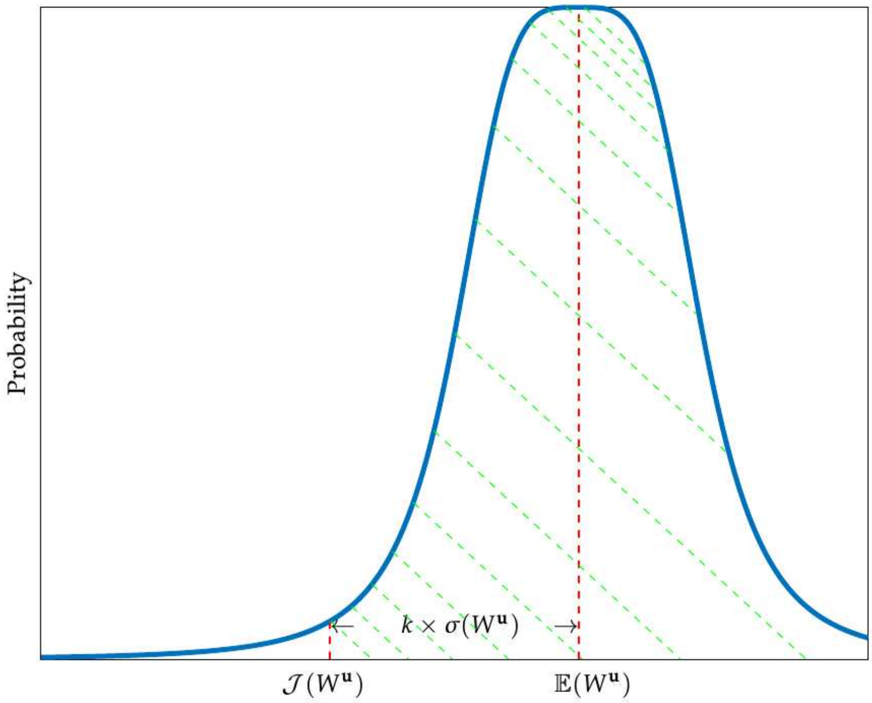

Figure 1: Investor's Risk Characterization

Proof. The first three properties can be trivially verified. For concavity, we can either follow the argument in Landsman (2008) (see pp319 - 320) or in Owadally (2012) (see p4435).

□

The first two properties are not surprising since we have mentioned in the Introduction, that under the assumption that returns follow a (joint) elliptical distribution, optimizing a risk measure from the TIPH risk measure class is equivalent to optimizing the single period MSD selection criterion. One may note that by definition the negative of a single period selection criterion is a risk measure. For different members of the TIPH risk measure class, the form of single MSD selection criterion only varies through the parameter κ. As shown in Landsman and Makov (2011), for example, for VaR we have $\kappa = F^{-1}(q)$ for some $q \in [0, 1]$, where F^{-1} is the inverse of the distribution function of a standard univariate elliptical random variable. The parameter κ characterizes the investor's risk tolerance. The larger the κ, the more risk averse the investor is. For any $\kappa > 0$, $\mathcal{J}(W^{\boldsymbol{u}})$ represents a quantile value, that is κ standard deviations of the wealth away from the investor's expected terminal wealth $\mathbb{E}(W^{\boldsymbol{u}})$ to the left (see Figure 1). This quantile value in turn corresponds to a probability p, where

$$p = \mathbb{P}\Big(W^{\boldsymbol{u}} \geq \mathcal{J}(W^{\boldsymbol{u}})\Big).$$

Thus, the investor will have with probability p at least $\mathcal{J}(W^{\boldsymbol{u}})$ at the end of the investment period (the green area in Figure 1). This provides a confidence level to the investor which is somewhat inherited from VaR. Hence, it is possible to choose an appropriate risk aversion parameter, provide the returns are reasonable, so that his wealth is above zero with a high probability.

The scaling property and the concavity form an important aspect of the single period MSD selection criterion. Unlike the single period MV selection criterion, the MSD selection criterion does not possess a nice quadratic structure, however, these two nice properties make it a suitable candidate in both single and in multiperiod portfolio selection.

3.2 Optimal Strategy and Value Function

Theorem 3.2. *For any given time $n = 0, ..., N-1$, a market state $\theta_n = i \in S$, and $x \in (0, \infty)$, assume that $\kappa_n(i) > \hat{\kappa}_n(i)$, where*

$$\hat{\kappa}_n(i) \quad = \quad \sqrt{g_n(i)(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i))^2}, \tag{11}$$

the optimal strategy is given by

$$\boldsymbol{u}_n^*(i) \quad = \quad \frac{\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)\big) f_n(i)}{\kappa_n(i)} \left(\boldsymbol{\Sigma}_n^{-1}(i) \boldsymbol{M}_n(i) - \frac{b_n(i) \boldsymbol{\Sigma}_n^{-1}(i) \boldsymbol{1}}{a_n(i)} \right) + \frac{\boldsymbol{\Sigma}_n^{-1}(i) \boldsymbol{1}}{a_n(i)}, \tag{12}$$

and the corresponding value function is given by

$$\begin{aligned} V(n, x, i) \quad &= \quad x A_n(i) + C_n(i) \big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\big) + \sum_{m=n+1}^{N-1} \overline{\boldsymbol{Q}^{m-(n+1)} \boldsymbol{Q}_{\boldsymbol{C}_m}}(i) \\ &+ \sum_{m=n+1}^{N-2} \overline{\boldsymbol{Q}^{m-(n+1)} \boldsymbol{Q}_{\boldsymbol{C}_m} Q_{\boldsymbol{A}_{m+1}}}(i), \end{aligned}$$

where

$$\begin{aligned} & f_n(i) = \sqrt{\frac{\frac{1}{a_n(i)}}{1 - \frac{g_n(i)}{\kappa_n^2(i)} \big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)\big)^2}} = \sqrt{\frac{\frac{1}{a_n(i)}}{1 - \frac{\hat{\kappa}_n^2(i)}{\kappa_n^2(i)}}}, \\ & A_N = 0, \\ & A_n(i) = \kappa_n(i) f_n(i) \left(\frac{\hat{\kappa}_n^2(i)}{\kappa_n^2(i)} - 1 \right) + \big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)\big) \frac{b_n(i)}{a_n(i)}, \\ & \boldsymbol{A}_n = \Big(A_n(1), ..., A_n(k) \Big)^T, \boldsymbol{C}_n = \Big(C_n(1), ..., C_n(k) \Big)^T. \end{aligned}$$

Proof. **Step 1**: For $n = N-1$, we have the following static optimization problem

$$\begin{aligned} \max_{\boldsymbol{u}_{N-1}(i)} \quad & \left(\mathbb{E}_{N-1,x,i}(W_N) - \kappa_{N-1}(i) \sqrt{Var_{N-1,x,i}(W_N)} \right), \\ \text{s.t.} \quad & W_N = W_{N-1} \boldsymbol{R}_N^T(i) \boldsymbol{u}_{N-1}(i) + C_{N-1}(i), \\ & \boldsymbol{1}^T \boldsymbol{u}_{N-1}(i) = 1. \end{aligned}$$

Since x and $C_{N-1}(i)$ are known at $n = N-1$, simple calculations yield the conditional expectation

$$\mathbb{E}_{N-1,x,i}(W_N) = x\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}(i) + C_{N-1}(i), \tag{13}$$

and the conditional variance

$$Var_{N-1,x,i}(W_N) = x^2\boldsymbol{u}_{N-1}^T(i)\boldsymbol{\Sigma}_{N-1}(i)\boldsymbol{u}_{N-1}(i). \tag{14}$$

Let

$$S_{N-1}(i) \quad = \quad S_{N-1}(i, \boldsymbol{u}_{N-1}(i)) := \sqrt{\boldsymbol{u}_{N-1}^T(i)\boldsymbol{\Sigma}_{N-1}(i)\boldsymbol{u}_{N-1}(i)}. \tag{15}$$

Thus, we have

$$\begin{aligned} & \mathbb{E}_{N-1,x,i}(W_N) - \kappa_{N-1}(i)\sqrt{Var_{N-1,x,i}(W_N)} \\ = \quad & x\Big(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}(i) - \kappa_{N-1}(i)S_{N-1}(i)\Big) + C_{N-1}(i). \end{aligned} \tag{16}$$

Then, it is easy to see that optimizing the above function is equivalent to optimizing

$$\Big(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}(i) - \kappa_{N-1}(i)S_{N-1}(i)\Big),$$

(which is really the **Scaling Property** in Lemma 3.1).

Thus, we obtain an equivalent optimization problem:

$$\begin{aligned} \max_{\boldsymbol{u}_{N-1}(i)} \quad & \Big(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}(i) - \kappa_{N-1}(i)S_{N-1}(i)\Big), \\ \text{s.t.} \quad & \boldsymbol{1}^T\boldsymbol{u}_{N-1}(i) = 1. \end{aligned}$$

Let $\lambda_{N-1}(i)$ be the Lagrange multiplier. Since $x > 0$, the **Concavity** in Lemma 3.1 holds. Thus, the following first order conditions yield a unique global optimum, if there exists a solution:

$$\boldsymbol{M}_{N-1}(i) - \kappa_{N-1}(i)\frac{\boldsymbol{\Sigma}_{N-1}(i)\boldsymbol{u}_{N-1}(i)}{S_{N-1}(i)} - \lambda_{N-1}(i)\boldsymbol{1} = 0, \tag{17}$$

$$\boldsymbol{1}^T\boldsymbol{u}_{N-1}(i) = 1. \tag{18}$$

From (17), we obtain

$$\boldsymbol{u}_{N-1}(i) \quad = \quad \frac{S_{N-1}(i)}{\kappa_{N-1}(i)}\boldsymbol{\Sigma}_{N-1}^{-1}(i)\Big(\boldsymbol{M}_{N-1}(i) - \lambda_{N-1}(i)\boldsymbol{1}\Big). \tag{19}$$

Substituting (19) into (18) we find $\lambda_{N-1}(i)$ as

$$\lambda_{N-1}(i) = \frac{\frac{S_{N-1}(i)}{\kappa_{N-1}(i)}\mathbf{1}^T\boldsymbol{\Sigma}_{N-1}^{-1}(i)\boldsymbol{M}_{N-1}(i) - 1}{\frac{S_{N-1}(i)}{\kappa_{N-1}(i)}\mathbf{1}^T\boldsymbol{\Sigma}_{N-1}^{-1}(i)\mathbf{1}}. \tag{20}$$

Substituting (20) into (19) gives the optimal strategy

$$\boldsymbol{u}_{N-1}^*(i) = \frac{S_{N-1}^*(i)}{\kappa_{N-1}(i)}\left(\boldsymbol{\Sigma}_{N-1}^{-1}(i)\boldsymbol{M}_{N-1}(i) - \frac{b_{N-1}(i)\boldsymbol{\Sigma}_{N-1}^{-1}(i)\mathbf{1}}{a_{N-1}(i)}\right) + \frac{\boldsymbol{\Sigma}_{N-1}^{-1}(i)\mathbf{1}}{a_{N-1}(i)}, \tag{21}$$

where $S_{N-1}^*(i) = S_{N-1}(i, \boldsymbol{u}_{N-1}^*(i))$.

One can obtain $S_{N-1}^*(i)$ by substituting (21) into (15). Thus we have

$$S_{N-1}^*(i) = \sqrt{\frac{\frac{1}{a_{N-1}(i)}}{1 - \frac{h_{N-1}(i)}{\kappa_{N-1}^2(i)} + \frac{b_{N-1}^2(i)}{\kappa_{N-1}^2(i)a_{N-1}(i)}}} := f_{N-1}(i), \tag{22}$$

provided that $\kappa_{N-1}(i) > \sqrt{g_{N-1}(i)} := \hat{\kappa}_{N-1}(i)$. Substituting $S_{N-1}^*(i)$ into (21), we obtain the desired form of $\boldsymbol{u}_{N-1}^*(i)$.

One may note that we always have $g_{N-1}(i) \geq 0$, by Cauchy-Schwartz inequality, and the fact that $\boldsymbol{\Sigma}_{N-1}^{-1}(i)$ is positive definite (which is a consequence of the assumption that $\boldsymbol{\Sigma}_{N-1}(i)$ is positive definite). Indeed, we see that

$$\begin{aligned} g_{N-1}(i) &= h_{N-1}(i) - \frac{b_{N-1}^2(i)}{a_{N-1}(i)} \\ &= \frac{1}{\mathbf{1}^T\boldsymbol{\Sigma}_{N-1}^{-1}(i)\mathbf{1}}\Big(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{\Sigma}_{N-1}^{-1}(i)\boldsymbol{M}_{N-1}(i)\mathbf{1}^T\boldsymbol{\Sigma}_{N-1}^{-1}(i)\mathbf{1} \\ &\quad -\big(\mathbf{1}^T\boldsymbol{\Sigma}_{N-1}^{-1}(i)\boldsymbol{M}_{N-1}(i)\big)^2\Big) \geq 0. \end{aligned}$$

Next, let us calculate the value function $V(N-1, x, i)$ at time $N-1$. By (21) and (22), we see that

$$\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}^*(i) = \frac{f_{N-1}(i)g_{N-1}(i)}{\kappa_{N-1}(i)} + \frac{b_{N-1}(i)}{a_{N-1}(i)},$$

and

$$\sqrt{(\boldsymbol{u}_{N-1}^*)^T(i)\boldsymbol{\Sigma}_{N-1}(i)\boldsymbol{u}_{N-1}^*(i)} = S_{N-1}^*(i) = f_{N-1}(i).$$

Thus, we have

$$\begin{aligned} V(N-1, x, i) &= x\Big(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}^*(i) - \kappa_{N-1}(i)S_{N-1}^*(i)\Big) + C_{N-1}(i) \\ &= xA_{N-1}(i) + C_{N-1}(i). \end{aligned} \tag{23}$$

Step 2: For $n = N-2$, we have the following static optimization problem:

$$\begin{aligned}\max_{\boldsymbol{u}_{N-2}(i)} \quad & \Bigg(\mathbb{E}_{N-2,x,i}(W_{N-1}) - \kappa_{N-2}(i)\sqrt{Var_{N-2,x,i}(W_{N-1})} \\ & + \mathbb{E}_{N-2,x,i}\Big(V(N-1, W_{N-1}, \theta_{N-1})\Big)\Bigg), \\ \text{s.t.} \quad & W_{N-1} = W_{N-2}\boldsymbol{R}^T_{N-1}(i)\boldsymbol{u}_{N-2}(i) + C_{N-2}(i), \\ & \boldsymbol{1}^T\boldsymbol{u}_{N-2}(i) = 1. \end{aligned} \tag{24}$$

Similarly, as in (13) and (14), we calculate expressions for $\mathbb{E}_{N-2,x,i}(W_{N-1})$ and $\sqrt{Var_{N-2,x,i}(W_{N-1})}$, which yield

$$\mathbb{E}_{N-2,x,i}(W_{N-1}) = x\boldsymbol{M}^T_{N-2}(i)\boldsymbol{u}_{N-2}(i) + C_{N-2}(i), \tag{25}$$

$$\sqrt{Var_{N-2,x,i}(W_{N-1})} = x\sqrt{\boldsymbol{u}^T_{N-2}(i)\boldsymbol{\Sigma}_{N-2}(i)\boldsymbol{u}_{N-2}(i)}. \tag{26}$$

Moreover, by using (23) and (24), we find an expression for $\mathbb{E}_{N-2,x,i}\Big(V(N-1, W_{N-1}, \theta_{N-1})\Big)$. This gives

$$\begin{aligned} & \mathbb{E}_{N-2,x,i}\Big(V(N-1, W_{N-1}, \theta_{N-1})\Big) \\ = \;& x\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\boldsymbol{M}^T_{N-2}(i)\boldsymbol{u}_{N-2}(i) + C_{N-2}(i)\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i) + \overline{\boldsymbol{Q}}_{\boldsymbol{C}_{N-1}}(i). \end{aligned} \tag{27}$$

Combining these three expressions, and by recognizing that x and $C_{N-2}(i)\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i) + \overline{\boldsymbol{Q}}_{\boldsymbol{C}_{N-1}}(i)$ are known at $n = N-2$, thus we obtain an equivalent optimization problem:

$$\begin{aligned}\max_{\boldsymbol{u}_{N-2}(i)} \quad & \Big(\boldsymbol{M}^T_{N-2}(i)\boldsymbol{u}_{N-2}(i) - \kappa_{N-2}(i)S_{N-2}(i) + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\boldsymbol{M}^T_{N-2}(i)\boldsymbol{u}_{N-2}(i)\Big), \\ \text{s.t.} \quad & \boldsymbol{1}^T\boldsymbol{u}_{N-2}(i) = 1, \end{aligned}$$

where

$$S_{N-2}(i) = S_{N-2}(i, \boldsymbol{u}_{N-2}(i)) := \sqrt{\boldsymbol{u}^T_{N-2}(i)\boldsymbol{\Sigma}_{N-2}(i)\boldsymbol{u}_{N-2}(i)}. \tag{28}$$

We repeat the procedure from **Step 1**. Let $\lambda_{N-2}(i)$ be the corresponding Lagrange multiplier. The solution of the following system of equations, if exists, yields an unique global optimum:

$$\begin{aligned} & \big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big)\boldsymbol{M}_{N-2}(i) - \kappa_{N-2}(i)\frac{\boldsymbol{\Sigma}_{N-2}(i)\boldsymbol{u}_{N-2}(i)}{S_{N-2}(i)} - \lambda_{N-2}(i)\boldsymbol{1} = 0 \\ & \boldsymbol{1}^T\boldsymbol{u}_{N-2}(i) = 1. \end{aligned}$$

Solving this system yields

$$\boldsymbol{u}^*_{N-2}(i) = \frac{\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big)}{\kappa_{N-2}(i)}S^*_{N-2}(i)\Bigg(\boldsymbol{\Sigma}^{-1}_{N-2}(i)\boldsymbol{M}_{N-2}(i)$$

$$-\frac{b_{N-2}(i)\boldsymbol{\Sigma}_{N-2}^{-1}(i)\mathbf{1}}{a_{N-2}(i)}\Bigg) + \frac{\boldsymbol{\Sigma}_{N-2}^{-1}(i)\mathbf{1}}{a_{N-2}(i)},$$

where $S_{N-2}^*(i) = S_{N-2}(i, \boldsymbol{u}_{N-2}^*(i))$. Similarly to **Step 1**, we obtain an expression of $S_{N-2}^*(i)$ by using (28) and (29). This yields

$$S_{N-2}^*(i) = \sqrt{\frac{\frac{1}{a_{N-2}(i)}}{1 - \frac{g_{N-2}(i)}{\kappa_{N-2}^2(i)}\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big)^2}} := f_{N-2}(i),$$

provided that $\kappa_{N-2}(i) > \sqrt{g_{N-2}(i)\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big)^2}$. This gives the desired form of optimal strategy at $n = N-2$. Arguing in the same way as in **Step 1**, we see that $g_{N-2}(i) \geq 0$.

By knowing the optimal strategy, we can easily obtain the value function $V(N-2, x, i)$ as in **Step 1**, in which we need (25), (26), and (27). Thus, simple algebra yields

$$V(N-2, x, i) = xA_{N-2}(i) + C_{N-2}(i)\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big) + \overline{\boldsymbol{Q}}_{\boldsymbol{C}_{N-1}}(i).$$

Step 3: For $n = N-2, ..., 0$, we use backward induction. Since we have proved the claim holds at $N-2$, let us assume it holds up to $n+1$. Over any time period $[n, n+1]$, we solve the following static optimization problem:

$$\begin{aligned}
&\max_{\boldsymbol{u}_n(i)} \quad \left(\mathbb{E}_{n,x,i}(W_{n+1}) - \kappa_n(i)\sqrt{Var_{n,x,i}(W_{n+1})} + \mathbb{E}_{n,x,i}\Big(V(n+1, W_{n+1}, \theta_{n+1})\Big)\right), \\
&\text{s.t.} \quad W_{n+1} = W_n \boldsymbol{R}_{n+1}^T(i)\boldsymbol{u}_n(i) + C_n(i), \\
&\qquad\quad \mathbf{1}^T\boldsymbol{u}_n(i) = 1.
\end{aligned}$$

We repeat the procedure in **Step 1** (or **Step 2**). Firstly, we calculate

$$\mathbb{E}_{n,x,i}(W_{n+1}) = x\boldsymbol{M}_n^T(i)\boldsymbol{u}_n(i) + C_n(i), \tag{29}$$

$$\sqrt{Var_{n,x,i}(W_{n+1})} = x\sqrt{\boldsymbol{u}_n^T(i)\boldsymbol{\Sigma}_n(i)\boldsymbol{u}_n(i)}. \tag{30}$$

By induction hypothesis and the proof of Lemma 3 in Wu and Li (2012), we obtain

$$\begin{aligned}
\mathbb{E}_{n,x,i}\Big(V(n+1, W_{n+1}, \theta_{n+1})\Big) &= x\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\boldsymbol{M}_n^T(i)\boldsymbol{u}_n(i) + C_n(i)\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i) \\
&\quad + \sum_{m=n+1}^{N-1} \overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}}(i) \\
&\quad + \sum_{m=n+1}^{N-2} \overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}\boldsymbol{Q}_{\boldsymbol{A}_{m+1}}}(i).
\end{aligned} \tag{31}$$

Then, we obtain the first order (necessary and sufficient) condition for the equivalent optimization problem with the Lagrange multiplier $\lambda_n(i)$:

$$
\begin{aligned}
&(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\boldsymbol{M}_n(i) - \kappa_n(i)\frac{\boldsymbol{\Sigma}_n(i)\boldsymbol{u}_n(i)}{S_n(i)} - \lambda_n(i)\mathbf{1} = 0,\\
&\mathbf{1}^T\boldsymbol{u}_n(i) = 1,
\end{aligned}
$$

where $S_n(i) = S_n(i, \boldsymbol{u}_n(i)) := \sqrt{\boldsymbol{u}_n^T(i)\boldsymbol{\Sigma}_n(i)\boldsymbol{u}_n(i)}$.

The unique solution of this system is then given by

$$
\boldsymbol{u}_n^*(i) \;=\; \frac{\left(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\right)}{\kappa_n(i)} S_n^*(i)\left(\boldsymbol{\Sigma}_n^{-1}(i)\boldsymbol{M}_n(i) - \frac{b_n(i)\boldsymbol{\Sigma}_n^{-1}(i)\mathbf{1}}{a_n(i)}\right) + \frac{\boldsymbol{\Sigma}_n^{-1}(i)\mathbf{1}}{a_n(i)},
$$

where $S_n^*(i) = S_n(i, \boldsymbol{u}_n^*(i))$.

We can obtain $S_n^*(i)$ in the same way as in **Step 1** (or **Step 2**). Thus, we have

$$
S_n^*(i) = \sqrt{\frac{\frac{1}{a_n(i)}}{1-\frac{g_n(i)}{\kappa_n^2(i)}\left(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\right)^2}} \;:= f_n(i),
$$

provided that $\kappa_n(i) > \sqrt{g_n(i)\left(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\right)^2}$.

Furthermore, by using the optimal strategy, (29), (30) and (31) we obtain the value function

$$
\begin{aligned}
V(n,x,i) \;=\; & xA_n(i) + C_n(i)\left(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\right) + \sum_{m=n+1}^{N-1}\overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}}(i)\\
& + \sum_{m=n+1}^{N-2}\overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}\boldsymbol{Q}_{\boldsymbol{A}_{m+1}}}(i).
\end{aligned}
$$

This completes the proof. □

3.3 The Multiperiod Portfolio Selection Scheme

From the proof of Theorem 3.2, we see that when the separable multiperiod selection criterion of MSD type is used for portfolio selection, there exists an optimal strategy if:

1. The investor's risk aversion parameter is above a given lower bound.
2. For all market states $i \in S$ and all periods $n = 0, ..., N-1$ the wealth of the investor is positive.

The first condition is crucial. The investor has to be risk averse enough in order to obtain an optimal strategy (recall that the larger the risk aversion parameter, the more risk averse the investor is). We note that for any $n = 0, ..., N-1$, one has to specify their future risk aversion $\kappa_{n+1}(\theta_{n+1}), ..., \kappa_{N-1}(\theta_{N-1})$ for all market states $\theta_{n+1}, ..., \theta_{N-1}$ in order to obtain the lower bound for the current risk aversion parameter κ_n. This reflects the time consistency idea, according to which the investor maintains a risk aversion consistently through time. Mathematically, we could remove the second condition, however, to reflect the reality, this is a must. Of course, there is no guarantee that the wealth of the investor always stays positive, although this may not be a big issue in practice (see Remark 8 in Wu (2013)). However, the whole point of portfolio selection is to make a decision. Depending on the risk aversion, the investor may still wish to take the risk and go for the investment, provided that his wealth stays above zero with a high probability (under our optimal strategy). This leads to **Algorithm 3.1**.

```
set abandon = false;
for n = N − 1, ..., 0 do
    for θ_n = 1, ..., k do
        set W_n = 1;
        calculate κ̂_n(θ_n) by using (11);
        choose an δ_n(θ_n) > 0;
        set κ_n(θ_n) = κ̂_n(θ_n) + δ_n(θ_n);
        calculate u_n(θ_n) by using (12);
        calculate p_n(u_n, θ_n) = P(W^u_{n+1} > 0);
        if p_n(u_n, θ_n) > 1 − exp(−δ_n) then
            keep the strategy u_n(θ_n);
        else
            abandon = true;
        end
    end
end
if abandon == false then
    take the investment;
else
    abandon the investment;
end
```

Algorithm 3.1: Multiperiod MSD Portfolio Selection Scheme

By **Algorithm 3.1**, our portfolio selection process proceeds in the following way. Over the period $[n, n+1]$, and given a market state θ_n, we pick a number $\delta_n(\theta_n) > 0$ in such a way that $\kappa_n(\theta_n) > \hat{\kappa}_n(\theta_n)$ holds. Next, with our strategy in (12), we calculate the probability $p_n(\boldsymbol{u}_n, \theta_n)$ that the wealth at the end of this period is positive for every 1 dollar which we invest at the beginning of this period. Thanks to the **Scaling Property**, this will be enough to determine whether to abandon the investment. Finally, we choose a threshold equal to $\big(1 - \exp(-\delta_n)\big)$. If the probability is larger than this threshold, we keep our strategy, otherwise we give up the investment. The choice of threshold is arbitrary. We only need an increasing function (since the larger the risk aversion parameter, the more risk averse the investor is) whose range is between 0 and 1.

3.4 Optimal Conditional Expectation and Conditional Variance of Terminal Wealth

Apart from the optimal strategy, we derive the optimal conditional expectation and conditional variance of the terminal wealth. For the purpose of this section, we introduce some additional definitions. For $n = 0, ..., N-1$, $i \in S$, define

$$D_n(i) = \frac{(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i))f_n(i)g_n(i)}{\kappa_n(i)} + \frac{b_n(i)}{a_n(i)}, \quad \hat{D}_n(i) = 2C_n(i)D_n(i), \quad \tilde{C}_n(i) = C_n^2(i),$$

$$\boldsymbol{D_n} = (D_n(1), ..., D_n(k)), \quad \boldsymbol{\hat{D}_n} = (\hat{D}_n(1), ..., \hat{D}_n(k)), \quad \boldsymbol{\tilde{C}_n} = (\tilde{C}_n(1), ..., \tilde{C}_n(k)),$$

$$G_n(i) = \frac{(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i))f_n(i)g_n(i)}{\kappa_n(i)}\left(\frac{(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i))f_n(i)}{\kappa_n(i)} + D_n(i) + \frac{b_n(i)}{a_n(i)}\right) + \frac{a_n(i)+b_n^2(i)}{a_n^2(i)}, \qquad \boldsymbol{G_n} = (G_n(1), ..., G_n(k)).$$

Proposition 3.3. *Given $W_0 = x > 0$, and $\theta_0 = i \in S$, the optimal conditional expectation and conditional second moments of the terminal wealth are given by*

$$\mathbb{E}_{0,x,i}(W_N^*) = \alpha(i)x + \beta(i), \tag{32}$$
$$\mathbb{E}_{0,x,i}\big((W_N^*)^2\big) = \gamma(i)x^2 + \delta(i)x + \eta(i), \tag{33}$$

where

$$\alpha(i) = D_0(i)\overline{\prod_{n=1}^{N-1}\boldsymbol{Q}_{\boldsymbol{D}_n}}(i), \quad \beta(i) = C_0(i)\overline{\prod_{n=1}^{N-1}\boldsymbol{Q}_{\boldsymbol{D}_n}}(i) + \overline{\sum_{m=1}^{N-1}\boldsymbol{Q}^{m-1}\boldsymbol{Q}_{\boldsymbol{C}_m}\prod_{n=m+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{D}_n}}(i),$$
$$\gamma(i) = G_0(i)\tau(i), \quad \delta(i) = \hat{D}_0(i)\tau(i) + D_0(i)\psi(i), \quad \eta(i) = \tilde{C}_0(i)\tau(i) + C_0(i)\psi(i) + \lambda(i),$$

$$\tau(i) = \overline{\prod_{n=1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i), \qquad \psi(i) = \sum_{\ell=1}^{N-1}\overline{\prod_{m=1}^{\ell-1}\boldsymbol{Q}_{\boldsymbol{D}_m}\boldsymbol{Q}_{\boldsymbol{\hat{D}_\ell}}\prod_{n=\ell+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i),$$

$$\lambda(i) = \overline{\sum_{m=1}^{N-1}\boldsymbol{Q}^{m-1}\boldsymbol{Q}_{\boldsymbol{\tilde{C}}_m}\prod_{n=m+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i) + \sum_{\ell=1}^{N-2}\sum_{m=\ell+1}^{N-1}\overline{\boldsymbol{Q}^{\ell-1}\boldsymbol{Q}_{\boldsymbol{C}_\ell}\prod_{t=\ell+1}^{m-1}\boldsymbol{Q}_{\boldsymbol{D}_t}\boldsymbol{Q}_{\boldsymbol{\hat{D}}_m}\prod_{n=m+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i).$$

Proof. For every $n = 0, ..., N-1$, we have

$$W_{n+1}^* = W_n^*\boldsymbol{R}_{n+1}^T(\theta_n)\boldsymbol{u}_n^*(\theta_n) + C_n(\theta_n).$$

Taking conditional expectation with respect to the condition $(W_0 = x, \theta_0 = i, \theta_1, ..., \theta_n)$, and by taking into account (12) together with the conditional independence between W_n^* and $\boldsymbol{R}_{n+1}$, we obtain

$$\begin{aligned}
& \mathbb{E}\Big(W_{n+1}^*|W_0=x,\theta_0=i,\theta_1,...,\theta_n\Big) \\
= \; & D_n(\theta_n)\mathbb{E}\Big(W_n^*|W_0=x,\theta_0=i,\theta_1,...,\theta_{n-1}\Big)+C_n(\theta_n).
\end{aligned} \tag{34}$$

Using (34) and backward recursion, we find that

$$\mathbb{E}\Big(W_N^*|W_0=x,\theta_0=i,\theta_1,...,\theta_{N-1}\Big) = x\prod_{n=0}^{N-1}D_n(\theta_n)+\sum_{m=0}^{N-1}\Big(C_m(\theta_m)\prod_{n=m+1}^{N-1}D_n(\theta_n)\Big).$$

Iterated conditional expectation then implies

$$\begin{aligned}
\mathbb{E}_{0,x,i}(W_N^*) = \; & xD_0(i)\mathbb{E}_{0,x,i}\Big(\prod_{n=1}^{N-1}D_n(\theta_n)\Big)+\sum_{m=1}^{N-1}\mathbb{E}_{0,x,i}\Big(C_m(\theta_m)\prod_{n=m+1}^{N-1}D_n(\theta_n)\Big) \\
& +C_0(i)\mathbb{E}_{0,x,i}\Big(\prod_{n=1}^{N-1}D_n(\theta_n)\Big).
\end{aligned}$$

By Lemma 3 in Wu and Li (2012), we have

$$\mathbb{E}_{0,x,i}\Big(\prod_{n=1}^{N-1}D_n(\theta_n)\Big) = \overline{\prod_{n=1}^{N-1}\boldsymbol{Q}_{\boldsymbol{D}_n}}(i),$$

and for all $m=1,...,N-1$ we have

$$\mathbb{E}_{0,x,i}\Big(C_m(\theta_m)\prod_{n=m+1}^{N-1}D(\theta_n)\Big) = \overline{\boldsymbol{Q}^{m-1}\boldsymbol{Q}_{\boldsymbol{C}_m}\prod_{n=m+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{D}_n}}(i).$$

This then yields (32).

To compute the conditional second moments, we compute the square of the optimal wealth process:

$$(W_{n+1}^*)^2 = \Big(W_n^*\boldsymbol{R}_{n+1}^T(\theta_n)\boldsymbol{u}_n^*(\theta_n)+C_n(\theta_n)\Big)^2.$$

Again, we take conditional expectation with respect to the condition $(W_0=x,\theta_0=i,\theta_1,...,\theta_n)$. By (12) and a similar independence argument as in calculating the conditional expectation we obtain

$$\begin{aligned}
& \mathbb{E}\Big((W_{n+1}^*)^2|W_0=x,\theta_0=i,\theta_1,...,\theta_n\Big) \\
= \; & G_n(\theta_n)\mathbb{E}\Big((W_n^*)^2|W_0=x,\theta_0=i,\theta_1,...,\theta_{n-1}\Big) \\
& +\hat{D}_n(\theta_n)\mathbb{E}\Big(W_n^*|W_0=x,\theta_0=i,\theta_1,...,\theta_{n-1}\Big)+\tilde{C}_n(\theta_n).
\end{aligned} \tag{35}$$

Using (35) and backward recursion, we find that

$$
\begin{aligned}
& \mathbb{E}\Big((W_N^*)^2|W_0=x,\theta_0=i,\theta_1,...,\theta_{N-1}\Big) \\
= \quad & x^2\prod_{n=0}^{N-1}G_n(\theta_n)+x\Bigg(\hat{D}_0(\theta_0)\prod_{n=1}^{N-1}G_n(\theta_n) \\
& +D_0(\theta_0)\sum_{\ell=1}^{N-1}\Bigg(\prod_{m=1}^{\ell-1}D_m(\theta_m)\hat{D}_\ell(\theta_\ell)\prod_{n=\ell+1}^{N-1}G_n(\theta_n)\Bigg)\Bigg) \\
& +\tilde{C}_0(\theta_0)\prod_{n=1}^{N-1}G_n(\theta_n)+\sum_{m=1}^{N-1}\Bigg(\tilde{C}_m(\theta_m)\prod_{n=m+1}^{N-1}G_n(\theta_n)\Bigg) \\
& +C_0(\theta_0)\sum_{\ell=1}^{N-1}\Bigg(\prod_{m=1}^{\ell-1}D_m(\theta_m)\hat{D}_\ell(\theta_\ell)\prod_{n=\ell+1}^{N-1}G_n(\theta_n)\Bigg) \\
& +\sum_{\ell=1}^{N-2}\Bigg(C_\ell(\theta_\ell)\sum_{m=\ell+1}^{N-1}\hat{D}_m(\theta_m)\prod_{t=\ell+1}^{m-1}D_t(\theta_t)\prod_{n=m+1}^{N-1}G_n(\theta_n)\Bigg).
\end{aligned}
$$

In the above expression, we may arrange and combine terms in different ways. However, we have chosen this form deliberately, so that when we calculate this conditional expectation later, we can directly apply Lemma 3 in Wu and Li (2012). Next, by iterated conditional expectation, we have

$$
\begin{aligned}
\mathbb{E}_{0,x,i}\Big(\big(W_N^*\big)^2\Big) \quad = \quad & x^2G_0(i)\mathbb{E}_{0,x,i}\Bigg(\prod_{n=1}^{N-1}G_n(\theta_n)\Bigg)+x\hat{D}_0(i)\mathbb{E}_{0,x,i}\Bigg(\prod_{n=1}^{N-1}G_n(\theta_n)\Bigg) \\
& +xD_0(i)\sum_{\ell=1}^{N-1}\mathbb{E}_{0,x,i}\Bigg(\prod_{m=1}^{\ell-1}D_m(\theta_m)\hat{D}_\ell(\theta_\ell)\prod_{n=\ell+1}^{N-1}G_n(\theta_n)\Bigg) \\
& +\tilde{C}_0(i)\mathbb{E}_{0,x,i}\Bigg(\prod_{n=1}^{N-1}G_n(\theta_n)\Bigg)+\sum_{m=1}^{N-1}\mathbb{E}_{0,x,i}\Bigg(\tilde{C}_m(\theta_m)\prod_{n=m+1}^{N-1}G_n(\theta_n)\Bigg) \\
& +C_0(i)\sum_{\ell=1}^{N-1}\mathbb{E}_{0,x,i}\Bigg(\prod_{m=1}^{\ell-1}D_m(\theta_m)\hat{D}_\ell(\theta_\ell)\prod_{n=\ell+1}^{N-1}G_n(\theta_n)\Bigg) \\
& +\sum_{\ell=1}^{N-2}\sum_{m=\ell+1}^{N-1}\mathbb{E}_{0,x,i}\Bigg(C_\ell(\theta_\ell)\prod_{t=\ell+1}^{m-1}D_t(\theta_t)\hat{D}_m(\theta_m)\prod_{n=m+1}^{N-1}G_n(\theta_n)\Bigg). \quad (36)
\end{aligned}
$$

Similarly, as in calculating $\mathbb{E}_{0,x,n}(W_N^*)$, we apply Lemma 3 in Wu and Li (2012), which then yields

$$
\begin{aligned}
& \mathbb{E}_{0,x,i}\Bigg(\prod_{n=1}^{N-1}G_n(\theta_n)\Bigg)=\overline{\prod_{n=1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}(i)}, \quad \text{and} \\
& \mathbb{E}_{0,x,i}\Bigg(\prod_{m=1}^{\ell-1}D_m(\theta_m)\hat{D}_\ell(\theta_\ell)\prod_{n=\ell+1}^{N-1}G_n(\theta_n)\Bigg)=\overline{\prod_{m=1}^{\ell-1}\boldsymbol{Q}_{\boldsymbol{D}_m}\boldsymbol{Q}_{\hat{\boldsymbol{D}}_\ell}\prod_{n=\ell+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_m}(i)},
\end{aligned}
$$

and for $m = 1, ..., N-1$,

$$\mathbb{E}_{0,x,i}\left(\tilde{C}_m(\theta_m)\prod_{n=m+1}^{N-1} G_n(\theta_n)\right) = \overline{\boldsymbol{Q}^{m-1}\boldsymbol{Q}_{\tilde{\boldsymbol{C}}_m}\prod_{n=m+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i),$$

and for $m = \ell+1, ..., N-1, \ell = 1, ..., N-2$,

$$\mathbb{E}_{0,x,i}\left(C_\ell(\theta_\ell)\prod_{t=\ell+1}^{m-1} D_t\hat{D}_m(\theta_m)\prod_{n=m+1}^{N-1} G_n(\theta_n)\right) = \overline{\boldsymbol{Q}^{\ell-1}\boldsymbol{Q}_{\boldsymbol{C}_\ell}\prod_{t=\ell+1}^{m-1}\boldsymbol{Q}_{\boldsymbol{D}_t}\boldsymbol{Q}_{\hat{\boldsymbol{D}}_m}\prod_{n=m+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i).$$

Finally, by substituting the above results into (36), we obtain (33). This completes the proof. □

A direct consequence of this proposition leads to

$$\begin{aligned} Var_{0,x,i}(W_N^*) &= \mathbb{E}_{0,x,i}\big((W_N^*)^2\big) - \Big(\mathbb{E}_{0,x,i}(W_N^*)\Big)^2 \\ &= \big(\gamma(i) - \alpha^2(i)\big)x^2 + \big(\delta(i) - 2\alpha(i)\beta(i)\big)x \\ &\quad + \big(\eta(i) - \beta(i)^2\big). \end{aligned}$$

4 Numerical Illustrations

We collect (weekly) stock prices of ANZ, BHP, and Telstra, which traded on Australian Securities Exchange [1], during two periods of time [2]. We calculate their expected returns and the corresponding covariance matrices. Thus, in all examples that follow, we assume that there are two market states ($S = 1, 2$) and three risky assets ($d = 3$). We further assume that the investment horizon consists of five periods ($N = 5$), and for $n = 0, ..., N-1$, the expected returns $\mathbf{m}_n$ and covariance matrices $\mathbf{\Sigma}_n$ are equal. Thus, we drop the dependence on time, and for each market state we simply write

State 1:

$$\mathbf{m}(1) = \begin{pmatrix} -0.000566 \\ 0.000180 \\ -0.002364 \end{pmatrix}, \quad \mathbf{\Sigma}(1) = \begin{pmatrix} 0.002203 & 0.000848 & 0.000330 \\ 0.000848 & 0.002971 & 0.000248 \\ 0.000330 & 0.000248 & 0.000884 \end{pmatrix},$$

and

State 2:

$$\mathbf{m}(2) = \begin{pmatrix} 0.002425 \\ -0.000633 \\ 0.003943 \end{pmatrix}, \quad \mathbf{\Sigma}(2) = \begin{pmatrix} 0.000537 & 0.000261 & 0.000195 \\ 0.000261 & 0.000730 & 0.000105 \\ 0.000195 & 0.000105 & 0.000311 \end{pmatrix}.$$

[1] Data obtained from Yahoo Finance https://au.finance.yahoo.com/

[2] Two periods are 01 01 2008 - 23 05 2011 and 02 01 2012 - 25 05 2015.

Moreover, without loss of generality, we will assume that the initial wealth is 1 dollar, i.e., $W_0 = 1$.

We use three examples to illustrate our model. In the first example, we investigate the effect of different risk aversion parameters. In the second example, we investigate the effect of cash injections. In the last example, we investigate the difference between investment strategy with and without cash injections and offtakes under multiple market states.

Example 4.1. *We assume that there is only one market state, which we take to be State 2. There are no cash injections or offtakes.*

Now, we apply **Algorithm 3.1**. We choose four sequences of $\delta_n(2)$ such that we have: a constant $\kappa_n(2)$ through time, an increasing sequence of $\kappa_n(2)$ through time, a decreasing sequence of $\kappa_n(2)$ through time, and a random sequence of $\kappa_n(2)$ through time. This is summarized in Table 1.

	1st period	2nd period	3rd period	4th period	last period
constant $\kappa_n(2)$ through time					
$\kappa_n(2)$	1	1	1	1	1
$\hat{\kappa}_n(2)$	0.789058	0.630508	0.472666	0.315436	0.158743
increasing $\kappa_n(2)$ through time					
$\kappa_n(2)$	1	1.1	1.6	2	2.2
$\hat{\kappa}_n(2)$	0.781398	0.623197	0.467012	0.312385	0.158743
decreasing $\kappa_n(2)$ through time					
$\kappa_n(2)$	2.2	2	1.6	1.1	1
$\hat{\kappa}_n(2)$	0.784313	0.628606	0.472401	0.315436	0.158743
random $\kappa_n(2)$ through time					
$\kappa_n(2)$	1.6	1	2	2.2	1.1
$\hat{\kappa}_n(2)$	0.782971	0.624450	0.469305	0.315180	0.158743

Table 1: Risk Aversions and its Lower Bounds

It seems that the influence on the actual value of the lower bound $\hat{\kappa}_n(2)$ is quite small, and the general trend remains the same for all four cases. Thus, from now on we will consider only a constant $\kappa_n(2)$ through time, and will simply write $\kappa(2)$ instead. One may wonder whether we should take this investment. In order to make this decision, we need to specify the distribution of the return $\boldsymbol{r}(2)$. For example, if we assume $\boldsymbol{r}(2) \sim N(\mathbf{m}(2), \mathbf{\Sigma}(2))$ i.e., normally distributed with mean $\mathbf{m}(2)$ and covariance matrix $\mathbf{\Sigma}(2)$, **Algorithm 3.1** then implies that we will take this investment under each of these four cases. From now on, we will always assume that we are given a distribution of return such that we will take the given investment, and focus on the analysis of our model.

Now, fix two risk aversion parameters, $\kappa(2) = 1$ and $\kappa(2) = 3$. The former represents a more risky choice than the latter. For each case, we compute the optimal strategies (see Table 2), and the optimal conditional expectations and conditional variances of the investor's terminal wealth for different time length of investment (see Table 3).

$\kappa(2) = 1$					
	1st period	2nd period	3rd period	4th period	last period
ANZ	0.124591	0.127184	0.128699	0.129819	0.130761
BHP	-0.502347	-0.242418	-0.090490	0.021798	0.116286
Telstra	1.377756	1.115234	0.961791	0.848383	0.752952
$\kappa(2) = 3$					
	1st period	2nd period	3rd period	4th period	last period
ANZ	0.130190	0.130494	0.130788	0.131073	0.131353
BHP	0.059000	0.089497	0.118914	0.147541	0.175632
Telstra	0.810810	0.780009	0.750298	0.721386	0.693015

Table 2: Optimal Strategies ($\kappa(2) = 1$ and $\kappa(2) = 3$)

$\kappa(2) = 1$					
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
$\mathbb{E}_{0,1,2}(W_N^*)$	1.006053	1.010941	1.015149	1.018850	1.022123
$Var_{0,1,2}(W_N^*)$	0.000672	0.001105	0.001448	0.001749	0.002031
$\kappa(2) = 3$					
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
$\mathbb{E}_{0,1,2}(W_N^*)$	1.003475	1.006822	1.010044	1.013144	1.016123
$Var_{0,1,2}(W_N^*)$	0.000271	0.000540	0.000807	0.001073	0.001341

Table 3: Optimal Conditional Expectations and Conditional Variances of Investor's Terminal Wealth

In addition, we plot the optimal strategies for each case (see Figure 2 - Figure 3). As expected, in the less risky case (i.e., $\kappa(2) = 3$), investor's optimal strategy is more conservative along the way. This is reflected in the fact that he keeps a large and steady proportion in the less risky asset (i.e., Telstra). We can also see this by plotting the single period optimal expectation of the portfolio value and its variance for each time (see Figure 4 - Figure 5). The more risky case (i.e., $\kappa(2) = 1$) dominates the less risky case (i.e., $\kappa(2) = 3$) through time in the sense that it always has a higher expected return and a higher standard deviation.

Example 4.2. *We assume that there is only one market state which we take to be State 2. Assume that there is a constant cash injection of \$0.1 at each periods. The risk aversion parameter is assumed to be $\kappa(2) = 3$.*

As we have seen in Theorem 3.2, the presence of cash (in the form of our model) does not affect the optimal strategies, however, it affects the optimal conditional expectation and the conditional variance of investor's terminal wealth. We calculate investor's optimal conditional expectation and conditional variance of the terminal wealth, which yields

$$E_{0,1,2}(W_5^*) = 1.519203, \quad Var_{0,1,2}(W_5^*) = 0.001943.$$

Let us compare this result with the case of no cash injections (the last column of Table 3 when $\kappa(2) = 3$). After subtracting the extra cash injections and by ignoring the time value of money, we see that the optimal conditional expectation and variance of investor's wealth are higher

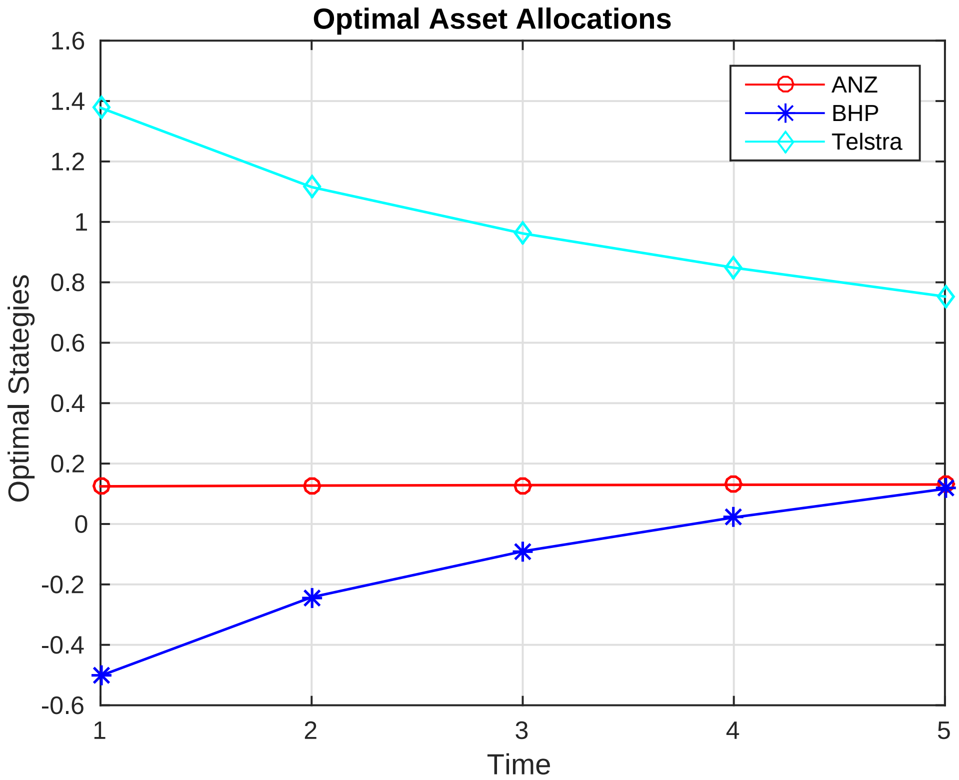

Figure 2: Optimal Strategies ($\kappa(2) = 1$)

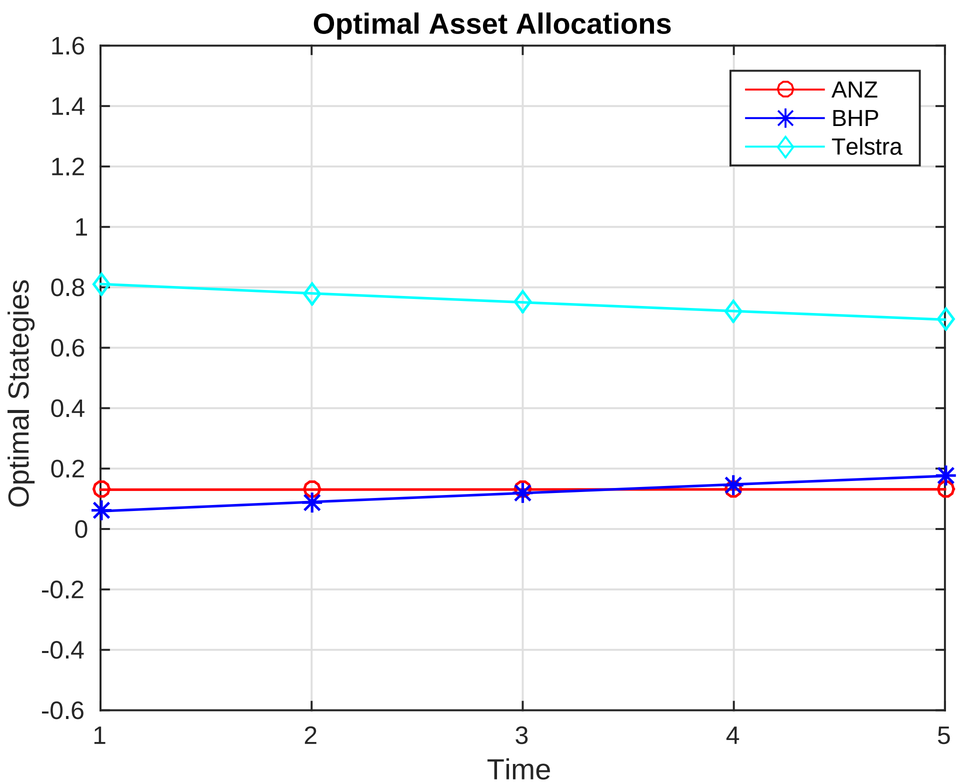

Figure 3: Optimal Strategies ($\kappa(2) = 3$)

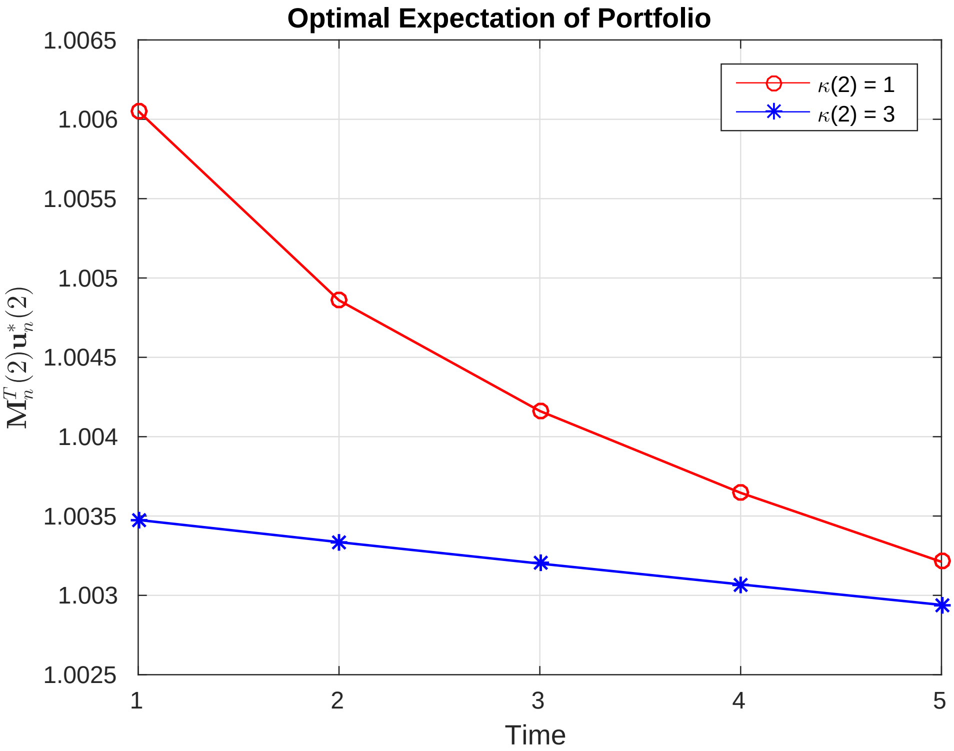

Figure 4: Single Period Optimal Expectations of Portfolio ($\kappa(2) = 1$, and $\kappa(2) = 3$)

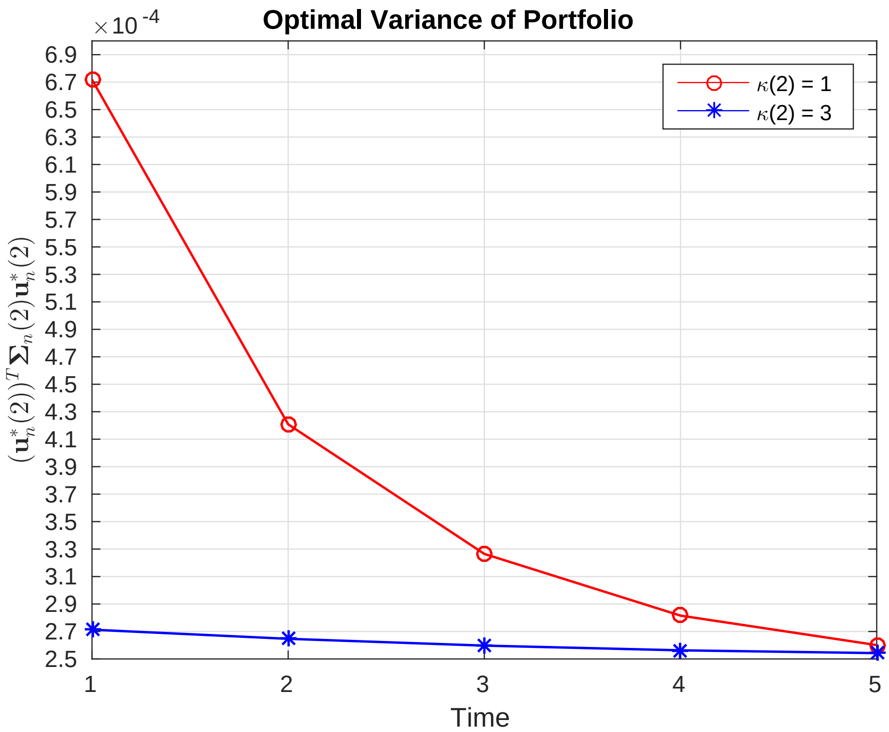

Figure 5: Single Period Optimal Variances of Portfolio ($\kappa(2) = 1$, and $\kappa(2) = 3$)

if there are cash injections. This makes sense in this example. As he injects extra amount into his portfolio he will invest more in these risky assets. On one hand, this increases the expectation of his wealth and on the other hand, he also exposes himself to uncertain environment.

It is worth noting that in this example we only consider cash injections but no takeoffs. It becomes an interesting question if the investor has a plan at the beginning of his investment horizon to withdraw certain (deterministic) amount of money at some future time. The question is how much he is able to withdraw. It may happen that the investor wishes to withdraw large amount, but it turns out that he does not have enough money in his portfolio (at the time he wishes to withdraw). In Table 3, we have calculated the optimal conditional expectation and conditional variance of investor's terminal wealth for different time length of investment. Thus, we can then calculate the corresponding MSD values of the wealth position. This is summarized in Table 4. Similarly to the single period MSD selection criterion (see Section 3.1), for every risk aversion parameter $\kappa(2)$, we can attach a probability p such that

$$p = \mathbb{P}\Big(W_N^* \geq E_{0,1,2}(W_N^*) - \kappa(2) * \sqrt{Var_{0,1,2}(W_N^*)}\Big).$$

Thus, for any given distribution of asset returns, one can calculate such p. This provides some confidence level to the investor about the amount he would be able to withdraw without going bankrupt.

$\kappa(2) = 3$					
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
$\mathbb{E}_{0,1,2}(W_N^*) - \kappa(2) * \sqrt{Var_{0,1,2}(W_N^*)}$	0.954089	0.937108	0.924821	0.914874	0.906264

Table 4: Optimal Conditional MSD Values of Investor's Terminal Wealth

State 1					
	1st period	2nd period	3rd period	4th period	last period
ANZ	0.184722	0.180831	0.176960	0.173108	0.169264
BHP	0.158328	0.153187	0.148075	0.142987	0.137910
Telstra	0.656950	0.665982	0.674965	0.683906	0.692827
State 2					
	1st period	2nd period	3rd period	4th period	last period
ANZ	0.130198	0.130500	0.130791	0.131075	0.131353
BHP	0.059815	0.090056	0.119259	0.147704	0.175632
Telstra	0.809986	0.779444	0.749950	0.721221	0.693015

Table 5: Optimal Strategies (State 1 and State 2)

Example 4.3. *We assume that there are two market states. If the market is in State 1 ("bad state")*[3]*, we take out* 0.1 *dollar, and if the market is in State 2 ("good state"), we add* 0.1 *dollar. The risk aversion is assumed to be* 3 *for both states and all time. We also assume the*

[3] This has been classified as a "bad state" since majority of the assets in this state have less expected return and all assets have larger standard deviation than in State 2".

transition matrix to be given by

$$\boldsymbol{Q} = \begin{pmatrix} 0.1 & 0.9 \\ 0.15 & 0.85 \end{pmatrix}.$$

For each market state, we calculated the optimal strategies which we summarized in Table 5. Let us have a look how market transitions affect the choice of the optimal strategy. Given we are in State 1 at the beginning (i.e., $n = 0$), we follow the corresponding optimal strategy for State 1. When we move to the second period, if the market state switches to the State 2, we use the corresponding optimal strategy for State 2 (by treating the initial state as State 2 and dealing with a four period problem). We continue this process until we select the optimal strategy for each time period. To give a concrete example, let us assume that the market has the following transitions.

$$\text{State 1} \rightarrow \text{State 2} \rightarrow \text{State 2} \rightarrow \text{State 1} \rightarrow \text{State 2}.$$

The corresponding optimal strategy will be

$$\boldsymbol{u}_0^*(1) = \begin{pmatrix} 0.184722 \\ 0.158328 \\ 0.656950 \end{pmatrix} \quad \text{(1st period of State 1)}$$

$$\rightarrow \quad \boldsymbol{u}_1^*(2) = \begin{pmatrix} 0.130500 \\ 0.090056 \\ 0.779444 \end{pmatrix} \quad \text{(2nd period of State 2)}$$

$$\rightarrow \quad \boldsymbol{u}_2^*(2) = \begin{pmatrix} 0.130791 \\ 0.119259 \\ 0.749950 \end{pmatrix} \quad \text{(3rd period of State 2)}$$

$$\rightarrow \quad \boldsymbol{u}_3^*(1) = \begin{pmatrix} 0.173108 \\ 0.142987 \\ 0.683906 \end{pmatrix} \quad \text{(4th period of State 1)}$$

$$\rightarrow \quad \boldsymbol{u}_4^*(2) = \begin{pmatrix} 0.131353 \\ 0.175632 \\ 0.693015 \end{pmatrix} \quad \text{(last period of State 2).}$$

Next, we calculate the optimal conditional expectations and conditional variances of the investor's terminal wealth with cash injections and offtakes (as described in Example 4.3):

$$E_{0,1,1}(W_5^*) = 1.202311, \quad Var_{0,1,1}(W_5^*) = 0.020094.$$
$$E_{0,1,2}(W_5^*) = 1.399611, \quad Var_{0,1,2}(W_5^*) = 0.021796,$$

and without cash injections and offtakes:

$$E_{0,1,1}(W_5^*) = 1.008384, \quad Var_{0,1,1}(W_5^*) = 0.002028.$$
$$E_{0,1,2}(W_5^*) = 1.013296, \quad Var_{0,1,2}(W_5^*) = 0.001612.$$

By taking extra positions during the "good" market state and reducing positions during the "bad" state, we see that he obtains a higher expected wealth (as it can be checked, by ignoring the time value of money, this holds even after subtracting the expected cash injections). However, like in Example 4.2, this has created more variations (variance has increased significantly).

5 Conclusion

In this paper, we develop a portfolio selection scheme where a multiperiod selection criterion of MSD type is considered. We perform the analysis in a market of risky assets and obtain a closed form optimal strategy in which market transitions and intermediate cash injections and offtakes are allowed. This model forms a good base to further study multiperiod portfolio selection problem in which a multiperiod selection criterion is of a type from the TIPH risk measure class. It is also interesting to see the effect of short selling and transaction costs to our model. These questions are left for future research.

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