ON FINITE PRESENTATIONS OF INVERSE SEMIGROUPS WITH ZERO HAVING POLYNOMIAL GROWTH

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Abstract. We study growth of inverse semigroups defined by finite presentations. Let S be a finitely presented Rees quotient of a free inverse semigroup given by an irredundant presentation with n generators and m relators. We show that if S has polynomial growth, then $m \geq n^2 - 1$ and this estimate is sharp. For any positive integer n, we also find, up to isomorphism, syntactic descriptions of all presentations that achieve this sharp lower bound. As part of the process, we describe all irredundant presentations of finite Rees quotients of free inverse semigroups having rank n, with the smallest number, namely n^2, of relators.

1. Introduction

In the 1950s and 1960s, Shvarts [45] and independently Milnor [26] introduced the notion of the growth function of a finitely generated group and established connections between geometry of manifolds and growth of their fundamental groups. The extensive study of the growth of groups, semigroups and other algebraic systems in fact began after the publication of Milnor's article, bringing forth a number of striking and deep results regarding possible types of growth of algebras and connections between their asymptotic behavior and abstract properties. New waves of interest were stimulated by Gromov [15] who proved that every finitely generated group having polynomial growth is virtually nilpotent, and by Grigorchuk [13], who exhibited the first examples of groups having intermediate growth. We refer the reader to the monographs by Ufnarovsky [48], Krause and Lenagan [19], de la Harpe [8], Mann [25], Sapir [34] and the survey by Grigorchuk [14] for a bibliography on results and methods of this intensively developing area of modern algebra.

One of the important notions in combinatorial group theory related to asymptotic behaviour is the *deficiency* of a finitely presented group G, which is the maximal difference between the number of generators and relations over all possible presentations of G. It is well-known that every group or semigroup of deficiency greater than zero is infinite. B.H. Neumann, in a pioneering article [28], explored connections between the deficiency and rank of finite semigroups. In particular, he constructed examples of finite semigroups of an arbitrary rank n given by a presentation with n generators and n relations. The first author [39] showed that every monoid given by a presentation with $n+k$ generators and n relations contains a free submonoid, freely generated by k of the generators, which can be found effectively. B. Baumslag and S. Pride [3] showed that if the deficiency of a finitely presented group is at least two then the group has a subgroup of finite index that admits an epimorphism onto a noncyclic free group, so in particular has exponential growth. That

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growth is exponential in these cases also follows from a result of Romanovskii [33]. Stöhr extended the result of B. Baumslag and Pride to groups with deficiency one for which one of the relators is a proper power. By contrast, it follows from work of Bieri [5] (communication with D. Osin) that groups of deficiency one with polynomial growth are either free abelian of rank one or two or isomorphic to the Baumslag-Solitar group $BS(1,-1)$. By further contrast, Wilson [50] proves that solvable groups of deficiency one are isomorphic to Baumslag-Solitar groups of the form $BS(1,n)$. The situation for semigroups of deficiency one has been fully analysed by the first author [38], [40], where he gives an algorithmic description of semigroups of deficiency one that do not have free nonmonogenic subsemigroups. Such semigroups satisfy nontrivial identities and a bounded height criterion used by Wolf [51] and Bass [4] (a weaker form of a height criterion due to Shirshov [37], see [41]), and therefore have polynomial growth. This leads to a classification of cancellative semigroups of deficiency one having polynomial growth [42].

In 1996, the authors [43] initiated the study of asymptotic behaviour of finitely presented Rees quotients of free inverse semigroups, which form a class of semigroups referred to as $\mathfrak{M}_{FI}$ (and formally defined below in terms of presentations as inverse semigroups with zero). In particular, it was shown that every semigroup S from $\mathfrak{M}_{FI}$ has polynomial or exponential growth, and there exists an algorithm to determine the type of growth. This gives an exact analogue of the well-known Ufnarovsky theorem [47], [48] for finitely presented monomial algebras (also following from the Gilman article [11]). However, in contrast with finitely presented monomial algebras, where languages of nonzero words may be described using finite state automata (see [47], [48]), the language of nonzero geodesic words of a semigroup S in the class $\mathfrak{M}_{FI}$, with respect to its natural presentation, is rational, that is, accepted by a finite state automaton, if and only if S is finite (see Proposition 2.4 below). Furthermore, Lau [20] and Brazil (unpublished) prove that the (Hilbert) growth series of a nonmonegenic free inverse semigroup with respect to its natural generating set is irrational. Lau [20], [21], [22] also showed that every semigroup from $\mathfrak{M}_{FI}$ having polynomial growth has a rational growth series and obtained results regarding the Gelfand-Kirillov dimensions. It is also shown in [43] that in the case of polynomial growth, S satisfies nontrivial semigroup identities, which are a modification of Adjan's celebrated identity [1] holding in the bicyclic monoid $\langle a, b \mid ab = 1\rangle$. The results of [43] were developed further in a series of articles [9], [44], [10] by the authors, giving various geometric and algebraic criteria for polynomial growth and applying them to investigate the growth of semigroups from $\mathfrak{M}_{FI}$ given by a small number of relators. In particular, these methods were able to produce in [10] a sequence of two-generator, three-relator semigroups whose Gelfand-Kirillov dimensions form an infinite set, namely $\{4, 5, 6, \ldots\}$, and deduced that inverse semigroups defined by one relation have exponential growth, under the condition that the word trees of both sides of the relation contain more than one edge. This constraint on word trees is necessary however, because the first author exhibits in [42] the first known example of a one relation nonmonogenic inverse semigroup with polynomial growth, and uses this example to build a tower of inverse semigroups of deficiency one and arbitrarily large rank, all of which have exponential growth yet satisfy quasi-solvable identities (in the sense of Piochi [30]). In [10, Theorem 9.1], the authors find, in terms of a fixed number of generators, a sharp lower bound in the number of relators that is necessary for polynomial growth for Rees quotients of free inverse semigroups, under the condition that none of the generators are nilpotent.

Let n be a positive integer. The present article has two goals. The first is to establish the smallest possible number of relators over all irredundant presentations of semigroups from $\mathfrak{M}_{FI}$ having rank n and polynomial growth. The second goal is to describe all presentations that achieve this lower bound. The next two theorems provide the solution to both problems. The first aim is realised by the following result (that appears below as Theorem 5.13):

Theorem 1.1. *If $S = \mathrm{Inv}\langle A \mid c_i = 0 \text{ for } i = 1, \ldots, L\rangle$ is an irredundant presentation defining an inverse semigroup S with zero having polynomial growth, where A is an alphabet of size $n \geq 2$, then $L \geq n^2 - 1$.*

The second aim is achieved by the following description (that appears below, in separate cases, as Theorems 7.6, 7.8 and 7.11):

Theorem 1.2. *Let $S = \mathrm{Inv}\langle A \mid c_i = 0 \text{ for } i = 1, \ldots, L\rangle$ be an irredundant presentation defining an inverse semigroup S, where $A = \{a_1, \ldots, a_n\}$ is an alphabet of size $n \geq 2$ and $L = n^2 - 1$. Then S has polynomial growth if and only if the generators may be reordered and the relators rewritten, up to $\mathcal{J}$-equivalence, such that either condition (1) below holds, where all but one of the generators is nilpotent, or condition (2) below holds, with two alternatives according to whether all generators, or all but two of the generators, are nilpotent:*

(1) (a) *the inverse subsemigroup $\mathrm{Inv}(a_1)$ is infinite monogenic (with a presentation that uses no relators)*
    (b) *the inverse subsemigroup $S_{\text{fin}} = \mathrm{Inv}(a_2, \ldots, a_n)$ is finite given by a presentation using $(n-1)^2$ relators;*
    (c) *$\{a_1\}S_{\text{fin}} = \{0\}$, using $2n-2$ relators $a_1 a_j = a_1 a_j^{-1} = 0$ for $2 \leq j \leq n$.*

(2) (a) *the inverse subsemigroup $\mathrm{Inv}(a_1, a_2)$ has one of the following presentations, using three relators:*
        (i) *$\mathrm{Inv}\langle a_1, a_2 \mid a_1^2 = a_2^2 = a_1 a_2^{-1} = 0\rangle$;*
        (ii) *$\mathrm{Inv}\langle a_1, a_2 \mid a_1 a_2 = a_1 a_2^{-1} = a_2^{\gamma} a_1 a_1^{-1} a_2^{\delta} = 0\rangle$ for some $\gamma, \delta \geq 0$ such that $\gamma + \delta > 0$;*
    (b) *the inverse subsemigroup $S_{\text{fin}} = \mathrm{Inv}(a_3, \ldots, a_n)$ is finite given by a presentation using $(n-2)^2$ relators (interpreted as $S_{\text{fin}} = \{0\}$ if $n = 2$);*
    (c) *$\{a_1, a_2\}S_{\text{fin}} = \{0\}$, using $4n-8$ relators $a_i a_j = a_i a_j^{-1} = 0$ for $i = 1, 2$ and $3 \leq j \leq n$.*

Though cases (1)(b) and (2)(b) do not explicitly list relators, nevertheless, a general description is given below, in Section 6, of finite Rees quotients of free inverse semigroups using n generators and n^2 relators, and then the classification given by the previous theorem becomes complete.

It should also be noted, in part (ii) of case (2)(a) of this theorem, that the following simplifications take place:

If $\gamma = 1$ and $\delta = 0$ then

$$\mathrm{Inv}(a_1, a_2) \;\cong\; \langle a_1, a_2 \mid a_1 a_2 = a_1 a_2^{-1} = a_2 a_1 = 0\rangle\ .$$

If $\gamma = 0$ *and* $\delta = 1$ *then*

$$\mathrm{Inv}(a_1, a_2) \;\cong\; \langle a_1, a_2 \mid a_1 a_2 = a_1 a_2^{-1} = a_1^{-1} a_2 = 0 \rangle \ .$$

Both of these special cases yield the same semigroups up to isomorphism, since

$$\langle a_1, a_2 \mid a_1 a_2 = a_1 a_2^{-1} = a_2 a_1 = 0 \rangle \;\cong\; \langle a_1, a_2 \mid a_1 a_2 = a_1 a_2^{-1} = a_1^{-1} a_2 = 0 \rangle \ .$$

The article is as self-contained as possible. Section 2 provides all of the necessary background, definitions, and explicit statements or summaries of results and techniques developed or refined in [10], [43] and [44]. This section also introduces new techniques and terminology, in particular the notion of a two-standard presentation, elaborated upon in Section 3, emphasising notions of even and odd pairs of relators, domination from the left and right for generators, and graphical interpretations. Examples are included in Section 4, with graphical descriptions in special cases, especially to orient the reader and to illustrate and represent the three classes of examples in the main classification that appears in Section 7. Section 5 develops a sequence of lemmas, culminating in establishing the sharp lower bound ($n^2 - 1$ where n is the number of generators) for the number of relators that are necessary for polynomial growth of inverse semigroups from our class. The proofs provide detailed fine-grained numerical information that is later relied upon in the main classification in Section 7. Section 6 establishes the sharp lower bound (n^2 where n is the number of generators) for the number of relators that are necessary for finiteness of inverse semigroups from our class, and provides a complete description of inverse semigroups for which the sharp lower bound is achieved. As a stepping stone, the corresponding simpler result for semigroups is described. In Section 7, we give a complete description of presentations of inverse semigroups from our class having polynomial growth and using the sharp lower bound for the number of relators. The description involves three main classes, and for each class we precede the main description by considering the special case involving two-standard presentations. In Section 8, the final section, we apply our results to deduce connections between the growth of arbitrary finitely presented inverse semigroups and the number of relations.

2. Preliminaries

We assume familiarity with the basic definitions and elementary results from the theory of semigroups, which can be found in any of [6], [16], [17] or [23]. Throughout let A be a finite alphabet containing at least two letters and put

$$B \;=\; A \cup A^{-1}$$

where the elements of A^{-1} are formal inverses of corresponding elements of A and vice-versa (so A and A^{-1} are disjoint and any a in A may also be denoted by $(a^{-1})^{-1}$). Let L be a positive integer and suppose that $c_1, \ldots, c_L \in B^+$. Consider the inverse semigroup S with zero given by the following finite presentation:

$$S \;=\; \mathrm{Inv}\langle\, A \mid c_i = 0 \;\text{ for }\; i = 1, \ldots, L \,\rangle \, .$$

In this paper we only consider presentations within the class of inverse semigroups. Because presentations of this form occur so often in this paper we abbreviate the notation slightly to write

$$S \ = \ \langle\, A \mid c_i = 0 \ \text{ for } \ i = 1, \ldots, L \,\rangle\,. \tag{1}$$

The words $c_1, \ldots, c_L$ are called (*zero*) *relators*. Observe that S may be regarded as (isomorphic to) the Rees quotient of the free inverse semigroup FI_A generated by A with respect to the ideal generated by the relators. The class of finitely presented inverse semigroups with zero defined by presentations (1) may now be formally referred to as $\mathfrak{M}_{FI}$.

The *content* of a word $w \in B^*$, denoted by content(w), is the set of letters from A that appear in w or w^{-1}. If $w_1, \ldots, w_n \in B^+$ then denote by $(w_1, \ldots, w_n)$ the subsemigroup of B^+ generated by $w_1, \ldots, w_n$, which we may regard as a subset of FI_A or of S in context. In contrast, denote by $\mathrm{Inv}(w_1, \ldots, w_n)$ the inverse subsemigroup of S generated by $w_1, \ldots, w_n$. We use the symbol $\overset{\circ}{=}$ to denote literal equality of words, that is, $w_1 \overset{\circ}{=} w_2$ means that words w_1 and w_2 coincide letter by letter. If $v, w \in B^*$ and $w \overset{\circ}{=} xvy$ for some $x, y \in B^*$ then we call v a *subword* (or *factor*) of w. Recall that w is *reduced* if w does not contain xx^{-1} as a subword for any letter $x \in B$, and that w is *cyclically reduced* if w and w^2 are both reduced (whence all powers of w are reduced).

Reference to Green's relation $\mathcal{J}$ throughout will be with respect to FI_A. Call a word u a *divisor* of a word v if the equation $v = sut$ holds in FI_A for some $s, t \in B^*$. Recall that elements of FI_A may be regarded as birooted word trees (introduced for the first time in [27] and referred to also as *Munn trees*), the terminology and theory of which are explained in [16] (see also [43, Section 2]). As in [43], denote the word tree of a word w over B by $T(w)$. Two words are $\mathcal{J}$-related if and only if their word trees are identical. If u and v are words, then $T(u)$ is a subtree of $T(v)$ if and only if u divides v. If X and Y are sets of words, regarded as subsets of FI_A, then we write $X =_{\mathcal{J}} Y$ if there is a bijection between X and Y that respects $\mathcal{J}$. Recall also that an element s of a semigroup S with zero is *nilpotent* if some power of s is zero.

Any given element w of FI_A may also be expressed as

$$w = u_1 u_1^{-1} u_2 u_2^{-1} \ldots u_r u_r^{-1} \overline{w}$$

for some nonnegative integer r and reduced words $u_1, \ldots, u_r, \overline{w}$, and we call $\overline{w}$ the *reduced part* of w. If r is as small as possible, so that no u_i can be an initial segment of u_j for $i \neq j$, then the previous expression for w is called the *Schein (left) canonical form of* w (see [35]), which is unique up to order of idempotents. It follows from the Schein canonical form that a Rees quotient of a free inverse semigroup is finite if and only if there are finitely many reduced words that are nonzero, and this occurs, in the case that the Rees quotient comes from the class $\mathfrak{M}_{FI}$, if and only if all monogenic subsemigroups of the Rees quotient generated by reduced words are nil. In particular, any infinite subsemigroup of a Rees quotient of a free inverse semigroup from the class $\mathfrak{M}_{FI}$ contains a free monogenic subsemigroup.

The following fact is used implicitly, where we may exchange some letters with their formal inverses as generators. Suppose that S is given by the presentation (1) and write $A = \{a_1, a_2, \ldots, a_n\}$ where $|A| = n$. Put $A' = \{a_1^{\varepsilon_1}, a_2^{\varepsilon_2}, \ldots, a_n^{\varepsilon_n}\}$ where $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$. Suppose further that $c_i' \ \mathcal{J} \ c_i$ for $i = 1, \ldots, L$. Then, interpreting formal inversion of generators in the usual way, it is clear that

$$B \;=\; A' \cup (A')^{-1}$$

and

$$S \;=\; \mathrm{Inv}\langle\, A' \mid c_i' = 0 \;\text{ for }\; i = 1, \ldots, L\,\rangle\,.$$

When writing about or using presentations of the form (1) in the text of this paper, we make the following underlying assumptions:

(i) The alphabet A is finite and $|A| \geq 2$.

(ii) The number L of relators is at least one.

(iii) No relator is $\mathcal{J}$-equivalent to a single letter from A. (In particular, this guarantees that S is not a free monogenic inverse semigroup with zero.)

(iv) No relator $\mathcal{J}$-divides any other relator in the presentation for S. (If this were not the case then we could delete a relator without changing the Rees quotient.)

(v) At least one relator is $\mathcal{J}$-equivalent to a reduced word.

These assumptions may be referred to collectively as the *irredundancy* of the presentation. Condition (v) is included, because if it failed then there would exist at least two letters $a, b \in A$ that generate a noncyclic free subsemigroup (see remarks following Theorem 2.1 of [44]), so that the growth of S would become exponential for a trivial reason, and the presentation would not be interesting from our point of view.

Condition (iv) is useful because of the following simple proposition that is used repeatedly and implicitly in what follows:

Proposition 2.1. *Let S_1 and S_2 be semigroups described by irredundant presentations*

$$S_1 \;=\; \langle\, A_1 \mid c_i = 0 \;\text{ for }\; i = 1, \ldots, L\,\rangle$$

and

$$S_2 \;=\; \langle\, A_2 \mid c_i' = 0 \;\text{ for }\; i = 1, \ldots, L'\,\rangle\,.$$

Then S_1 and S_2 are isomorphic if and only if $L = L'$ and there is a bijection $\beta : A_1 \cup A_1^{-1} \to A_2 \cup A_2^{-1}$, which induces a bijection between words over the respective extended alphabets, and a permutation π of $\{1, \ldots, L\}$ such that

$$c_i\beta \;\mathcal{J}\; c_{i\pi}' \qquad \textit{for } i = 1, \ldots, L.$$

Call a presentation of the form

$$S' \;=\; \langle\, A \mid c_i' = 0 \;\text{ for }\; i = 1, \ldots, L'\,\rangle \tag{2}$$

two-standard if it is irredundant and each c_i' has the form a^2, ab, ab^{-1} or $a^{-1}b$ for some $a, b \in A$, that is, all relators are reduced words of length two belonging to distinct $\mathcal{J}$-classes and one or both letters in any given relator belong to A.

A useful consequence of condition (iii) of irredundancy is that every relator contains a reduced subword of length two and, in particular, no relator can have the form aa^{-1} or $a^{-1}a$ for $a \in A$. Thus, if an inverse semigroup S is given by an irredundant presentation of the form (1), then one may form a two-standard presentation of the form (2) where $L' \leq L$, representing a homomorphic image of S, by replacing each relator by a reduced subword of length two, removing any duplicates up to $\mathcal{J}$-equivalence, and making sure each of these reduced subwords of length two contains at least one letter from A (by inverting a word, if necessary). The two-standard presentations that arise in this way are not necessarily unique and the corresponding homomorphic images of S may be non-isomorphic.

Consider a semigroup S given by (1), where $A = \{a_1, \ldots, a_n\}$ is fixed:

$$S \;=\; \langle\, a_1, \ldots, a_n \mid c_i = 0 \;\text{ for }\; i = 1, \ldots, L\,\rangle\,. \tag{3}$$

Inverse subsemigroups of S generated by subsets of A have presentations that are inherited from (3). We make this precise using the following notation. Let X be a nonempty subset of A of size m, and write

$$X \;=\; \{a_{i_1}\,,\; \ldots\;,\; a_{i_m}\}\;,$$

where $i_1, \ldots, i_m$ are distinct elements of $\{1, \ldots, n\}$ (typically written in increasing order). Put $\mathcal{C} = \{c_1, \ldots, c_L\}$ and

$$\mathcal{D} \;=\; \{c \in \mathcal{C} \mid \text{content}(c) \subseteq X\}\;.$$

If $\mathcal{D}$ is empty then $\text{Inv}(a_{i_1}, \ldots, a_{i_m})$ is free and we define

$$S_X \;=\; \text{Inv}(a_{i_1}, \ldots, a_{i_m}) \,\dot\cup\, \{0\}\;,$$

which is a free inverse semigroup of rank m with zero adjoined. Suppose that $\mathcal{D}$ is nonempty (the case that typically concerns us). Then $0 \in \text{Inv}(a_{i_1}, \ldots, a_{i_m})$, and we now simply put

$$S_X \;=\; \text{Inv}(a_{i_1}, \ldots, a_{i_m})\;.$$

We also write

$$S_{i_1,\ldots,i_m} \;=\; S_X\;.$$

Suppose that $\mathcal{D}$ is of size $\ell > 0$, and write

$$\mathcal{D} \;=\; \{d_1, \ldots, d_\ell\}\;.$$

Now consider the semigroup $\widehat{S}_X$ defined by the following irredundant presentation, as an inverse semigroup with zero:

$$\widehat{S}_X \;=\; \text{Inv}\langle\, a_{i_1}, \ldots, a_{i_m} \mid d_1 = \ldots = d_\ell = 0\,\rangle\;.$$

The nonzero multiplication of elements inside S_X may be identified with the same multiplication regarded as elements of FI_X (identified as a subset of FI_A), and a product of words becomes zero in S_X precisely when a relator from $\mathcal{D}$ divides it. Hence the natural identification of nonzero elements of S_X with elements of FI_X induces an isomorphism from S_X to $\widehat{S}_X$, regarding the latter as a Rees quotient of FI_X. Hence there is no confusion, throughout this paper, by making the identification

$$S_X \;=\; \widehat{S}_X \;=\; \text{Inv}\langle a_{i_1}, \ldots, a_{i_m} \mid d_1 = \ldots = d_\ell = 0\rangle\;. \tag{4}$$

We recall some terminology concerning growth of semigroups. Consider a semigroup T generated by a finite subset X. The *length* $\ell(t)$ of an element $t \in T$ (with respect to X) is the least number of factors in all representations of t as a product of elements of X, and

$$g_T(m) = \big|\,\{\, t \in T \mid \ell(t) \le m\,\}\,\big|$$

is called the *growth function* of T. Recall that T has *polynomial growth* if there exist natural numbers q and d such that $g_T(m) \le qm^d$ for all natural numbers m, and *exponential* *growth* if there exists a real number $\alpha > 1$ such that $g_T(m) \geq \alpha^m$ for all sufficiently large m.

We recall the well-known notion of *bounded height* (first introduced by Shirshov [37]). Let X be a subset of a semigroup T. Denote by (X) the subsemigroup of T generated by X. If $s \in (X)$ can be expressed as a product $s = h_1^{\alpha_1} \dots h_k^{\alpha_k}$ for some $h_1, \dots, h_k \in X$ and positive integers $\alpha_1, \dots, \alpha_k$, and k is as small as possible, then we say the *height of s with respect to X* is k. We say that a subset K of T has *height bounded by k* if there exists a finite subset X of K such that $K \subseteq (X)$ and the height of elements of K with respect to X is at most k. One of the main results of [44] is that a semigroup from the class $\mathfrak{M}_{FI}$ has polynomial growth if and only if it has bounded height.

We recall a graphical technique of central importance, explained in detail in the next paragraph, that is a modification of an idea due to Ufnarovsky [33] [34] (see also [29, Chapter 24]) in the setting of monomial algebras. This idea has wide applicability and arises in other settings (see, for example, De Bruijn [7] and [24, Chapter 1] where the terminology De Bruijn graph is introduced, and Rauzy [31], where subgraphs of De Bruijn graphs are introduced, related to the combinatorics of infinite words). A related construction is used by Gilman [11] for calculating degrees of growth and solving a word problem in a class of groups and monoids given by certain finite presentations.

Consider an irredundant presentation (1) for an inverse semigroup S. We recall the technical definition, modified for our particular context, of the *Ufnarovsky graph* $\Gamma = \Gamma_S$ *of* S (depending on the presentation), which is the key tool used in [43], modified again slightly in [44], and used extensively in [10]. Put $d+1 = \max\{\, \ell(c_i) \mid i = 1, \dots, k \,\}$ and

$$\overline{d} + 1 = \max\{\, \ell(c) \mid c \text{ is a reduced word } \mathcal{J}\text{−equivalent to some relator} \,\} .$$

Then $\overline{d}$ exists by condition (v) of irredundancy, and may be found by inspecting word trees of relators. By condition (iii) of irredundancy, no word is $\mathcal{J}$-equivalent to a single letter, so $d \geq \overline{d} \geq 1$. Vertices of $\Gamma = \Gamma_S$ are defined to be reduced words of length $\overline{d}$ that are nonzero in S. If v_1 and v_2 are vertices then a directed edge from v_1 to v_2 is defined in Γ if there exist letters $g, h \in A \cup A^{-1}$ such that $v_1 g$ is a reduced word that is nonzero in S and $v_1 g \mathrel{\dot{=}} h v_2$. We regard the letter g as a label for this edge. Paths in Γ may then be labelled by reduced words that are nonzero in S. Conversely if $w \mathrel{\dot{=}} vu \mathrel{\dot{=}} u'v'$ is any nonzero reduced word where v and v' have length $\overline{d}$ then u labels a path in Γ emanating from v and terminating at v'. By a *cycle* in Γ we mean a path that starts and finishes at the same vertex. By a *loop at a vertex v* we mean a cycle that begins at v using no other vertex more than once. Recall from Section 3 of [43] that (z, P) is an *adjacent pair* if z is a reduced word that labels a loop in Γ at a vertex v and P is a letter labelling an edge that emanates from v and terminates outside the loop. Combining Theorems 2.1, 3.3 and 4.3 of [44] and Lemma 3.2 of [43], we have the following criteria for polynomial growth:

Theorem 2.2. *Let S be given by an irredundant presentation* (1)*. Then the following conditions are equivalent:*

(a) *S has polynomial growth.*
(b) *S does not contain any noncyclic free subsemigroups.*

(c) *The set of reduced words that are nonzero in* S *has bounded height and all reduced words that are not cyclically reduced are nilpotent (with index of nilpotency* $\le d+1$*).*

(d) (i) Γ_S *has no vertex contained in different cycles; and*
(ii) *if* (z,P) *is an adjacent pair in* Γ_S *then* $z^{d+1}PP^{-1}z^{d+1}=0$ *in* S.

A sufficient condition for polynomial growth (which becomes necessary if every relator is $\mathcal{J}$*-related to a reduced word) is*

(e) (i) Γ_S *has no vertex contained in different cycles; and*
(ii) *if* (z,P) *is any adjacent pair then* (z^{-1},P) *is not adjacent.*

The substance of the following result was remarked upon at the end of Section 3 of [43]:

Proposition 2.3. *Let* S *be given by an irredundant presentation* (1)*. Then the following conditions are equivalent:*

(a) *The semigroup* S *is finite.*
(b) *Only finitely many reduced words are nonzero in* S*.*
(c) *The graph* Γ_S *has no cycles.*
(d) *Every monogenic subsemigroup of* S *generated by a reduced word is nil.*

The following observation was noted in the third paragraph of the Introduction and is proved here:

Proposition 2.4. *Let* S *be given by an irredundant presentation* (1)*. Then the language* $\mathcal{L}$ *of geodesic words that are nonzero in* S *is rational if and only if* S *is finite.*

Proof. If S is finite then $\mathcal{L}$ is finite so clearly rational. Suppose then that S is infinite, so that Γ_S has at least one loop. Let u be the label of this loop. Then u is a cyclically reduced (and primitive) word over the alphabet $B=A\cup A^{-1}$. Thus, u^n is nonzero in S, for any integer n, and u is a geodesic word in S and also in FI_A. Let $H=\mathrm{Inv}(u)$ be the inverse subsemigroup of S generated by u. Since u is not an idempotent in FI_A, then H is isomorphic to the free monogenic inverse semigroup $\mathcal{F}_1$ of FI_A generated by u (see [32]). Furthermore, since u is cyclically reduced, every word over B which is geodesic in H can be uniquely written in one of the following forms:

$$u^{-\alpha}u^{\alpha}u^{\theta}u^{\beta}u^{-\beta}\,,\quad u^{\beta}u^{-\beta}u^{-\theta}u^{-\alpha}u^{\alpha}\quad(\alpha,\beta,\theta\ge 0\,,\ \ \alpha+\beta+\theta\ne 0)\,, \tag{5}$$

exhausting all possible (canonical) geodesic forms for elements of $\mathcal{F}_1$. Clearly, all of these words are nonzero in S. Let ρ be the syntactic congruence on the language $\mathcal{L}$. We show that there are infinitely many distinct ρ-classes. Indeed, let $\mathcal{L}_1$ be the language of elements of H that take the geodesic forms in (5). Put $U_n \mathrel{\mathring{=}} u^{-n-1}u$ and $X_n \mathrel{\mathring{=}} u^{n+2}u^{-1}$, for each odd positive integer n. Clearly, all U_n, X_n, and U_nX_n are words in $\mathcal{L}_1$. Consider positive odd integers i and n with $i<n$. Then, using the fact that $n-i-1\ge 1$, and calculating in FI_A, noting that all words are nonzero in S, we have

$$\begin{aligned} U_nX_i &\mathrel{\mathring{=}} u^{-n-1}uu^{i+2}u^{-1} \mathrel{\mathring{=}} u^{-n-1}u^{i+2}uu^{-1} \mathrel{\mathring{=}} u^{-n+i+2}u^{-1}(u^{-i-2}u^{i+2})(uu^{-1}) \\ &= u^{-n+i+2}u^{-1}(uu^{-1})(u^{-i-2}u^{i+2}) = u^{-n+i+2}u^{-1}u^{-i-2}u^{i+2} \mathrel{\mathring{=}} u^{-n-1}u^{i+2}\,. \end{aligned}$$

This shows that the word U_nX_i is equal in S to a word of lesser length, and so cannot be geodesic. Thus, in contrast with $U_iX_i\in\mathcal{L}_1\subseteq\mathcal{L}$, we have $U_nX_i\notin\mathcal{L}$. This shows that the ρ-congruence classes of U_i and U_n are distinct. Since n can be made arbitrarily large, this shows that there are infinitely many syntactic congruence classes with respect to the language $\mathcal{L}$, completing the proof that $\mathcal{L}$ is not rational. □

Consider an irredundant presentation (1) for S with generating set A. Let X be a nonempty subset of A. We call X *left orthogonal* if $xy^{-1} = 0$ in S for all distinct $x, y \in X$. In particular, X is (trivially) left orthogonal if $|X| = 1$. Note also that if $X = \{x_1, \ldots, x_k\}$, where $|X| = k \geq 2$, then X is left orthogonal if $x_i x_j^{-1} = 0$ in S for $1 \leq i < j \leq k$. In particular, if a and b are distinct letters and $ab^{-1} = 0$ in S then $\{a, b\}$ is left orthogonal. The following two lemmas are used repeatedly below in the proofs of the main theorems in Section 7.

Lemma 2.5. *Suppose that S is given by an irredundant presentation* (1) *with generating set A which is the disjoint union $A = A_1 \cup A_2$ of nonempty subsets A_1 and A_2 such that A_1 is left orthogonal and $A_1(A_2 \cup A_2^{-1}) = \{0\}$ in S. Then the following hold:*

(a) *If the sets of nonzero reduced words in* $\mathrm{Inv}(A_1)$ *and* $\mathrm{Inv}(A_2)$ *respectively have bounded height then the set of reduced words in S has bounded height.*

(b) *If* $\mathrm{Inv}(A_1)$ *has polynomial growth and* $\mathrm{Inv}(A_2)$ *is finite then S has polynomial growth.*

Proof. Put $T_0 = (A_1)$, $T_1 = \mathrm{Inv}(A_1)$ and $T_2 = \mathrm{Inv}(A_2)$. From the earlier discussion linking presentations (3) and (4), we may identify T_1 and T_2 with the presentations S_{A_1} and S_{A_2} respectively. By hypothesis, we have

$$T_0 T_2 \ = \ \{0\} \ . \tag{6}$$

Consider reduced words $s \in T_1$ and $u, v \in T_2$ that are nonzero in S. It follows from (6) and left orthogonality of A_1 that (i) $sv \neq 0$ in S implies $s \in T_0^{-1}$, and (ii) $us \neq 0$ in S implies $s \in T_0$. In particular,

$$usv \ = \ 0 \quad \text{in } S. \tag{7}$$

Hence, every reduced word w that is nonzero in S has a factorisation

$$w \ = \ svt \tag{8}$$

for some $s \in T_0^{-1} \cup \{1\}$, $t \in T_0 \cup \{1\}$ and $v \in T_2 \cup \{1\}$, where not all of s, t, v are empty. Thus, if the sets of reduced words that are nonzero in T_1 and T_2 respectively have bounded height, it is immediate that the set of nonzero reduced words in S also has bounded height. This proves part (a).

We now prove part (b). Suppose that T_1 has polynomial growth and T_2 is finite. By part (c) of Theorem 2.2, the set of reduced words that are nonzero in T_1 has bounded height, and reduced words that are not cyclically reduced are nilpotent. We verify that S has the same properties. Trivially, by finiteness, the set of nonzero reduced words in T_2 has bounded height. By part (a), the set of nonzero reduced words in S has bounded height.

It remains to verify that all reduced words in S that are not cyclically reduced are nilpotent. Let w be a reduced word that is nonzero in S and is not cyclically reduced. Then $w = svt$ has a factorisation given by (8). If v is empty then $w = st$ is a reduced but not cyclically reduced word in T_1, so is nilpotent. Hence we may suppose that v is nonempty. Observe that

$$w^2 \ = \ sv(ts)vt \ .$$

Let z be the reduced part of ts, that is $z = \overline{ts}$ (the result of evaluating ts in the free group), so that vzv divides w^2. If z is nonempty, then $vzv = 0$ in S by (7), so that $w^2 = 0$ in S, whence w is nilpotent. Hence we may suppose that z is empty, so that $w = svs^{-1}$. But v is nilpotent, by part (d) of Proposition 2.3, since T_2 is finite. It follows that w is nilpotent also, since powers of v always divide corresponding powers of svs^{-1} in the free inverse semigroup. This completes the proof that all reduced words in S that not cyclically reduced are nilpotent. Hence S has polynomial growth, by part (c) of Theorem 2.2, completing the proof of part (b) of this lemma. □

Lemma 2.6. *Suppose that S is a semigroup with polynomial growth given by an irredundant presentation* (1) *with generating set A that is the disjoint union $A = A_1 \cup A_2$ of nonempty subsets A_1 and A_2. Suppose that there exists some letter $a \in A_1$ such that whenever c is a relator in the presentation* (1) *such that*

$$a \in \mathrm{content}(c) \qquad \textit{and} \qquad \mathrm{content}(c) \cap A_2 \neq \emptyset \ ,$$

then there exists some letter $b \in A_2$ such that

$$c \ \mathcal{J} \ ab \qquad \textit{or} \qquad c \ \mathcal{J} \ ab^{-1} \ .$$

Then the inverse subsemigroup $\mathrm{Inv}(A_2)$ *of S is finite.*

Proof. Put $T = \mathrm{Inv}(A_2)$. Suppose that T is infinite. By part (d) of Proposition 2.3, there exists a reduced word $v \in T$ that generates an infinite monogenic subsemigroup of T. Put $w = a^{-1}va$. Then w is a reduced word that is not cyclically reduced, so w is nilpotent by part (c) of Theorem 2.2. Hence $w^\gamma = 0$ in S for some positive integer γ. Thus there is some relator c that divides w^γ. If $a \notin \mathrm{content}(c)$ then c must divide v^γ, so that $v^\gamma = 0$ in T, contradicting that v generates an infinite monogenic subsemigroup. Hence $a \in \mathrm{content}(c)$.

We first show that $\mathrm{content}(c) \neq \{a\}$. Suppose to the contrary that $\mathrm{content}(c) = \{a\}$, so that c is $\mathcal{J}$-related to some power of a. By inspection, the only powers of a that divide w^γ are $a^{\pm 1}$. Hence $c \ \mathcal{J} \ a$. But this contradicts condition (iii) of irredundancy of (1). This shows that $\mathrm{content}(c) \neq \{a\}$, so that $\mathrm{content}(c) \cap A_2 \neq \emptyset$. Hence, by hypothesis, there is some letter $b \in A_2$ such that $c \ \mathcal{J} \ ab$ or $c \ \mathcal{J} \ ab^{-1}$. But, by inspection, neither ab nor ab^{-1} can divide w^γ, which yields a contradiction. This proves that T is finite. □

The following results from [10], Theorems 5.2, 6.1 and 9.1 respectively, are also foundational for the main arguments in this article, so are reproduced here for ease of reference, with very slight modifications, suitable for our context:

Theorem 2.7. *If S is given by an irredundant presentation* (1) *with L relators and S contains no noncyclic free subsemigroups then $L \geq 3$.*

Theorem 2.8. *If S is given by an irredundant presentation* (1) *with $L = 3$ relators then S has polynomial growth if and only if*

$$S \ \cong \ \langle\, a, b \mid a^2 = b^2 = ab = 0 \,\rangle \qquad \textit{or} \qquad S \ \cong \ \langle\, a, b \mid ab = a^{-1}b = C = 0 \,\rangle \ ,$$

where C divides $a^\gamma b^{-1} b a^\gamma$ for some positive integer γ.

Theorem 2.9. *If S has polynomial growth and is given by an irredundant presentation* (1) *with L relators and none of the generators (elements of A) are nilpotent then $L \geq \frac{3}{2}n(n-1)$.*

3. Notation for Two-standard Presentations

We introduce some notation pertaining to two-standard presentations that will be useful in developing lemmas and theorems in the next section. Suppose that S is given by a two-standard presentation

$$S \;=\; \langle\, A \mid c_i = 0 \;\text{ for }\; i = 1, \ldots, L \,\rangle$$

where $A = \{a_1, \ldots, a_n\}$. Let $i, j \in \{1, \ldots, n\}$, with $i \neq j$. Put

$$\rho_{i,j} \;=\; \{c_k \mid k \in \{1, \ldots, L\} \;\text{ and }\; \text{content}(c_k) = \{a_i, a_j\}\}$$

and

$$R_{i,j} \;=\; |\rho_{i,j}|\;.$$

Recall that if $X, Y \subseteq S\backslash\{0\}$ then we write $X =_{\mathcal{J}} Y$ when there is a one-one correspondence between X and Y such that corresponding elements are $\mathcal{J}$-related. Note that if w is any word over B then $w \;\mathcal{J}\; w^{-1}$.

Because all relators are reduced words of length 2, it is clear that each of $c_1, \ldots, c_L$ with content $\{a_i, a_j\}$ is $\mathcal{J}$-equivalent to one of

$$a_i a_j, \quad a_j a_i, \quad a_i a_j^{-1} = (a_j a_i^{-1})^{-1} \quad \text{or} \quad a_j^{-1} a_i = (a_i^{-1} a_j)^{-1}\;.$$

Thus, condition (iv) of irredundancy guarantees that $\rho_{i,j} =_{\mathcal{J}} X$ for some subset X of

$$\{\, a_i a_j,\; a_j a_i,\; a_i a_j^{-1},\; a_j^{-1} a_i \,\}\;.$$

In particular, if $R_{i,j} = 2$ then there are exactly six possibilities for $\rho_{i,j}$, determined up to $\mathcal{J}$-equivalence. If

$$\rho_{i,j} \;=_{\mathcal{J}}\; \{a_i a_j, a_j a_i\} \quad \text{or} \quad \rho_{i,j} \;=_{\mathcal{J}}\; \{a_i a_j^{-1}, a_j^{-1} a_i\}\;,$$

then we say that $\rho_{i,j}$ is an *even pair*. When $\rho_{i,j}$ is an even pair then there is a pair of cycles in the subgraph of Γ_S involving the vertices $a_i^{\pm 1}, a_j^{\pm 1}$, the two possibilities depicted in Figures 1 and 2.

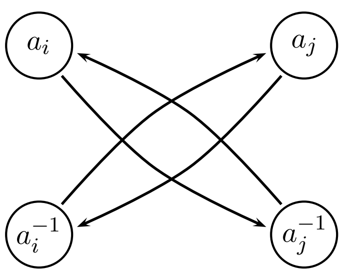

Figure 1

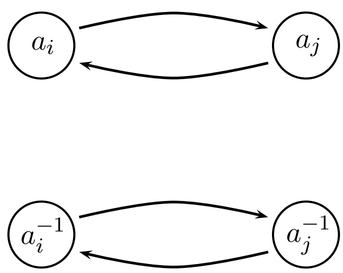

Figure 2

If $R_{i,j} = 2$ and $\rho_{i,j}$ is not even, then we say that $\rho_{i,j}$ is an *odd pair*, in which case

$$\rho_{i,j} \;=_{\mathcal{J}}\; \{a_i a_j, a_i a_j^{-1}\}\,, \quad \{a_i a_j, a_i^{-1} a_j\}\,, \quad \{a_j a_i, a_j a_i^{-1}\} \quad \text{or} \quad \{a_j a_i, a_j^{-1} a_i\}\;.$$

When $\rho_{i,j}$ is an odd pair then there are no cycles in the subgraph of Γ_S involving the vertices $a_i^{\pm 1}, a_j^{\pm 1}$, and, in fact, one vertex behaves like a source and the vertex corresponding to its inverse behaves as a sink, the four possibilities depicted in Figures 3, 4, 5 and 6.

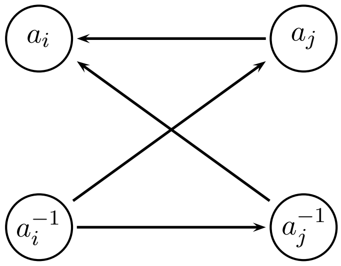

FIGURE 3

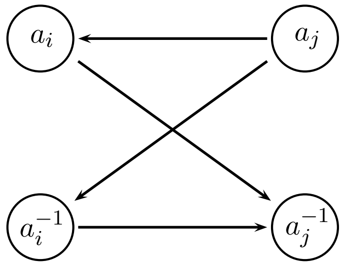

FIGURE 4

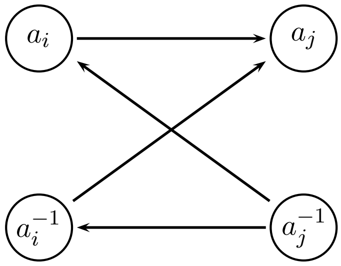

FIGURE 5

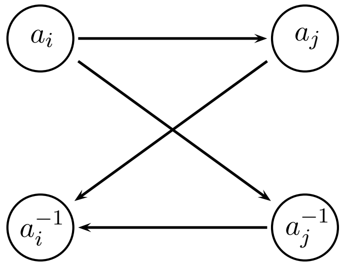

FIGURE 6

In the case that

$$\rho_{i,j} \ =_{\mathcal{J}} \ \{a_i a_j, a_i a_j^{-1}\} \ =_{\mathcal{J}} \ \{a_j a_i^{-1}, a_j^{-1} a_i^{-1}\}$$

then we say a_i *dominates* a_j *from the left* and also a_i^{-1} *dominates* a_j *from the right.* The notion of domination from the left is useful, because if w is any word over the alphabet B for which a_i is the last letter of w, and a_i dominates a_j from the left, then the words $wa_j^{\pm 1}$ and $a_j^{\pm 1} w^{-1}$ are zero in S. Similarly, if a_i dominates a_j from the right, and w is a word beginning with a_i then the words $a_j^{\pm 1} w$ and $w^{-1} a_j^{\pm 1}$ are zero in S. Later, in the context of polynomial growth, we set up chains of domination, and domination from one side can become a transitive relation.

Observe, in each of the diagrams corresponding to the four odd pairs, there is a unique vertex that behaves like a sink, and this is represented by the unique generator that dominates from the left: in Figure 3, a_i dominates a_j from the left; in Figure 4, a_j^{-1} dominates a_i from the left (so a_j dominates a_i from the right); in Figure 5, a_j dominates a_i from the left; in Figure 6, a_i^{-1} dominates a_j from the left (so a_i dominates a_j from the right).

4. EXAMPLES

The following examples illustrate some important special cases of the three types of phenomena described in Theorem 1.2 above and Section 7 below. Some graphs of particular presentations using three generators are also given here, which may assist the reader in digesting the general analysis that takes place later in the article.

Example 4.1. This is an example of the phenomenon in part (2) of Theorem 1.2 where case (a)(i) occurs, so that all generators are nilpotent. Let $n \geq 3$ and consider any choice of fixed integers $p_3, \ldots, p_n \geq 2$. Consider the semigroup

$$S \; = \; \langle\, a_1, \ldots, a_n \mid a_1^2 = a_2^2 = a_j^{p_j} = a_1 a_2^{-1} = a_1 a_j = a_1 a_j^{-1} = a_i a_j = a_i a_j^{-1} = 0\,,\; 2 \leq i < j \leq n \,\rangle\,.$$

Note that $n^2 - 1$ relators appear in this presentation. Also, if $p_3 = \ldots = p_n = 2$, then S is isomorphic to the semigroup described in Example 9.4 of [10]. The graph Γ_S is pictured in Figure 7, in the special case that $n = 3$ and $p_3 = 2$:

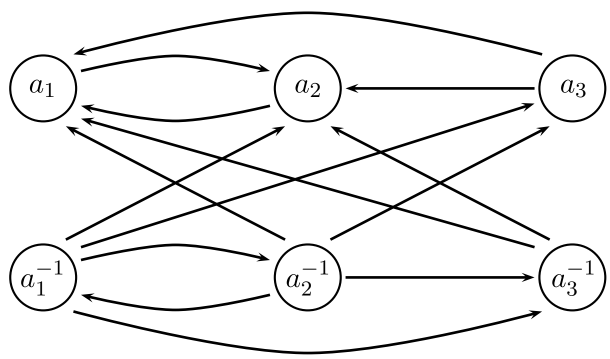

FIGURE 7

Note that, in tracing the word that labels a path in the graph of Figure 7, the label of any given edge is the letter denoting the target vertex. We describe explicitly the language of all reduced words that are nonzero in S. First put

$$\mathscr{L}_1 \; = \; a_1(a_2a_1)^*(a_2 \cup 1) \qquad \text{and} \qquad \mathscr{L}_2 \; = \; a_2(a_1a_2)^*(a_1 \cup 1)\,,$$

which are the languages consisting of reduced words over the alphabet $\{a_1, a_2\}$ that are nonzero in S and begin with a_1 and a_2 respectively. Put

$$X \; = \; \{(i_3, \ldots, i_k) \mid 0 \leq i_3 \leq p_3\,, \ldots, 0 \leq i_n \leq p_n\}\;.$$

Now put

$$\mathscr{K} \; = \; \{w \mid w \text{ is nonempty and reduced of the form } w = a_3^{-i_3} \ldots a_n^{-i_n} a_n^{j_n} \ldots a_3^{j_3} \text{ for some } (i_3, \ldots, i_n), (j_3, \ldots, j_n) \in X\}\;.$$

Note that, in the above form for $w \in \mathscr{K}$, since w is reduced, if one of i_n or j_n is nonzero then the other is zero, and if $i_n = j_n = 0$ and one of i_{n-1} or j_{n-1} is nonzero then the other is zero, and so on. Then, because a_i dominates a_j from the left, whenever $3 \leq i < j \leq n$, we have that $\mathscr{K}$ is the finite language of all reduced words over the alphabet $\{a_3^{\pm 1}, \ldots, a_n^{\pm 1}\}$ that are nonzero in S. Since $a_1^{-1}a_2$ and $a_2^{-1}a_1$ are nonzero in S, and since a_1 and a_2 dominate a_j from the left for $j \geq 3$, the language $\mathscr{L}$ of reduced words that are nonzero in $S \cup \{1\}$ can be described by the rational expression

$$\mathscr{L} \; = \; (\mathscr{L}_1^{-1} \cup \mathscr{L}_2^{-1} \cup 1)\mathscr{K}(\mathscr{L}_1 \cup \mathscr{L}_2 \cup 1) \cup (\mathscr{L}_1^{-1} \cup 1)(\mathscr{L}_2 \cup 1) \cup (\mathscr{L}_2^{-1} \cup 1)(\mathscr{L}_1 \cup 1)\;.$$

Then $\mathscr{L}$ has height bounded by seven, relative to $\mathscr{K} \cup \{a_1^{\pm 1}, a_2^{\pm 1}, (a_1a_2)^{\pm 1}, (a_2a_1)^{\pm 1}\}$ (since $\mathscr{L}_1$ and $\mathscr{L}_2$ have height bounded by three relative to $\{a_1^{\pm 1}, a_2^{\pm 1}, (a_1a_2)^{\pm 1}, (a_2a_1)^{\pm 1}\}$). By an argument based on the proof of part (b) of Lemma 2.5, all reduced but not cyclically reduced words are nilpotent, so S has polynomial growth, by part (c) of Theorem 2.2.

Example 4.2. This is an example of the phenomenon in part (1) of Theorem 1.2 where all but one of the generators are nilpotent. Let $n \geq 3$ and consider any choice of fixed integers $p_2, \ldots, p_n \geq 2$. Consider the semigroup

$$S \;=\; \langle\, a_1, \ldots, a_n \mid a_2^{p_2} = \ldots = a_n^{p_n} = a_i a_j = a_i a_j^{-1} = 0\,,\; 1 \leq i < j \leq n \,\rangle\,.$$

Again, $n^2 - 1$ relators appear in this presentation. The graph Γ_S is pictured in Figure 8, in the special case that $n = 3$ and $p_2 = p_3 = 2$:

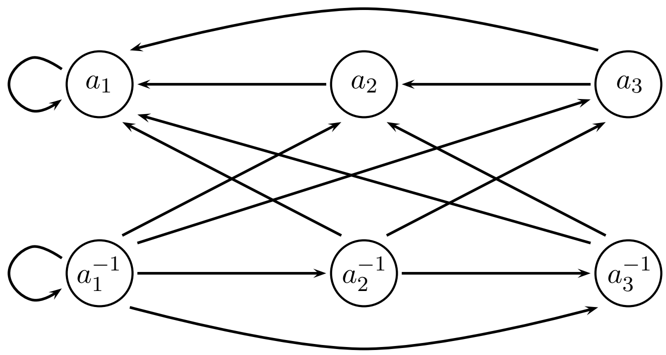

Figure 8

To describe the language $\mathscr{L}$ of reduced words that are nonzero in $S \cup \{1\}$, now put

$$X \;=\; \{(i_2, \ldots, i_n) \mid 0 \leq i_2 \leq p_2\,, \ldots, 0 \leq i_n \leq p_n\}\,.$$

and

$$\mathscr{K} \;=\; \{w \mid w \text{ is nonempty and reduced of the form } w = a_2^{-i_2} \ldots a_n^{-i_n} a_n^{j_n} \ldots a_2^{j_2} \text{ for some } (i_2, \ldots, i_n), (j_2, \ldots, j_n) \in X\}\,.$$

Again $\mathscr{K}$ is finite and now, by inspection,

$$\mathscr{L} \;=\; (a_1^{-1})^* \mathscr{K} a_1^* \cup (a_1^{-1})^* \cup a_1^*\,.$$

Clearly $\mathscr{L}$ has height bounded by three, relative to $\mathscr{K} \cup \{a_1^{\pm 1}\}$. Again, by an argument based on the proof of part (b) of Lemma 2.5, all reduced but not cyclically reduced words are nilpotent, so S has polynomial growth, by part (c) of Theorem 2.2.

Example 4.3. This is an example of the phenomenon in part (2) of Theorem 1.2 where case (a)(ii) occurs, so that all but two of the generators are nilpotent. Let $n \geq 3$ and consider any choice of fixed positive integers p_1 and p_2 and integers $p_3, \ldots, p_n \geq 2$. Consider the semigroup

$$S \;=\; \langle\, a_1, \ldots, a_n \mid a_3^{p_3} = \ldots = a_n^{p_n} = a_2^{p_1} a_1 a_1^{-1} a_2^{p_2} = a_i a_j = a_i a_j^{-1} = 0\,,\; 1 \leq i < j \leq n \,\rangle\,.$$

Note again that n^2-1 relators appear in this presentation, but, by contrast with the previous two examples, one of the relators is not $\mathcal{J}$-equivalent to a reduced word. If we remove that relator we get the following presentation in which all relators are reduced:

$$S_0 \ = \ \langle\, a_1,\dots,a_n \mid a_3^{p_3} = \dots = a_n^{p_n} = a_i a_j = a_i a_j^{-1} = 0\,,\ 1 \le i < j \le n \,\rangle\ .$$

Both S and S_0 have identical languages of nonzero reduced words. The graph Γ_{S_0} is pictured in Figure 9, in the special case that $n=3$ and $p_3=2$:

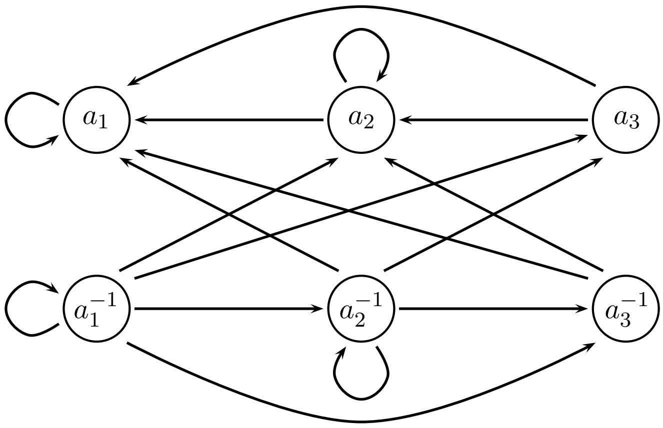

FIGURE 9

Note that the graph Γ_{S_0} is identical to the graph displayed in Figure 8, for Example 4.2, except for the addition of loops at the vertices a_2 and a_2^{-1}. It follows, by Theorem 1.1 (proved below as Theorem 5.13), that S_0 has exponential growth, since the presentation uses fewer than n^2-1 relators. One can see this directly by part (d)(ii) of Theorem 2.2, since (a_2,a_1) is an adjacent pair but $a_2a_1a_1^{-1}a_2$ is nonzero in S.

As in Example 4.1, put

$$X \ = \ \{(i_3,\dots,i_n) \mid 0 \le i_3 \le p_3\,,\dots,0 \le i_n \le p_n\}\ .$$

and

$$\mathscr{K} \ = \ \{w \mid w \text{ is nonempty and reduced of the form } w = a_3^{-i_3}\dots a_n^{-i_n}a_n^{j_n}\dots a_3^{j_3}$$
$$\text{for some } (i_3,\dots,i_n),(j_3,\dots,j_n) \in X\}\ .$$

Then $\mathscr{K}$ is the finite language of all reduced words over the alphabet $\{a_3^{\pm1},\dots,a_n^{\pm1}\}$ that are nonzero in S_0. By inspection, the language $\mathscr{L}$ of reduced words that are nonzero in $S_0\cup\{1\}$ (and therefore also in $S\cup\{1\}$) can be described by the rational expression

$$\mathscr{L} \ = \ (a_1^{-1})^*(a_2^{-1})^*\mathscr{K}a_2^*a_1^* \cup (a_1^{-1})^*\big((a_2^{-1})^* \cup a_2^* \cup \big((a_2^{-1})^+ \cup a_2^+\big)\big)a_1^*\big)\ .$$

Clearly $\mathscr{L}$ has height bounded by five, relative to $\mathscr{K}\cup\{a_1^{\pm1},a_2^{\pm1}\}$. Again, by an argument based on the proof of part (b) of Lemma 2.5, all reduced but not cyclically reduced words are nilpotent in S (though certainly not in S_0), so S has polynomial growth, by part (c) of Theorem 2.2.

5. Lower bounds for polynomial growth

In the following proofs, graphical criteria for determining polynomial or exponential growth in Theorem 2.2 will be used so often that they may be applied without explicit reference. The first lemma severely constrains the nature of relators in a two-standard presentation, for a semigroup with polynomial growth, when the relators come in pairs with the same content.

Lemma 5.1. *Suppose that the presentation for S is in two-standard form. Let i, j be distinct positive integers and suppose that a_i is not nilpotent. If* (i) *$\rho_{i,j}$ is an even pair, or* (ii) *$\rho_{i,j}$ is an odd pair such that a_j dominates a_i from the right or left, then S has exponential growth.*

Proof. If (i) holds then a_i is a vertex of Γ_S contained in two cycles, one labelled by a_i (since a_i is not nilpotent) and another by $a_j a_i$ or $a_j^{-1} a_i$ (since $\rho_{i,j}$ is an even pair), so S has exponential growth. If (ii) holds, then, without loss of generality, we may suppose $\rho_{i,j} = \{a_i a_j, a_i^{-1} a_j\}$, so that (a_i, a_j^{-1}) and (a_i^{-1}, a_j^{-1}) are adjacent pairs with respect to Γ_S, so S has exponential growth, by part (e) of Theorem 2.2, noting that all relators are reduced, since the presentation is in two-standard form. □

The next lemma shows that, under certain conditions, domination from one side is transitive.

Lemma 5.2. *Suppose that the presentation for S is in two-standard form. Let i, j, k be distinct positive integers such that a_i is not nilpotent and $R_{i,j} = R_{i,k} = R_{j,k} = 2$. Suppose that $\rho_{i,j}$ and $\rho_{j,k}$ are odd pairs such that a_i dominates a_j from the left, and a_j dominates a_k from the left. If S has polynomial growth then $\rho_{i,k}$ is an odd pair and a_i dominates a_k from the left.*

Proof. Suppose that S has polynomial growth. By part (i) of Lemma 5.1, $\rho_{i,k}$ is an odd pair, and by part (ii), a_k does not dominate a_i from the right or left. Therefore, a_i must dominate a_k from the right or left. By hypothesis, a_i^2, $a_j a_i$ and $a_k^{-1} a_j$ are nonzero in S. If a_i dominates a_k from the right then the word $a_i a_k^{-1}$ is also nonzero in S, so that the vertex a_i is a vertex of Γ_S contained in two cycles, one labelled by a_i and another by $a_i a_k^{-1} a_j$, contradicting that S has polynomial growth. Hence a_i dominates a_k from the left. □

Lemma 5.3. *Suppose that the presentation for S is in two-standard form. Let i, j, k be distinct positive integers. If $\rho_{i,j}$ and $\rho_{j,k}$ are even pairs then S has exponential growth.*

Proof. Without loss of generality, we may suppose that $\rho_{i,j} = \{a_i a_j, a_j a_i\}$ and either

$$\text{(i)}\ \ \rho_{j,k} = \{a_j a_k, a_k a_j\} \qquad \text{or} \qquad \text{(ii)}\ \ \rho_{j,k} = \{a_j a_k^{-1}, a_k^{-1} a_j\}\ .$$

In both cases, $a_i^{-1} a_j$ labels a cycle at the vertex a_j in Γ_S. But $a_k^{-1} a_j$ and $a_k a_j$ label another cycle at a_j, in cases (i) and (ii) respectively. In both cases, S has exponential growth. □

Corollary 5.4. *Suppose that the presentation for S is in two-standard form. Let i, j, k be distinct positive integers. If either* (i) *$R_{i,j} \le 1$ and $R_{j,k} \le 1$, or* (ii) *$R_{i,j} = 1$ and $\rho_{j,k}$ is an even pair, then S has exponential growth.*

Proof. We can add at least two relators in case (i), and one relator in case (ii), to form a homomorphic image of S for which $\rho_{i,j}$ and $\rho_{j,k}$ become even pairs. This image has exponential growth, by Lemma 5.3, so that S has exponential growth. □

Lemma 5.5. *Suppose that the presentation for S is in two-standard form. Let i, j, k be distinct positive integers. Suppose that $\rho_{i,j}$ and $\rho_{i,k}$ are odd pairs such that a_i dominates a_j and a_k from the left. If $\rho_{j,k}$ is an even pair then S has exponential growth.*

Proof. Suppose that $\rho_{j,k}$ is an even pair. Since a_i dominates a_j and a_k from the left, we have

$$\rho_{i,j} =_{\mathcal{J}} \{a_i a_j, a_i a_j^{-1}\} \qquad \text{and} \qquad \rho_{i,k} =_{\mathcal{J}} \{a_i a_k, a_i a_k^{-1}\} .$$

Since $\rho_{j,k}$ is even, we have either

$$\text{(i)} \;\; \rho_{j,k} =_{\mathcal{J}} \{a_j a_k, a_k a_j\}, \;\; \text{or} \;\; \text{(ii)} \;\; \rho_{j,k} =_{\mathcal{J}} \{a_j a_k^{-1}, a_j^{-1} a_k\}.$$

In case (i), $(a_k^{-1} a_j, a_i)$ and $(a_j^{-1} a_k, a_i)$ form adjacent pairs, and, in case (ii), $(a_k a_j, a_i)$ and $(a_j^{-1} a_k^{-1}, a_i)$ form adjacent pairs with respect to Γ_S. In both cases, S has exponential growth, by part (e) of Theorem 2.2, noting that all relators are reduced, since the presentation is in two-standard form. □

The next lemma plays a crucial role in the development of our main theorems below.

Lemma 5.6. *Suppose that S is a semigroup having a two-standard presentation, with $n \geq 3$ generators, such that*

$$\rho_{1,2} =_{\mathcal{J}} \{a_1 a_2^{-1}\} ,$$

and that j is an integer such that $2 < j \leq n$ and $R_{1,j} = R_{2,j} = 2$. If S has polynomial growth then

$$\rho_{1,j} =_{\mathcal{J}} \{a_1 a_j, a_1 a_j^{-1}\} \qquad \textit{and} \qquad \rho_{2,j} =_{\mathcal{J}} \{a_2 a_j, a_2 a_j^{-1}\} ,$$

so that a_1 and a_2 both dominate a_j from the left.

Proof. Suppose that S has polynomial growth. Note that, by hypothesis, the words $a_1 a_2$, $a_1^{-1} a_2$ and $a_2 a_1$ are nonzero in S. Without loss of generality, we may assume $j = 3$. By part (ii) of Corollary 5.4, both $\rho_{1,3}$ and $\rho_{2,3}$ are odd pairs.

We first prove that $a_3 a_1$ is not $\mathcal{J}$-related to an element of $\rho_{1,3}$ (that is, neither $a_3 a_1$ nor $a_1^{-1} a_3^{-1}$ is an element of $\rho_{1,3}$). Suppose to the contrary that $a_3 a_1$ is $\mathcal{J}$-related to an element of $\rho_{1,3}$. Because $\rho_{1,3}$ is odd, we have that $a_1 a_3$ is nonzero in S. Because $\rho_{2,3}$ is odd, we have that $a_2 a_3^{-1}$ is nonzero in S, or $a_3^{-1} a_2$ is nonzero in S (but not both). If $a_2 a_3^{-1}$ is nonzero in S then

$$a_2 a_1 \qquad \text{and} \qquad a_3 a_2^{-1} a_1$$

both label cycles at the vertex a_1 of Γ_S, contradicting that S has polynomial growth. If $a_3^{-1} a_2$ is nonzero in S then

$$(a_2 a_1, a_3) \qquad \text{and} \qquad (a_1^{-1} a_2^{-1}, a_3)$$

are adjacent pairs with respect to Γ_S, again contradicting that S has polynomial growth. This completes the proof that $a_3 a_1$ is not $\mathcal{J}$-related to an element of $\rho_{1,3}$.

We now prove that $a_1^{-1} a_3$ is not $\mathcal{J}$-related to an element of $\rho_{1,3}$. Suppose to the contrary that $a_1^{-1} a_3$ is $\mathcal{J}$-related to an element of $\rho_{1,3}$. Because $\rho_{1,3}$ is odd, we have that $a_1 a_3^{-1}$ is nonzero in S. Because $\rho_{2,3}$ is odd, we have that a_2a_3 is nonzero in S, or a_3a_2 is nonzero in S (but not both). If a_2a_3 is nonzero in S then

$$a_1a_2 \qquad \text{and} \qquad a_3a_1^{-1}a_2$$

label cycles at the vertex a_2 of Γ_S, contradicting that S has polynomial growth. If a_3a_2 is nonzero in S then

$$(a_2a_1, a_3^{-1}) \qquad \text{and} \qquad (a_1^{-1}a_2^{-1}, a_3^{-1})$$

are adjacent pairs with respect to Γ_S, again contradicting that S has polynomial growth. This completes the proof that $a_1^{-1}a_3$ is not $\mathcal{J}$-related to an element of $\rho_{1,3}$.

Because $\rho_{1,3}$ is odd, the previous observations prove that $\rho_{1,3} =_{\mathcal{J}} \{a_1a_3, a_1a_3^{-1}\}$. By the same argument, interchanging the roles of a_1 and a_2, noting that $a_2a_1^{-1} = (a_1a_2^{-1})^{-1}$, we have $\rho_{2,3} =_{\mathcal{J}} \{a_2a_3, a_2a_3^{-1}\}$, and the lemma is proved. □

Lemma 5.7. *Suppose that S is a semigroup having a two-standard presentation, with $n \geq 3$ generators. Let i, j, k be distinct positive integers such that $R_{i,j} = R_{i,k} = 2$ and $R_{j,k} \leq 3$. If a_i is not nilpotent, and a_i dominates a_j from the left and a_k from the right, then S has exponential growth.*

Proof. Suppose that a_i is not nilpotent and dominates a_j from the left and a_k from the right. Then a_i^2, a_ja_i, $a_j^{-1}a_i$, a_ia_k and $a_ia_k^{-1}$ are all nonzero in S. Because $R_{j,k} \leq 3$, at least one of the following words is nonzero in S:

$$w_1 = a_ja_k\,, \quad w_2 = a_ka_j\,, \quad w_3 = a_ja_k^{-1}\,, \quad w_4 = a_k^{-1}a_j\ .$$

Put

$$w \;=\; \begin{cases} a_k^{-1}a_j^{-1}a_i & \text{if } w_1 \text{ is nonzero in } S,\\ a_ka_ja_i & \text{if } w_2 \text{ is nonzero in } S,\\ a_ka_j^{-1}a_i & \text{if } w_3 \text{ is nonzero in } S,\\ a_k^{-1}a_ja_i & \text{if } w_4 \text{ is nonzero in } S. \end{cases}$$

In each case a_i and w label different cycles at the vertex a_i in Γ_S, so that S has exponential growth. □

Lemma 5.8. *Let S be a semigroup having a two-standard presentation, with at least four generators, such that $R_{1,2} = R_{3,4} = 1$. If S has polynomial growth then*

$$R_{1,3} + R_{2,3} + R_{1,4} + R_{2,4} \;\geq\; 10\ .$$

Proof. Suppose that S has polynomial growth, and, by way of contradiction, that the conclusion fails. By part (i) of Corollary 5.4, we have $R_{1,3}, R_{2,3}, R_{1,4}, R_{2,4} \geq 2$, so that

$$8 \;\leq\; R_{1,3} + R_{2,3} + R_{1,4} + R_{2,4} \;\leq\; 9\ .$$

It follows that at least three of $R_{1,3}, R_{2,3}, R_{1,4}, R_{2,4}$ must be exactly 2. Without loss of generality we may suppose

$$\rho_{1,2} = \{a_1a_2^{-1}\}\,, \quad \rho_{3,4} = \{a_3a_4^{-1}\}\,, \quad R_{1,3} = R_{2,3} = R_{1,4} = 2\ .$$

By the first half of the conclusion of Lemma 5.6, we have $\rho_{1,3} =_{\mathcal{J}} \{a_1a_3, a_1a_3^{-1}\}$. But by Lemma 5.6 again, applied now to a_3 in place of a_1, a_4 in place of a_2, and a_1 in place of a_3, we have $\rho_{1,3} =_{\mathcal{J}} \{a_3a_1, a_3a_1^{-1}\}$. But a_1a_3 is not $\mathcal{J}$-related to a_3a_1 or $a_3a_1^{-1}$, so we get a contradiction, and the lemma is proved. □

Lemma 5.9. *Let S be a semigroup having a two-standard presentation, with at least three generators, such that a_1 and a_2 are nilpotent, a_3 is not nilpotent and $R_{1,2} = 1$. If S has polynomial growth then*

$$R_{1,3} + R_{2,3} \ \geq \ 6 \ .$$

Proof. Suppose that S has polynomial growth, and, by way of contradiction, that the conclusion fails. Then $R_{1,3} + R_{2,3} \leq 5$. By part (i) of Corollary 5.4, we have $R_{1,3}, R_{2,3} \geq 2$, so that either (i) $R_{1,3} = R_{2,3} = 2$, (ii) $R_{1,3} = 3$ and $R_{2,3} = 2$, or (iii) $R_{1,3} = 2$ and $R_{2,3} = 3$. Without loss of generality we may suppose $\rho_{1,2} = \{a_1a_2^{-1}\}$.

If (i) holds, then, $\rho_{2,3} =_{\mathcal{J}} \{a_2a_3, a_2a_3^{-1}\}$, by Lemma 5.6, so that

$$(a_3, a_2) \qquad \text{and} \qquad (a_3^{-1}, a_2)$$

are adjacent pairs with respect to Γ_S, so that S has exponential growth, yielding a contradiction.

Suppose (ii) holds. By part (ii) of Corollary 5.4, $\rho_{2,3}$ is odd. If $\rho_{2,3} =_{\mathcal{J}} \{a_2a_3, a_2a_3^{-1}\}$ or $\{a_3a_2, a_3^{-1}a_2\}$ then (a_3, a_2) and (a_3^{-1}, a_2) are adjacent pairs, or (a_3, a_2^{-1}) and (a_3^{-1}, a_2^{-1}) are adjacent pairs, respectively, with respect to Γ_S, yielding exponential growth, which is impossible. Hence one of the following holds:

$$\text{(a)} \quad \rho_{2,3} =_{\mathcal{J}} \{a_2a_3, a_2^{-1}a_3\} \qquad \text{or} \qquad \text{(b)} \quad \rho_{2,3} =_{\mathcal{J}} \{a_3a_2, a_3a_2^{-1}\} \ .$$

Because $R_{1,3} = 3$, one of the following words is nonzero in S:

$$w_1 \ = \ a_1a_3 \ , \ \ w_2 \ = \ a_3a_1 \ , \ \ w_3 \ = \ a_1a_3^{-1} \quad \text{or} \quad w_4 \ = \ a_1^{-1}a_3 \ .$$

If $w_1 \neq 0$ then, in case (a), a_3 and $a_2a_1a_3$ label cycles at the vertex a_3 in Γ_S, and, in case (b), (a_2a_1, a_3) and $(a_1^{-1}a_2^{-1}, a_3)$ are adjacent pairs with respect to Γ_S. If $w_2 \neq 0$ then, in case (a), (a_1a_2, a_3^{-1}) and $(a_2^{-1}a_1^{-1}, a_3^{-1})$ are adjacent pairs, and, in case (b), a_3 and $a_1a_2a_3$ label cycles at the vertex a_3 of Γ_S. If $w_3 \neq 0$ then, in case (a), (a_2a_1, a_3^{-1}) and $(a_1^{-1}a_2^{-1}, a_3^{-1})$ are adjacent pairs, and, in case (b), a_3 and $a_1^{-1}a_2^{-1}a_3$ label cycles at the vertex a_3 of Γ_S. If $w_4 \neq 0$ then, in case (a), a_3 and $a_2^{-1}a_1^{-1}a_3$ label cycles at the vertex a_3, and, in case (b), (a_1a_2, a_3) and $(a_2^{-1}a_1^{-1}, a_3)$ are adjacent pairs with respect to Γ_S. All of these cases lead to exponential growth, which is a contradiction.

Similarly, (iii) leads to a contradiction, completing the proof of the lemma. □

Theorem 5.10. *Let S be a semigroup having a two-standard presentation, with $n \geq 2$ generators and L relators. Let $p \geq 0$ denote the number of integer pairs (i, j) such that $i < j$ and $R_{i,j} = 1$. If S has polynomial growth and all generators are nilpotent then $2p \leq n$ and*

$$L \ \geq \ n^2 + p(p-2) \ .$$

Proof. Suppose that S has polynomial growth and all generators are nilpotent. Certainly $R_{i,j} \geq 1$ for all $i < j$, by Theorem 2.7. If $p = 0$ then $R_{i,j} \geq 2$ for all $i < j$, so that

$L \geq n + 2\binom{n}{2} = n^2$, verifying the theorem in this case. If $p = 1$ then, without loss of generality, $R_{1,2} = 1$ and $R_{i,j} \geq 2$ for all $i < j$ such that $(i,j) \neq (1,2)$, so that

$$L \ \geq\ n + 2\left(\binom{n}{2} - 1\right) + 1 \ =\ n^2 - 1\ ,$$

verifying the theorem in this case also.

Suppose that $p > 1$. If $R_{i,j} = R_{k,\ell} = 1$ for some $i < j$ and $k < \ell$ such that $(i,j) \neq (k,\ell)$ and $\{i,j\} \cap \{k,\ell\} \neq \emptyset$ then S has exponential growth, by part (i) of Corollary 5.4, which is impossible. Hence $n \geq 2p$ and, without loss of generality, we may assume

$$R_{1,2} \ =\ R_{3,4} \ =\ \ldots \ =\ R_{2p-1,2p} \ =\ 1\ .$$

Thus $R_{i,j} \geq 2$ for all $i < j$ such that $(i,j) \notin \{(1,2), \ldots, (2p-1, 2p)\}$. By Lemma 5.8, for each s, t such that $1 \leq s < t \leq p$, we have

$$R_{2s-1,2t-1} + R_{2s-1,2t} + R_{2s,2t-1} + R_{2s,2t} \ \geq\ 10\ .$$

A simple count now yields

$$L \ \geq\ n + p + 10\binom{p}{2} + 2\left(\binom{n}{2} - \binom{2p}{2}\right) \ =\ n^2 + p^2 - 2p\ .$$

This completes the proof of the theorem. □

Corollary 5.11. *Suppose that $S = \langle A \mid c_i = 0$ for $i = 1, \ldots, L\rangle$ has polynomial growth, where A is an alphabet of size $n \geq 2$, and all of the generators (elements of A) are nilpotent. Then $L \geq n^2 - 1$.*

Proof. Certainly S has a homomorphic image S' given by a presentation in two-standard form with $L' \leq L$ relators, and S' has polynomial growth. By Theorem 5.10, there is a nonnegative integer p such that

$$L \geq L' \geq n^2 + p(p-2) \geq n^2 - 1\ ,$$

and the corollary is proved. □

We shall see shortly (Theorem 5.13 below) that we can remove the condition that all generators are nilpotent in Corollary 5.11.

Theorem 5.12. *Suppose that S has polynomial growth and is given by a presentation in two-standard form with $n \geq 2$ generators and L relators such that m generators are not nilpotent. Let $p \geq 0$ denote the number of pairs (i,j) such that $i < j$ and $R_{i,j} = 1$. Then $2p \leq n - m$ and*

$$L \ \geq\ \begin{cases} \frac{3n^2-5n+4}{2} & \text{if } m = n-1\ , \\ n^2 + p(p+2m-2) + \frac{m(m-3)}{2} & \text{if } m \neq n-1\ . \end{cases}$$

Proof. Note that if $R_{i,j} = 1$ for $i < j$ then a_i and a_j are nilpotent, because the inverse subsemigroup of S generated by a_i and a_j has polynomial growth and, by Theorem 2.7, at least three relators are required in the presentation restricted to words with content contained in $\{a_i, a_j\}$. In particular, if $m = n$ or $m = n - 1$ then $p = 0$.

If $m = n$ then, by Theorem 2.9, we have $L \geq \frac{3m(m-1)}{2} = n^2 + \frac{m(m-3)}{2}$, which agrees with the second formula in this case.

Suppose $m = n - 1$. Without loss of generality a_1 is the only nilpotent generator. By Theorem 2.7, we have $R_{1,j} \geq 2$ for all $j \geq 2$ and $R_{i,j} \geq 3$ whenever $2 \leq i < j$. Thus

$$L \;\geq\; 1 + 2(n-1) + 3\binom{n-1}{2} \;=\; \frac{3n^2 - 5n + 4}{2} \;.$$

Suppose now that $m < n - 1$. Without loss of generality, we may assume $a_1, \ldots, a_{n-m}$ are the only nilpotent generators. Let L' and L'' denote the number of relators with content contained in $\{a_1, \ldots, a_{n-m}\}$ and $\{a_{n-m+1}, \ldots, a_n\}$ respectively. By Theorem 5.10 above, and by Theorem 2.9, we have, respectively,

$$L' \;\geq\; (n-m)^2 + p(p-2) \qquad \text{and} \qquad L'' \;\geq\; \frac{3m(m-1)}{2} \;.$$

If $p = 0$ then $R_{i,j} \geq 2$ for all $i < j$, so that $L \geq L' + L'' + 2(n-m)m$, whence

$$L \;\geq\; (n-m)^2 + \frac{3m(m-1)}{2} + 2(n-m)m \;=\; n^2 + \frac{m(m-3)}{2} \;,$$

verifying the last part of the formula of the theorem in this case. Suppose henceforth that $p \geq 1$. As in the proof of Theorem 5.10, we have $2p \leq n - m$, and we may assume

$$R_{1,2} \;=\; R_{3,4} \;=\; \ldots \;=\; R_{2p-1,2p} \;=\; 1 \;.$$

Thus $R_{i,j} \geq 2$ for $i \in \{1, \ldots, n-m\}$ and $j \in \{n-m+1, \ldots, n\}$. By Lemma 5.9, for each $s \in \{1, \ldots, p\}$ and $t \in \{n-m+1, \ldots, n\}$, we have

$$R_{2s-1,t} + R_{2s,t} \;\geq\; 6 \;.$$

Thus $L \geq L' + L'' + 2(n-m-2p)m + 6pm$, so that

$$\begin{aligned} L \;&\geq\; (n-m)^2 + p(p-2) + \frac{3m(m-1)}{2} + 2(n-m-2p)m + 6pm \\ &=\; n^2 + p(p+2m-2) + \frac{m(m-3)}{2} \;. \end{aligned}$$

This completes the proof of the theorem. □

We can now remove the condition that all generators are nilpotent in Corollary 5.11.

Theorem 5.13. *Suppose that $S = \langle A \mid c_i = 0$ for $i = 1, \ldots, L\rangle$ has polynomial growth, where A is an alphabet of size $n \geq 2$. Then $L \geq n^2 - 1$.*

Proof. As before, S has a homomorphic image S' given by a presentation in two-standard form with $L' \leq L$ relators, and S' has polynomial growth. By Theorem 5.12, there are nonnegative integers p and m such that

$$L \;\geq\; L' \;\geq\; \begin{cases} \frac{3n^2-5n+4}{2} & \text{if } m = n-1 \;, \\ n^2 + p(p+2m-2) + \frac{m(m-3)}{2} & \text{if } m \neq n-1 \;. \end{cases}$$

But it is easy to check that each alternative is always at least $n^2 - 1$, and the theorem is proved. □

We deduce the following corollary for Rees quotients of finitely generated free inverse semigroups, where these Rees quotients need not be finitely presented (which will be applied in the final section):

Corollary 5.14. *Let S be the Rees quotient of FI_A where $A = \{a_1, \ldots, a_n\}$ for $n \geq 2$ given by the presentation*

$$S \ = \ \mathrm{Inv}\langle A \mid c_i = 0 \ \textit{ for } i \in I \rangle \ ,$$

where I is a nonempty indexing set (possibly infinite) and c_i is a word over $A \cup A^{-1}$ that is not $\mathcal{J}$-equivalent to a single letter, for each $i \in I$. Then there exists a finite set $D = \{d_1, \ldots, d_m\}$ of smallest size m satisfying the following conditions:

(i) *each d_j, for $1 \leq j \leq m$, is a reduced word of length 2 over $A \cup A^{-1}$;*
(ii) *the ideal generated by D in FI_A contains all c_i for $i \in I$.*

If $m < n^2 - 1$ then S contains a noncyclic free subsemigroup and therefore has exponential growth.

Proof. Clearly D exists, since the set of all reduced words of length 2 over $A \cup A^{-1}$ is finite and satisfies conditions (i) and (ii). Suppose that $m < n^2 - 1$, and put

$$T \ = \ \langle A \mid d_1 = \ldots = d_m = 0 \rangle \ .$$

Then the presentation for T is irredundant by the minimality of m, so is two-standard. Hence T contains a noncyclic free semigroup by Theorems 5.13 and 2.2. The corollory now follows because T is a morphic image of S. ☐

Scholium 5.15. From the proof of Theorem 5.13, the lower bound $n^2 - 1$ is achievable only in the following cases, where n is the number of generators, m the number of generators that are not nilpotent and p the number of pairs (i, j) such that $i < j$ and $R_{i,j} = 1$ in any two-standard presentation for a homomorphic image:

(i) $n \geq 2$, $m = 1$ and $p = 0$;
(ii) $n \geq 2$, $m = 2$ and $p = 0$;
(iii) $n \geq 2$, $m = 0$ and $p = 1$.

Let $S = \langle a_1, \ldots, a_n \mid c_1, \ldots, c_L \rangle$ be a two-standard presentation with $n \geq 2$ generators and $L = n^2 - 1$ relators, and suppose that S has polynomial growth. Observe that $R_{i,j} > 0$ for all $i \neq j$, by Theorem 2.7. Thus, the generators and relators may be reordered, if necessary, such that one of the following corresponding conditions hold:

(i)′ $a_2^2 = \ldots = a_n^2 = 0$ and $R_{i,j} = 2$ for each $i \neq j$;
(ii)′ $a_3^2 = \ldots = a_n^2 = 0$, $R_{1,2} = 3$ and $R_{1,j} = R_{i,j} = 2$ for $2 \leq i < j \leq n$;
(iii)′ $a_1^2 = \ldots = a_n^2 = 0$, $R_{1,2} = 1$ and $R_{1,j} = R_{i,j} = 2$ for $2 \leq i < j \leq n$.

6. Finite Rees quotients

In this section we prove that n^2 is the least number of relators necessary for finiteness of the Rees quotient of a free inverse semigroup given by an irredundant presentation as an inverse semigroup with zero using n generators. We give a precise description, up to isomorphism, of all irredundant presentations that achieve this sharp n^2 lower bound.

To put these results in historical context, it follows from the Golod-Shafarevich Theorem [12] and a result of Vinberg [49] that every finitely generated associative algebra over a field given by n generators and $\leq \frac{n^2}{4}$ relators is infinite dimensional. Anick [2] conjectured that for any $d, n \in \mathbb{N}$ with $d > \frac{n^2}{4}$ there exists a finite dimensional associative quadratic algebra with n generators and d relators. Iyudu and Shkarin [18] proved that every finitely generated quadratic semigroup algebra (that is, an algebra such that each relation is either a degree two monomial or a difference of degree two monomials) with n generators and $d \leq \frac{n^2+n}{4}$ relations is infinite dimensional and this estimate is sharp. Since the results of this section also give the complete description of all finite dimensional inverse semigroup algebras given by an irredundant presentation with n generators and n^2 relators that are words over the alphabet $A \cup A^{-1}$, this can be viewed as the proof of a generalised analogue of the Anick conjecture for Rees quotients of free inverse semigroups and their semigroup algebras. It may be noted that Kiyoshi Shirayanagi [36] found a classification of finite-dimensional monomial algebras in terms of word trees related to some partially ordered sets, in particular, showing that every finite dimensional monomial algebra has a unique irredundant presentation up to a permutation of generators. (This result is also related to Proposition 2.1 above).

We begin by giving a straightforward description of the corresponding result for Rees quotients of free semigroups.

Proposition 6.1. *Let S be a finitely presented Rees quotient of a free semigroup given by the following presentation as a semigroup with zero:*

$$S \; = \; \mathrm{Sgp}\langle A \mid c_1 = \ldots = c_k = 0 \rangle$$

with $|A| = n \geq 1$ generators and $k \geq 1$ relators, where each relator is a word of length at least two. If S is finite then $k \geq \frac{n(n+1)}{2}$. If $k = \frac{n(n+1)}{2}$ then S is finite if and only if

$$S \;\cong\; \mathrm{Sgp}\langle a_1, \ldots, a_n \mid a_i^{p_i} = c_{i,j} = 0\,, \;\; 1 \leq i < j \leq n \rangle$$

where the following conditions hold:

(i) $p_i \geq 2$ *for* $1 \leq i \leq n$*;*
(ii) $c_{i,j} = a_i a_j$ *for* $1 \leq i < j - 1 \leq n - 1$*;*
(iii) $c_{i,i+1} \in (a_i a_{i+1})^+ \cup (a_i a_{i+1})^+ a_i$ *for* $1 \leq i < n$*;*
(iv) $c_{i+1,i+2} = a_{i+1} a_{i+2}$ *if* $2 \leq i + 1 < n$ *and* $|c_{i,i+1}| > 2$*;*
(v) $p_{i+1} = 2$ *if* $1 \leq i < n$ *and* $|c_{i,i+1}| > 2$.
(vi) $p_i = 2$ *if* $1 \leq i < n$ *and* $|c_{i,i+1}| > 3$.

(Note that if $n = 1$ then the isomorphism is interpreted as $S \cong \mathrm{Sgp}\langle a_1 \mid a_1^{p_1} = 0 \rangle$ and conditions (ii)-(vi) *become vacuous.)*

Proof. We may suppose that $A = \{a_1, \ldots, a_n\}$. If S is finite then the presentation must include relators of length at least two, with distinct contents, that divide powers of a_i and $a_j a_\ell$ for $1 \leq i \leq n$ and $1 \leq j < \ell \leq n$, so that $k \geq n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Suppose that $k = \frac{n(n+1)}{2}$ and S is finite. By the previous observation there must be relators of the form $a_i^{p_i}$ for $p_i \geq 2$, when $1 \leq i \leq n$, establishing condition (i), and

$$c_{i,j} \;\in\; (a_i a_j)^+ \cup (a_i a_j)^+ a_i \cup (a_j a_i)^+ \cup (a_j a_i)^+ a_j \;, \tag{9}$$

when $1 \le i < j \le n$. Because $k = \frac{n(n+1)}{2}$, each relator is unique with respect to content. We claim that, up to reordering the generators and rewriting the relators, we may suppose that there is a total ordering

$$a_1 < \ldots < a_n$$

of the generators and a partition

$$A = A_1 \cup \ldots \cup A_{n-r}$$

into disjoint subsets consisting of r subsets of size two and $n-2r$ subsets of size one, where $0 \le r \le \frac{n}{2}$ such that

(a) $a_i < a_j$ if and only if $i < j$, which occurs if and only if $c_{i,j}$ begins with the letter a_i;
(b) $A_\alpha A_\beta = \{0\}$ whenever $1 \le \alpha < \beta \le n-r$;
(c) if $|A_\alpha| = 2$, for $1 \le \alpha \le n-r$, then A_α consists of a consecutive pair of generators, say $A_\alpha = \{a_j, a_{j+1}\}$, where $1 \le j \le n-2$, and $a_j a_{j+1} a_j$ is a prefix of $c_{j,j+1}$.

Note that these imply that $a_1 \in A_1$, $a_n \in A_{n-r}$, and, if $1 \le i \le n-1$ and $a_i \in A_\alpha$ and $a_{i+1} \in A_\beta$, then $\beta = \alpha$ or $\beta = \alpha + 1$.

Suppose this claim has been proved and that $1 \le i < j \le n$. By (a), $c_{i,j}$ begins with a_i, so that, by (9),

$$c_{i,j} \ \in \ (a_i a_j)^+ \cup (a_i a_j)^+ a_i \ .$$

Condition (iii) now follows. We have $a_i \in A_\alpha$ and $a_j \in A_\beta$ for some α and β. Note that $a_j a_i$ is nonzero in S, so that, by (b) and (c), either $\alpha = \beta$ and $j = i+1$, or $\alpha < \beta$ and $c_{i,j} = a_i a_j$. Conditions (ii) and (iv) now follow. If $|c_{i,i+1}| > 2$ then $A_\alpha = \{a_i, a_{i+1}\}$ and $a_{i+1} a_i a_{i+1}$ is nonzero in S, so that $p_{i+1} = 2$, for otherwise $\langle a_{i+1}^2 a_i \rangle$ would be an infinite subsemigroup of S, which is impossible. This verifies condition (v). If $|c_{i,i+1}| > 3$ then $a_i a_{i+1} a_i$ is nonzero in S, so that $p_i = 2$, for otherwise $\langle a_i^2 a_{i+1} \rangle$ would be an infinite subsemigroup of S, which is impossible. This verifies condition (vi).

It remains to prove the claim concerning the total ordering of the letters and the partition of A such that (a), (b) and (c) hold. First, it is convenient to extend the notation for the relators to make it symmetrical in the subscripts, by putting

$$c_{i,j} \ = \ c_{j,i}$$

if $1 \le j < i \le n$. Define a relation $<$ on A by, for $i, j \in \{1, \ldots, n\}$,

$$a_i \ < \ a_j \ \text{ if and only if } i \ne j \text{ and } a_i \text{ is the initial letter of } c_{ij},$$

in which case, by (9), $c_{i,j} \in (a_i a_j)^+ \cup (a_i a_j)^+ a_i$. Suppose that $i, j, k \in \{1, \ldots, n\}$ and

$$a_i \ < \ a_j \ < \ a_k \ .$$

We will show that $a_i < a_k$ and $c_{i,k} = a_i a_k$. We have $i \ne j \ne k$, $a_i a_j$ is a prefix of $c_{i,j}$ and $a_j a_k$ is a prefix of $c_{j,k}$. Note that $a_j a_i$ and $a_k a_j$ are nonzero in S. If $i = k$ then a_j is the initial letter of $c_{j,k} = c_{j,i} = c_{i,j}$, contradicting that a_i is the initial letter of $c_{i,j}$. Hence $i \ne k$. If a_i is not the initial letter of $c_{i,k}$ then $a_k a_i$ is a prefix of $c_{i,k}$, so that $a_i a_k$ is nonzero in S and $a_i a_k a_j$ generates an infinite cyclic subsemigroup of S, which is impossible. Hence a_i is the initial letter of $c_{i,k}$. This shows that $a_i < a_k$. If $a_i a_k a_i$ is a prefix of $c_{i,k}$ then again $a_i a_k$ is nonzero in S and we get a contradiction. Hence $c_{i,k} = a_i a_k$. In particular, the relation $<$

is transitive. The relation is total, for if $i, j \in \{1, \ldots, n\}$ and $i \neq j$ then either a_i or a_j is an initial letter of $c_{i,j}$, by (9). Hence $<$ is a total ordering of A.

Now define a relation $\sim$ on A by $a_i \sim a_j$ if and only if either

$$\text{(i) } i = j \text{ or (ii) } i \neq j \text{ and } a_i a_j a_i \text{ or } a_j a_i a_j \text{ is a prefix of } c_{i,j}.$$

Clearly $\sim$ is reflexive and symmetric. Suppose that $a_i \sim a_j \sim a_k$ for $i, j, k \in \{1, \ldots, n\}$. We show that $|\{i, j, k\}| \leq 2$. Suppose to the contrary that i, j, k are distinct. Then either $a_i a_j a_i$ or $a_j a_i a_j$ is a prefix of $c_{i,j}$, and either $a_j a_k a_j$ or $a_k a_j a_k$ is a prefix of $a_{j,k}$, giving rise to four possibilities. Suppose, firstly, that $a_i a_j a_i$ is a prefix of $c_{i,j}$ and $a_j a_k a_j$ is a prefix of $a_{j,k}$. Then $a_i < a_j < a_k$, so that $a_i < a_k$, by transitivity of $<$. In particular, $a_k a_i$ is nonzero in S. But $a_i a_j$ and $a_j a_k$ are also nonzero in S, so that $a_k a_i a_j$ generates an infinite cyclic subsemigroup of S, which is impossible. Suppose, secondly, that $a_i a_j a_i$ is a prefix of $c_{i,j}$ and $a_k a_j a_k$ is a prefix of $c_{j,k}$. Then $a_i a_j$, $a_j a_i$, $a_k a_j$ and $a_j a_k$ are all nonzero in S. Either $a_i a_k$ or $a_k a_i$ is nonzero in S, and it follows that either $a_i a_k a_j$ or $a_k a_i a_j$, respectively, generates an infinite cyclic subsemigroup of S, which is impossible. The remaining third and fourth possibilities similarly lead to contradictions. This completes the proof that $|\{i, j, k\}| \leq 2$. In particular at least two of i, j, k coincide, and it is immediate that $a_i \sim a_k$. This verifies that $\sim$ is transitive, so that $\sim$ is an equivalence relation. This also proves that all $\sim$-equivalence classes have size one or two.

Consider a $\sim$-equivalence class $\{a_j, a_k\}$ of size two, so that $j \neq k$ and we may suppose that $a_j a_k a_j$ is a prefix of $c_{j,k}$. In particular, $a_j < a_k$. If a_k does not cover a_j in the total order $<$ then, from above, $c_{j,k} = a_j a_k$, which is a contradiction. Hence a_k covers a_j in the total order. It follows that the partition of A corresponding to $\sim$ has the form

$$A \;=\; A_1 \cup \ldots \cup A_{n-r} \;,$$

where there are exactly $r \leq \frac{n}{2}$ equivalence classes of size 2 and $n - 2r$ equivalence classes of size 1, with the property that, if $1 \leq \alpha < \beta \leq n - r$ and $a_i \in A_\alpha$ and $a_j \in A_\beta$ then $a_i < a_j$. Further, if $a_i < a_j$ but it is not the case that $a_i \sim a_j$ then $c_{i,j} = a_i a_j$, so that $a_i a_j = 0$ in S. This shows that if $1 \leq \alpha < \beta \leq n - r$ then $A_\alpha A_\beta = \{0\}$ in S. Suppose the total ordering of the generators is given by

$$a_{i_1} < a_{i_2} < \ldots < a_{i_n} \;,$$

for some permutation $i_1, \ldots, i_n$ of $1, \ldots, n$. Our original claim about the existence of a total ordering of A subject to conditions (a), (b) and (c) now follows by reordering the generators and rewriting the relators by replacing a_{i_j} by a_j for $1 \leq j \leq n$.

This completes the proof of the 'only if' direction of the statement of the proposition.

Suppose, conversely that S is isomorphic to the semigroup given by the presentation in the statement of the theorem, satisfying conditions (i)-(vi). We may suppose S is given by the presentation. It follows from (i)-(vi) that a nonempty word w over the alphabet $\{a_1, \ldots, a_n\}$ is nonzero in S if and only if it is a product

$$w \;=\; w_n w_{n-1} \ldots w_1$$

where w_i is a (possibly empty) subword of $a_i^{p_i - 2} c_{i,i+1}$ such that $c_{i,i+1}$ is not a suffix of w_i, for $1 \leq i < n$, and w_n is a proper subword of $a_n^{p_n}$. There are only finitely many such words, so S is finite. Observe finally that the number of relators in the presentation is $\frac{n(n+1)}{2}$, completing the proof of the proposition. □

Corollary 6.2. *Let $S = \mathrm{Sgp}\langle A \mid c_1 = \ldots = c_k = 0\rangle$ be a finitely presented Rees quotient of a free semigroup given by a presentation using $n \geq 2$ generators and $k = \frac{n(n+1)}{2}$ relators, where each relator is a word of length at least two. Then S is finite if and only if all generators are nilpotent and there is a partition*

$$A \;=\; A_1 \cup \ldots \cup A_{n-r}$$

of A into $n-r$ disjoint subsets, consisting of r subsets of size two and $n-2r$ subsets of size one, where $0 \leq r \leq \frac{n}{2}$, such that

(i) *A_i generates a finite subsemigroup of S for each i, and*
(ii) *$A_iA_j = \{0\}$ for $1 \leq i < j \leq n-r$.*

In the next lemma and the two theorems that follow, we use often, and implicitly, the simple characterisation of finiteness of Rees quotients of free inverse semigroups, described in Proposition 2.3.

Lemma 6.3. *Let S be a finite inverse semigroup given by a two-standard presentation with generating set $A = \{a_1, \ldots, a_n\}$ where $n \geq 2$. Then the following hold:*

(i) *If $i, j \in \{1, \ldots, n\}$ are distinct and $R_{i,j} = 2$ then $\rho_{i,j}$ is an odd pair.*
(ii) *If $i, j, k \in \{1, \ldots, n\}$ are distinct and $R_{i,j} = R_{i,k} = R_{j,k} = 2$ such that a_i dominates a_j from the left, and a_j dominates a_k from the left, then a_i dominates a_k from the left.*
(iii) *If $i, j, k \in \{1, \ldots, n\}$ are distinct and $R_{i,j} = R_{i,k} = R_{j,k} = 2$ such that a_i dominates a_k from the left, and a_j dominates a_k from the left, then either a_i dominates a_j from the left, or a_j dominates a_i from the left.*

Proof. Part (i) follows, for if i, j are distinct, $R_{i,j} = 2$ and $\rho_{i,j}$ is an even pair, then a_i is a vertex in Γ_S contained in a cycle, contradicting that S is finite. Suppose $i, j, k \in \{1, \ldots, n\}$ are distinct. Suppose first that a_i dominates a_j from the left, and a_j dominates a_k from the left. By part (i), $\rho_{i,k}$ is an odd pair. Consider the word

$$w \;=\; \begin{cases} a_k a_j a_i & \text{if } a_k \text{ dominates } a_i \text{ from the left,} \\ a_k^{-1} a_j a_i & \text{if } a_k^{-1} \text{ dominates } a_i \text{ from the left,} \\ a_k^{-1} a_j a_i & \text{if } a_i^{-1} \text{ dominates } a_k \text{ from the left.} \end{cases}$$

In each of these cases, w labels a cycle at the vertex a_i in Γ_S, contradicting that S is finite. Hence a_i dominates a_k from the left, proving part (ii).

Suppose now that a_i dominates a_k from the left, and a_j dominates a_k from the left. By part (i), $\rho_{i,j}$ is an odd pair. If a_i^{-1} dominates a_j from the left, or a_j^{-1} dominates a_i from the left, then $a_j^{-1}a_k^{-1}a_i$ labels a cycle at the vertex a_i in Γ_S, contradicting that S is finite. Hence a_i dominates a_j from the left, or a_j dominates a_i from the left, proving part (iii). □

Theorem 6.4. *Let $S = \langle A \mid c_i = 0 \text{ for } i = 1, \ldots, L\rangle$ be a finitely presented Rees quotient of a free inverse semigroup given by a two-standard presentation using $n \geq 2$ generators and*

$L = n^2$ *relators. Then* S *is finite if and only if*

$$S \;\cong\; \langle\, a_1, \ldots, a_n \mid a_i^2 = a_i a_j = a_i a_j^{-1} = 0\,,\; 1 \le i < j \le n \,\rangle \;.$$

Proof. Sufficiency is clear because the graph of the presentation in the statement of the theorem has no cycles, so that S is finite, by Proposition 2.3.

To prove necessity, suppose that S is finite. We may suppose $A = \{a_1, \ldots, a_n\}$. Certainly all generators are nilpotent, so we may assume $c_i = a_i^2$ for $1 \le i \le n$. If $R_{i,j} < 2$ for any $i \ne j$ then the subsemigroup generated by a_i and a_j is infinite, since there must be cycles in the subgraph of Γ_S involving the vertices a_i and a_j, which is impossible. Hence $R_{i,j} \ge 2$ for all $i \ne j$. By a simple count, since $L = n^2$, we have $R_{i,j} = 2$ for all $i \ne j$. By part(i) of Lemma 6.3, $\rho_{i,j}$ is odd for distinct i, j. Since $\rho_{1,2}$ is odd, we have one of the following:

(i) a_1 dominates a_2 from the left;
(ii) a_1^{-1} dominates a_2 from the left;
(iii) a_2 dominates a_1 from the left; or
(iv) a_2^{-1} dominates a_1 from the left.

We may interchange a_1 with a_1^{-1}, a_2 and a_2^{-1} in cases (ii), (iii) and (iv) respectively, without changing S up to isomorphism, so we may suppose that case (i) holds. Then

$$S_{a_1,a_2} \;=\; \langle a_1, a_2 \mid a_1^2 = a_2^2 = a_1 a_2 = a_1 a_2^{-1} = 0\rangle \;,$$

which starts an induction. Suppose that $2 \le k < n$ and, as an inductive hypothesis, we may reorder the generators and rewrite the relators so that

$$S_{a_1,\ldots,a_k} \;=\; \langle\, a_1, \ldots, a_k \mid a_i^2 = a_i a_j = a_i a_j^{-1} = 0\,,\; 1 \le i < j \le k \,\rangle \;. \tag{10}$$

Since $\rho_{k,k+1}$ is odd, we have one of the following:

(i) a_k dominates a_{k+1} from the left;
(ii) a_k^{-1} dominates a_{k+1} from the left;
(iii) a_{k+1} dominates a_k from the left; or
(iv) a_{k+1}^{-1} dominates a_k from the left.

If case (i) holds, then, from (10), we have that a_i dominates a_k from the left for all $i \le k$, so that a_i also dominates a_{k+1} from the left, by part (ii) of Lemma 6.3, yielding

$$S_{a_1,\ldots,a_{k+1}} \;=\; \langle\, a_1, \ldots, a_{k+1} \mid a_i^2 = a_i a_j = a_i a_j^{-1} = 0\,,\; 1 \le i < j \le k+1 \,\rangle \;, \tag{11}$$

establishing the inductive step. In case (ii), we may interchange a_k and a_k^{-1}, so that (10) still holds, up to reordering generators and rewriting relators, and we are back in case (i).

Suppose that case (iii) holds, so that a_{k+1} dominates a_k from the left. Hence there is a smallest m such that $1 \le m \le k$ and a_{k+1} dominates a_m from the left. Suppose first that $m = 1$, so that a_{k+1} dominates a_1 from the left. If $2 \le j \le k$ then a_1 dominates a_j from the left, by (10), so that a_{k+1} also dominates a_j from the left, by part (ii) of Lemma 6.3. Thus

$$S_{a_1,\ldots,a_{k+1}} \;=\; \langle a_1, \ldots, a_{k+1} \mid a_i^2 = a_{k+1} a_1 = a_{k+1} a_1^{-1} = a_i a_j = a_i a_j^{-1} = 0\,,\; 1 \le i < j \le k\rangle \;,$$

which becomes (11), after a cyclic permutation of the generators and rewriting the relators, establishing the inductive step. Suppose now that $m > 1$ and consider ℓ such that $1 \le \ell \le m-1$. By (10), a_ℓ dominates a_k from the left. By part (iii) of Lemma 6.3, either a_{k+1} dominates ℓ from the left or a_ℓ dominates a_{k+1} from the left. The first alternative is excluded by the minimality of m. Hence a_ℓ dominates a_{k+1} from the left. If $m < j \le k$ then a_m dominates a_j from the left, by (11), so that a_{k+1} also dominates a_j from the left, by part (ii) of Lemma 6.3. Thus

$$\begin{aligned} S_{a_1,\dots,a_{k+1}} \;=\; \langle\, a_1,\dots,a_n \mid a_i^2 = a_ia_j = a_ia_j^{-1} = 0\,,\ 1 \le i < j \le k\,, \\ a_pa_{k+1} = a_pa_{k+1}^{-1} = a_{k+1}a_q = a_{k+1}a_q^{-1} = 0\,,\ 1 \le p < m \le q \le n \,\rangle\,, \end{aligned}$$

which becomes (11), after a cyclic permutation of the generators $a_m, \dots, a_{k+1}$ and rewriting the relators, establishing the inductive step. In case (iv), we may interchange a_{k+1} and a_{k+1}^{-1} without disturbing (10), and we are back in case (iii). This completes the induction and the proof of necessity. □

Corollary 6.5. *Let $S = \langle A \mid c_i = 0$ for $i = 1, \dots, L\rangle$ be a finitely presented Rees quotient of a free inverse semigroup given by a two-standard presentation using $n \ge 2$ generators and $L = n^2$ relators. Then S is finite if and only if $a^2 = 0$ in S for all $a \in A$ and $A \cup A^{-1}$ contains a subset A' of size n such that A' is totally ordered by domination from the left.*

Theorem 6.6. *Let S be a finitely presented Rees quotient of a free inverse semigroup given by an irredundant presentation*

$$S \;=\; \langle A \mid c_1 = \ldots = c_L = 0\rangle$$

with $|A| = n \ge 2$ generators. If S is finite then $L \ge n^2$. If $L = n^2$ then S is finite if and only if

$$S \;\cong\; \langle a_1, \dots, a_n \mid a_i^{p_i} = c_{i,j} = a_ia_j^{-1} = 0\,,\ \ 1 \le i < j \le n\rangle$$

where the following conditions hold:

(i) $p_i \ge 2$ *for* $1 \le i \le n$;
(ii) $c_{i,j} = a_ia_j$ *for* $1 \le i < j - 1 \le n - 1$;
(iii) $c_{i,i+1} \in (a_ia_{i+1})^+ \cup (a_ia_{i+1})^+a_i$ *for* $1 \le i < n$;
(iv) $c_{i+1,i+2} = a_{i+1}a_{i+2}$ *if* $2 \le i + 1 < n$ *and* $|c_{i,i+1}| > 2$;
(v) $p_{i+1} = 2$ *if* $1 \le i < n$ *and* $|c_{i,i+1}| > 2$;
(vi) $p_i = 2$ *if* $1 \le i < n$ *and* $|c_{i,i+1}| > 3$.

Proof. We may suppose that $A = \{a_1, \dots, a_n\}$. Suppose first that S is finite. In particular, $\langle a_1, \dots, a_n\rangle$ is finite, so by Proposition 6.1, at least $\frac{n(n+1)}{2}$ relators appear in the presentation for S using only words over A. But, since S is finite, the presentation must also include relators of length at least two, with distinct contents, that divide powers of $a_ia_j^{-1}$ for $1 \le i < j \le n$, contributing a further $\frac{n(n-1)}{2}$ relators. Hence $L \ge \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$.

Suppose now that $L = n^2$. To prove necessity, suppose that S is finite. By observations in the previous paragraph and from the proof of Proposition 6.1, we may assume, without any loss of generality, that

$$S \;=\; \langle a_1, \dots, a_n \mid a_i^{p_i} = c_{i,j} = \overline{c_{i,j}} = 0\,,\ \ 1 \le i < j \le n\rangle\,,$$

for some $p_i \ge 2$, for $1 \le i \le n$,

$$c_{i,j} \;\in\; (a_ia_j)^+ \cup (a_ia_j)^+a_i \cup (a_ja_i)^+ \cup (a_ja_i)^+a_j\;, \tag{12}$$

and

$$\overline{c_{i,j}} \;\in\; (a_ia_j^{-1})^+ \cup (a_ia_j^{-1})^+a_i \cup (a_j^{-1}a_i)^+ \cup (a_j^{-1}a_i)^+a_j^{-1}\;, \tag{13}$$

for $1 \le i < j \le n$. Let T be any homomorphic image of S given by a two-standard presentation. By Theorem 6.4, we may assume, after reordering generators from $A \cup A^{-1}$ and rewriting relators, if necessary, that

$$T \;=\; \langle\, a_1, \ldots , a_n \mid a_i^2 = a_i a_j = a_i a_j^{-1} = 0\,,\;\; 1 \le i < j \le n \,\rangle\,, \tag{14}$$

formed from the presentation for S, by choosing $a_i a_j$ and $a_i a_j^{-1}$ to be subwords of $c_{i,j}$ and $\overline{c_{i,j}}$ respectively, for $1 \le i < j \le n$.

Our next step is to prove the following:

$$c_{n-1,n} \;=\; a_{n-1} a_n \quad \text{if } \; |\overline{c_{n-1,n}}| > 2. \tag{15}$$

Suppose that $|\overline{c_{n-1,n}}| > 2$, so that $\overline{c_{n-1,n}}$ is a word of length at least three that alternates in a_{n-1} and a_n^{-1}, in some order. Suppose first that $|c_{n-1,n}| > 2$. Then $c_{n-1,n}$ is a word of length at least three that alternates in a_{n-1} and a_n, in some order. Put

$$w \;=\; a_n a_{n-1} a_n^{-1} a_{n-1}^{-1}\,.$$

Then no positive power of w is divided by any word of length three that alternates in a_{n-1} and a_n, or alternates in a_{n-1} and a_n^{-1}, so cannot be divided by any relator of S. Hence the subsemigroup generated by w is infinite, contradicting that S is finite. Hence $|c_{n-1,n}| = 2$, so $c_{n-1,n} \in \{a_{n-1}a_n, a_n a_{n-1}\}$. Since $a_n a_{n-1}$ does not appear as a relator in T, we conclude that $c_{n-1,n} = a_{n-1}a_n$. This completes the proof that (15) holds.

If $|\overline{c_{n-1,n}}| = 2$ then the following holds automatically (because $a_{n-1}a_n^{-1}$ must be a subword of $\overline{c_{n-1,n}}$, in order to be included as a relator in the presentation for T):

$$\overline{c_{n-1,n}} \;=\; a_{n-1} a_n^{-1}\,. \tag{16}$$

If $|\overline{c_{n-1,n}}| > 2$ then, by (15), we may interchange a_n and a_n^{-1} in the presentation of S, so that $c_{n-1,n}$ is transformed into $a_{n-1}a_n^{-1}$, so that (16) continues to hold, and $\overline{c_{n-1,n}}$ is transformed into an element of $(a_{n-1}a_n)^+ \cup (a_{n-1}a_n)^+ a_{n-1} \cup (a_n a_{n-1})^+ \cup (a_n a_{n-1})^+ a_n$, so that neither (12) nor (13) is disturbed after the transformation. Observe that interchanging a_n and a_n^{-1} has no material effect on T: all of the relators with content of size two are reproduced and the relator a_n^2 is replaced by a_n^{-2}, which is $\mathcal{J}$-equivalent to a_n^2. Thus we may assume (16) holds in all cases, (12) and (13) remain undisturbed, and T continues to have the presentation given by (14).

Our next step is to prove the following:

$$\overline{c_{i,j}} \;=\; a_i a_j^{-1} \quad \text{if } \; 1 \le i < j \le n \text{ and } i \ne n-1\,. \tag{17}$$

Suppose to the contrary that $\overline{c_{i_0,j_0}} \ne a_{i_0} a_{j_0}^{-1}$ for some i_0 and j_0 such that $1 \le i_0 < j_0 \le n$ and $i_0 \ne n-1$. Then $a_{j_0}^{-1} a_{i_0}$ is a subword of $\overline{c_{i_0,j_0}}$, so that we may form a new homomorphic image T' of S, modifying the presentation for T, replacing $a_{i_0} a_{j_0}^{-1}$ with $a_{j_0}^{-1} a_{i_0}$, to get the following two-standard presentation:

$$T' = \langle a_1, \ldots , a_n \,|\, a_i^2 = a_i a_j = a_i a_j^{-1} = a_{i_0} a_{j_0} = a_{j_0}^{-1} a_{i_0} = 0,\; 1 \le i < j \le n,\; (i,j) \ne (i_0, j_0)\rangle.$$

Put

$$w \;=\; \begin{cases} a_{i_0} a_{j_0}^{-1} a_n & \text{if } j_0 < n, \\ a_{i_0} a_n^{-1} a_{n-1} & \text{if } j_0 = n. \end{cases}$$

But, in each case, the subsemigroup of T' generated by w is infinite (since no positive power of w is divided by any relator in the presentation of T'), which contradicts that T' is finite. This completes the proof that (17) holds.

By (16) and (17), we have

$$\overline{c_{i,j}} \;=\; a_i a_j^{-1} \;\text{ for all } i,\, j \text{ such that } 1 \le i < j \le n. \tag{18}$$

By Proposition 6.1, we may further permute the generators from amongst A only, and rewrite relators, if necessary, so that conditions (i)-(vi) hold. Observe, by applying any permutation of A, and rewriting relators, (18) becomes

$$\overline{c_{i,j}} \;=\; a_j a_i^{-1} \;\text{ or }\; \overline{c_{i,j}} \;=\; a_i a_j^{-1} \;\text{ for all } i,\, j \text{ such that } 1 \le i < j \le n.$$

But, for each i and j, we have

$$a_j a_i^{-1} \;\mathcal{J}\; a_i a_j^{-1}$$

so that we can replace $a_j a_i^{-1}$ by $a_i a_j^{-1}$, if necessary, in the presentation for S, so that (18) still holds. Since conditions (i)-(vi) hold, this completes the proof of necessity.

To prove sufficiency, we may suppose, after rewriting generators and relators (see Proposition 2.1), that

$$S \;=\; \langle a_1, \ldots, a_n \mid a_i^{p_i} = c_{i,j} = a_i a_j^{-1} = 0\,,\;\; 1 \le i < j \le n \rangle$$

such that conditions (i)-(vi) hold. To prove S is finite, it suffices to show that there are only finitely many reduced words that are nonzero in S. It follows immediately from the relations that any nonempty reduced word w that is nonzero in S has a factorisation

$$w \;=\; u^{-1} v$$

where u and v are reduced words over the alphabet A that are nonzero in S and not both empty. But such nonempty words are described in the proof of Proposition 6.1, namely, products of the form

$$w_n w_{n-1} \ldots w_1$$

where w_i is a (possibly empty) subword of $a_i^{p_i - 2} c_{i,i+1}$ such that $c_{i,i+1}$ is not a suffix of w_i, for $1 \le i < n$, w_n is a proper subword of $a_n^{p_n}$, and not all of $w_1, \ldots, w_n$ are empty. There are only finitely many such words, and it follows that S is finite, completing the proof of sufficiency. □

Using the total ordering of the alphabet A, implemented in the proof of Proposition 6.1, we get the following inverse semigroup analogue of Corollary 6.2:

Corollary 6.7. *Let $S = \langle A \mid c_i = 0$ for $i = 1, \ldots, L\rangle$ be a finitely presented Rees quotient of a free inverse semigroup given by an irredundant presentation using $n \ge 2$ generators and $L = n^2$ relators. Then S is finite if and only if all generators are nilpotent and $A \cup A^{-1}$ contains a subset A' of size n such that there is a partition*

$$A' \;=\; A'_1 \cup \ldots \cup A'_{n-r}$$

into $n-r$ disjoint subsets, consisting of r subsets of size two and $n-2r$ subsets of size one, where $0 \le r \le \frac{n}{2}$, such that

(i) $\mathrm{Inv}(A'_i)$ *is a finite subsemigroup of S for each i, and*
(ii) $A'_i A'_j = \{0\}$ *for $1 \le i < j \le n-r$.*

7. Presentations Achieving the Sharp Lower Bound

In this section, a complete description is given of semigroups S from the class $\mathfrak{M}_{FI}$, such that S has polynomial growth and S is given by an irredundant presentation of the form (1) using $n \geq 2$ generators and $L = n^2 - 1$ relators. The arguments rely on the description in the previous section of finite inverse semigroups given by irredundant presentations involving $m \geq 2$ generators and m^2 relators. The section begins with three lemmas and a corollary, which are keys to the description that follows. The description is organised in three pairs of theorems, corresponding to the three classes of semigroups involving zero, one and two non-nilpotent generators respectively. In each of these pairs of theorems, the first is concerned with two-standard presentations and the second with general irredundant presentations.

Lemma 7.1. *Let $S = \langle A \mid c_1, \ldots, c_L = 0 \rangle$ be an irredundant presentation with $n \geq 2$ generators and $L = n^2 - 1$ relators. Suppose that $S' = \langle A \mid c'_1, \ldots, c'_{L'} = 0 \rangle$ is a two-standard presentation obtained from S by first taking, as relators, L reduced words of length two, in succession, that divide $c_1, \ldots, c_L$ respectively, and then removing duplicates up to $\mathcal{J}$-equivalence. If S has polynomial growth then*

(i) *$L' = L = n^2 - 1$;*
(ii) *for $i \neq j$, the relators c_i and c_j have no divisors in common that are reduced words of length two;*
(iii) *we may reorder the relators so that c'_i divides c_i for $i = 1, \ldots, L$.*

Proof. Suppose that S has polynomial growth. Then S' has polynomial growth, since S' is a homomorphic image of S. Certainly $L' \leq L$. By Theorem 5.13, $L' \geq n^2 - 1 = L$, so $L' = L$, verifying part (i).

To prove part (ii), assume to the contrary that there exist i and j such that $i \neq j$ and w is a reduced word of length two that divides both c_i and c_j. We may then form a two-standard presentation S' from S by first choosing w for both c_i and c_j and any other respective divisors for c_k where $k \neq i, j$. But then, to obtain irredundancy in finally forming this choice of S', at least one of the duplicates for w must be removed, so that $L' < L$, contradicting part (i). This proves part (ii), and then part (iii) is immediate. □

Lemma 7.2. *Let $S = \langle A \mid c_1, \ldots, c_L = 0 \rangle$ be an irredundant presentation with $n \geq 2$ generators and $L = n^2 - 1$ relators. If S has polynomial growth then the content of the relator c_i has size at most two for $1 \leq i \leq L$.*

Proof. Suppose that S has polynomial growth. We may write $A = \{a_1, \ldots, a_n\}$. Let S' be any homomorphic image of S that has a two-standard presentation obtained by choosing relators that are reduced words of length two that divide $c_1, \ldots, c_L$. Then S' has polynomial growth. By Lemma 7.1,

$$S' \; = \; \langle \, A \mid c'_1, \ldots, c'_L = 0 \, \rangle \tag{19}$$

where c'_i divides c_i for $i = 1, \ldots, L$. By part (ii) of Lemma 7.1, c_i and c_j have no divisors in common that are reduced words of length two, for $i \neq j$, a fact which is used implicitly in the argument below.

We argue by contradiction, and suppose that there exists a relator c_ℓ having content of size larger than two, for some $\ell \in \{1, \dots, L\}$. The presentation (19) can be of type (i), (ii) or (iii), as described in Scholium 5.15.

Case 1. Suppose that the presentation (19) is of type (i) in Scholium 5.15. Without loss of generality, a_1 is not nilpotent and, in view of (i)$'$, $R_{i,j} = 2$ for all $i \neq j$.

Assume first that c'_ℓ is the square of a letter. Without loss of generality, $c'_\ell = a_2^2$. Since $|\text{content}(c_\ell)| \geq 3$, there exist distinct $k, q \in \{1, \dots, n\}$, such that $k \neq 1, 2$, and a word $a_k^\varepsilon a_q^\delta$ that divides c_ℓ, for some $\varepsilon, \delta \in \{\pm 1\}$. Replacing $c'_\ell = a_2^2$ by $a_k^\varepsilon a_q^\delta$ in (19) does not alter $R_{1,2} = 2$, yet, in this new two-standard presentation, a_2 is not nilpotent, in addition to a_1, so that $R_{1,2} = 3$, by part (ii)$'$ of Scholium 5.15, which is a contradiction. This proves that c'_ℓ is not the square of a letter.

Hence $c'_\ell = a_i^\beta a_j^\gamma$ for some $i \neq j$ and $\beta, \gamma \in \{\pm 1\}$. Since $|\text{content}(c_\ell)| \geq 3$, there exist distinct k, q, such that $k \neq i, j$, and a word $v = a_k^\varepsilon a_q^\delta$ that divides c_ℓ, for some $\varepsilon, \delta \in \{\pm 1\}$. Replacing c'_ℓ by v in (19) does not alter the number of nilpotent generators (which remains steady at $n-1$), but $R_{i,j} = 1$ in this new presentation, which is impossible by part (i)$'$ of Scholium 5.15.

This proves that **Case 1** does not occur.

Case 2. Suppose that (19) is of type (ii) in Scholium 5.15. Without loss of generality, a_1 and a_2 are not nilpotent, $R_{1,2} = 3$ and $R_{1,j} = R_{i,j} = 2$ whenever $2 \leq i < j \leq n$.

Assume first that $c'_\ell = a_i^2$ for some $i \geq 3$. Since $|\text{content}(c_\ell)| \geq 3$, as in **Case 1**, there exist distinct k, q and a word $a_k^\varepsilon a_q^\delta$ that divides c_ℓ, for some $\varepsilon, \delta \in \{\pm 1\}$. Replacing c'_ℓ by $a_k^\varepsilon a_q^\delta$ in (19) produces a new two-standard presentation with more than two generators that are not nilpotent, contradicting Scholium 5.15. This proves that c'_ℓ is not the square of a letter.

Hence $c'_\ell = a_i^\beta a_j^\gamma$ for some $i \neq j$ and $\beta, \gamma \in \{\pm 1\}$. Since $|\text{content}(c_\ell)| \geq 3$, there exist distinct k, q, such that $k \neq i, j$, and a word $v = a_k^\varepsilon a_q^\delta$ that divides c_ℓ, for some $\varepsilon, \delta \in \{\pm 1\}$. Replacing c'_ℓ by v in (19) does not alter the nilpotent generators (which are $a_3, \dots, a_n$), but, in this new presentation,

$$R_{i,j} = \begin{cases} 1 & \text{if } \{i,j\} \neq \{1,2\} \\ 2 & \text{if } \{i,j\} = \{1,2\}, \end{cases}$$

which is impossible by part (ii)$'$ of Scholium 5.15.

This proves that **Case 2** does not occur.

Case 3. Suppose that (19) is of type (iii) in Scholium 5.15. Thus all generators are nilpotent and, without loss of generality, $R_{1,2} = 1$ and $R_{1,j} = R_{i,j} = 2$ whenever $2 \leq i < j \leq n$.

Assume first that $c'_\ell = a_i^2$ for some i. Since $|\text{content}(c_\ell)| \geq 3$, there exist distinct k, q, such that $\{k, q\} \neq \{1, 2\}$, and a word $a_k^\varepsilon a_q^\delta$ that divides c_ℓ, for some $\varepsilon, \delta \in \{\pm 1\}$. Replacing c'_ℓ by $a_k^\varepsilon a_q^\delta$ in (19) produces a new two-standard presentation with exactly $n-1$ nilpotent generators, but with $R_{k,q} = 3$, contradicting part (i)$'$ of Scholium 5.15. This proves that c'_ℓ is not the square of a letter.

Hence $c'_\ell = a_i^\beta a_j^\gamma$ for some $i \neq j$ and $\beta, \gamma \in \{\pm 1\}$. Since $|\text{content}(c_\ell)| \geq 3$, there exist distinct k, q, such that $k \neq i, j$, and a word $v = a_k^\varepsilon a_q^\delta$ that divides c_ℓ, for some $\varepsilon, \delta \in \{\pm 1\}$. Replacing c'_ℓ by v in (19) does not alter the nilpotent generators (which are all of $a_1, \dots, a_n$), but, in this new presentation,

$$R_{i,j} \;=\; \begin{cases} 1 & \text{if } \{i,j\} \neq \{1,2\} \\ 0 & \text{if } \{i,j\} = \{1,2\}, \end{cases}$$

which is impossible by part (iii)′ of Scholium 5.15.

This proves that **Case 3** does not occur, and completes the proof of the lemma. □

Lemma 7.3. *Let* $S = \langle a_1, \ldots, a_n \mid c_1, \ldots, c_L = 0\rangle$ *be given by an irredundant presentation using* $n \geq 3$ *generators. Suppose that the following hold:*

(i) *there is exactly one relator* c *with content* $\{a_1, a_2\}$ *and* $c \mathrel{\mathcal{J}} a_1 a_2^{-1}$*;*

(ii) *for* $3 \leq j \leq n$ *there are exactly two relators with content* $\{a_1, a_j\}$*, exactly two relators with content* $\{a_2, a_j\}$ *and no relators with content* $\{a_1, a_2, a_j\}$*.*

If S *has polynomial growth then, for each* $j \geq 3$ *there exist an integer* $p_j \geq 2$*, such that*

$$S_{1,2,j} \;\cong\; \langle\, a_1, a_2, a_j \mid a_1 a_2^{-1} = a_1 a_j = a_1 a_j^{-1} = a_2 a_j = a_2 a_j^{-1} = a_1^2 = a_2^2 = a_j^{p_j} = 0 \,\rangle \ .$$

Proof. Suppose that S has polynomial growth and $3 \leq j \leq n$. We may suppose that $c_1 = c = a_1 a_2^{-1}$, that the two relators with content $\{a_1, a_j\}$ are c_2 and c_3, and that the two relators with content $\{a_2, a_j\}$ are c_4 and c_5. Then

$$S_{1,2,j} \;=\; \langle a_1, a_2, a_3 \mid a_1 a_2^{-1} = c_2 = c_3 = c_4 = c_5 = c_6 = \ldots = c_{L'} = 0\rangle \tag{20}$$

where $c_6, \ldots, c_{L'}$, for some $L' \leq L$, denote all of the other relators in the presentation for S with content contained in $\{a_1, a_2, a_j\}$. But $S_{1,2,j}$ is irredundant and has polynomial growth. By Theorem 5.13, $L' \geq 8$. Also, $S_{1,2}$ has polynomial growth, so, by Theorem 2.8, rewriting generators, if necessary, by their inverses, we may suppose that

$$S_{1,2} \;=\; \langle a_1, a_2 \mid a_1^2 = a_2^2 = a_1 a_2^{-1} = 0\rangle \ ,$$

and that $c_6 = a_1^2$ and $c_7 = a_2^2$. By condition (ii), we have

$$c_8, \ldots, c_{L'} \;\subseteq\; \{a_1^{\gamma_1}, a_2^{\gamma_2}, a_j^{\gamma_3} \mid \gamma_1, \gamma_2, \gamma_3 \ \text{ are nonzero integers}\} \ .$$

By irredundancy of the presentation (20) it follows that $L' = 8$ and

$$\{c_8\} \;=_{\mathcal{J}}\; \{a_j^{p_j}\}$$

for some integer $p_j \geq 2$. There is no loss in generality therefore in supposing that (20) becomes

$$S_{1,2,j} \;=\; \langle a_1, a_2, a_j \mid a_1 a_2^{-1} = c_2 = c_3 = c_4 = c_5 = a_1^2 = a_2^2 = a_j^{p_j} = 0\rangle \ . \tag{21}$$

Let c_i' be any reduced word of length two that divides c_i for $2 \leq i \leq 5$. By part (ii) of Lemma 7.1, since the number of relators is $8 = 3^2 - 1$, it follows that

$$c_i' \;\notin\; \{a_1^{\pm 2}, a_2^{\pm 2}, a_j^{\pm 2}\}$$

for $2 \leq i \leq 5$. To prove the lemma, therefore, it suffices to show

$$\{c_2', c_3'\} \;=_{\mathcal{J}}\; \{a_1 a_j, a_1 a_j^{-1}\} \qquad \text{and} \qquad \{c_4', c_5'\} \;=_{\mathcal{J}}\; \{a_2 a_j, a_2 a_j^{-1}\} \ , \tag{22}$$

for then it follows that

$$\{c_2, c_3\} \;=_{\mathcal{J}}\; \{a_1 a_j, a_1 a_j^{-1}\} \qquad \text{and} \qquad \{c_4, c_5\} \;=_{\mathcal{J}}\; \{a_2 a_j, a_2 a_j^{-1}\} \ .$$

Put

$$S'_{1,2,j} \;=\; \langle a_1, a_2, a_j \mid a_1a_2^{-1} = c'_2 = c'_3 = c'_4 = c'_5 = a_1^2 = a_2^2 = a_j^2 = 0\rangle \;.$$

Then $S'_{1,2,j}$ is a two standard presentation formed from $S_{1,2,j}$, and $S'_{1,2,j}$ has polynomial growth. Observe that, with respect to $S'_{1,2,j}$, we have

$$\rho_{1,2} \;=\; \{a_1a_2^{-1}\}\,, \quad \rho_{1,j} \;=\; \{c'_2, c'_3\} \quad \text{and} \quad \rho_{2,j} \;=\; \{c'_4, c'_5\}\;.$$

Then (22) now follows, by Lemma 5.6, which completes the proof. □

The following corollary, which simplifies condition (ii) of the previous lemma, is immediate by Lemma 7.2:

Corollary 7.4. *Let $S = \langle a_1, \ldots, a_n \mid c_1, \ldots, c_L = 0\rangle$ be given by an irredundant presentation using $n \geq 3$ generators and $L = n^2 - 1$ relators. Suppose that the following hold:*

(i) *there is exactly one relator c with content $\{a_1, a_2\}$ and $c \;\mathcal{J}\; a_1a_2^{-1}$;*
(ii) *for $3 \leq j \leq n$ there are exactly two relators with content $\{a_1, a_j\}$ and exactly two relators with content $\{a_2, a_j\}$.*

If S has polynomial growth then, for each $j \geq 3$ there exists an integer $p_j \geq 2$, such that

$$S_{1,2,j} \;\cong\; \langle\, a_1, a_2, a_j \mid a_1a_2^{-1} = a_1a_j = a_1a_j^{-1} = a_2a_j = a_2a_j^{-1} = a_1^2 = a_2^2 = a_j^{p_j} = 0\,\rangle\;.$$

The following theorem generalises part of Theorem 2.8, in the case that all generators are nilpotent.

Theorem 7.5. *Let $S = \langle A \mid c_1, \ldots, c_L = 0\rangle$ be a semigroup with polynomial growth given by a two-standard presentation using $n \geq 2$ nilpotent generators and $L = n^2 - 1$ relators. If $n = 2$ then*

$$S \;\cong\; \langle\, a_1, a_2 \mid a_1^2 = a_2^2 = a_1a_2^{-1} = 0\,\rangle\;.$$

If $n \geq 3$ then S is isomorphic to

$$\langle\, a_1, \ldots, a_n \mid a_1^2 = \ldots = a_n^2 = a_1a_2^{-1} = a_1a_j = a_1a_j^{-1} = a_ia_j = a_ia_j^{-1} = 0\,,\; 2 \leq i < j \leq n\,\rangle\,.$$

Proof. We may assume that $A = \{a_1, \ldots, a_n\}$. By part (iii)′ of Scholium 5.15, we may assume that

$$c_i \;=\; a_i^2 \tag{23}$$

for $1 \leq i \leq n$, and that

$$R_{1,2} = 1 \qquad \text{and} \qquad R_{1,j} = R_{i,j} = 2$$

for $2 \leq i < j$. Since $R_{1,2} = 1$, by renaming generators, if necessary, there is no loss of generality in assuming that

$$\rho_{1,2} \;=\; \{a_1a_2^{-1}\}\;. \tag{24}$$

Hence

$$S_{1,2} \;=\; \langle a_1, a_2 \mid a_1^2 = a_2^2 = a_1a_2^{-1} = 0\rangle\;.$$

This proves the theorem in the case $n = 2$ (and also gives part of Theorem 2.8, where the generators are nilpotent).

We may henceforth assume that $n \geq 3$. Consider $3 \leq j \leq n$. Observe that $S_{1,2,j}$ has polynomial growth. Then a_1 and a_2 dominate a_j from the left, by Lemma 5.6. Hence, we may suppose, without loss in generality, that

$$\rho_{1,j} \;=\; \{a_1a_j\,,\; a_1a_j^{-1}\} \qquad \text{and} \qquad \rho_{2,j} \;=\; \{a_2a_j\,,\; a_2a_j^{-1}\}\,. \tag{25}$$

By Lemma 2.6, taking $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3, \ldots, a_n\}$ and $a = a_1$ (or $a = a_2$), it follows that $\mathrm{Inv}(a_3, \ldots, a_n)$ is finite. But the presentation for $S_{3,\ldots,n}$ uses $n-2$ generators and

$$L - 3 - 4(n-2) \;=\; n^2 - 1 - 4n + 5 \;=\; (n-2)^2$$

relators. Hence, by Corollary 6.5, $A_2 \cup A_2^{-1}$ contains a subset of size $n-2$ that is totally ordered by domination from the left. We may therefore rewrite A_2, so that, without loss of generality,

$$\rho_{i,j} \;=\; \{a_ia_j\,,\; a_ia_j^{-1}\}\,. \tag{26}$$

for $3 \leq i < j \leq n$. Note that this rewriting of A_2 does not disturb either (24) or (25), and does not disturb (23) up to $\mathcal{J}$-equivalence. By (23), (24), (25) and (26), the presentation for S in the statement of the theorem is proved, up to isomorphism. □

Theorem 7.6. *Let $S = \langle a_1, \ldots, a_n \mid c_1, \ldots, c_L = 0\rangle$ be given by an irredundant presentation using $n \geq 2$ nilpotent generators and $L = n^2 - 1$ relators. If $n = 2$ then S has polynomial growth if and only if S is isomorphic to*

$$\langle a, b \mid a^2 = b^2 = ab = 0\rangle \;\cong\; \langle a, b \mid a^2 = b^2 = ab^{-1} = 0\rangle\,.$$

If $n \geq 3$ then S has polynomial growth if and only if the generators may be reordered and the relators rewritten, up to $\mathcal{J}$-equivalence, such that the following hold:

1. $\mathrm{Inv}(a_1, a_2) = \langle a_1, a_2 \mid a_1^2 = a_2^2 = a_1a_2^{-1} = 0\rangle$*;*
2. $\mathrm{Inv}(a_3, \ldots, a_n)$ *is finite given by* $(n-2)^2$ *relators;*
3. $\{a_1, a_2\}\mathrm{Inv}(a_3, \ldots, a_n) = \{0\}$*, using* $4n - 8$ *relators* $a_ia_j = a_ia_j^{-1} = 0$ *for* $i = 1, 2$ *and* $3 \leq j \leq n$*.*

Proof. Observe that the claim of the theorem holds immediately for $n = 2$ by Theorem 2.8, in the case that the generators are nilpotent. Hence we may suppose that $n \geq 3$.

Put $A = \{a_1, \ldots, a_n\}$. To prove necessity, suppose that S has polynomial growth. We may assume that

$$c_1 \;=\; a_1^{p_1}\,,\; \ldots\,,\; c_n \;=\; a_n^{p_n}$$

for some integers $p_1, \ldots, p_n \geq 2$. Let S' be any homomorphic image of S that has a two-standard presentation obtained by choosing relators that are reduced words of length two that divide $c_1, \ldots, c_L$ respectively. By Lemma 7.1,

$$S' \;=\; \langle A \mid c_1', \ldots, c_L' = 0\rangle \tag{27}$$

where we may assume c_i' divides c_i for $i = 1, \ldots, L$. In particular, we may assume

$$c_1' = a_1^2\,,\; \ldots\,,\; c_n' = a_n^2\,.$$

Part (ii) of Lemma 7.1 guarantees that a_i^2 does not divide c_j for $1 \leq i \leq n$ and $j > n$. By Theorem 7.5, S' is isomorphic to

$$T_n \;=\; \langle\, A \,|\, a_1^2 = \ldots = a_n^2 = a_1a_2^{-1} = a_1a_j = a_1a_j^{-1} = a_ia_j = a_ia_j^{-1} = 0\,,\; 2 \leq i < j \leq n\,\rangle\,,$$

and there is no loss in generality in supposing

$$S' = T_n \; .$$

For $1 \le i < j \le n$, denote by $c_{i,j}$ and $\overline{c_{i,j}}$ the relators in the presentation of S for which the relators $a_i a_j$ and $a_i a_j^{-1}$, respectively, were chosen as divisors in forming the presentation $S' = T_n$. Thus the presentation for S may be rewritten as follows:

$$S \;=\; \langle\, A \mid a_1^{p_1} = \ldots = a_n^{p_n} = \overline{c_{1,2}} = c_{1,j} = \overline{c_{1,j}} = c_{i,j} = \overline{c_{i,j}} = 0 \;,\; 2 \le i < j \le n \,\rangle \; . \qquad (28)$$

By Lemma 7.2,

$$\text{content}(\overline{c_{1,2}}) = \{a_1, a_2\} \qquad \text{and} \qquad \text{content}(c_{i,j}) = \text{content}(\overline{c_{i,j}}) = \{a_i, a_j\},$$

for all $i < j$ where $j \ge 3$. Observe that

$$S_{1,2} \;=\; \langle\, a_1, a_2 \mid a_1^{p_1} = a_2^{p_2} = \overline{c_{1,2}} = 0 \,\rangle \;\cong\; \langle\, a, b \mid a^2 = b^2 = ab^{-1} \,\rangle \; .$$

The isomorphism implies that $p_1 = p_2 = 2$, and, without loss of generality, after reordering the generators and rewriting the relators, up to $\mathcal{J}$-equivalence, that $\overline{c_{1,2}} = a_1 a_2^{-1}$ and

$$\text{Inv}(a_1, a_2\} \;=\; S_{1,2} \;=\; \langle a_1, a_2 \mid a_1^2 = a_2^2 = a_1 a_2^{-1} = 0\rangle \; ,$$

completing the proof of part (1) of the theorem.

If $3 \le j \le n$ then, from (28), clearly conditions (i) and (ii) of Corollary 7.4 hold, so that, after rewriting generators and relators, up to $\mathcal{J}$-equivalence, we may suppose

$$S_{1,2,j} \;=\; \langle\, a_1, a_2, a_j \mid a_1 a_2^{-1} = a_1 a_j = a_1 a_j^{-1} = a_2 a_j = a_2 a_j^{-1} = a_1^{p_1} = a_2^{p_2} = a_j^{p_j} = 0 \,\rangle \; .$$

Thus, utilising $4n - 8$ relators, we now have

$$\{a_1, a_2\}\, \text{Inv}(a_3, \ldots, a_n) \;=\; \{0\} \; ,$$

completing the proof of part (3) of the theorem. By Lemma 2.6, taking $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3, \ldots, a_n\}$ and $a = a_1$ (or $a = a_2$), it follows that $\text{Inv}(a_3, \ldots, a_n)$ is finite. As before, the number of relators in the presentation for $S_{3,\ldots,n}$ is $(n-2)^2$, which completes the proof of part (2). This completes the proof of necessity.

To prove sufficiency, suppose that we have a semigroup S given by the presentation

$$S \;=\; \langle A \mid c_1, \ldots, c_L = 0\rangle$$

such that parts (1), (2) and (3) hold. We prove that S has polynomial growth. In particular, we may suppose that

$$a_1 a_2^{-1} \; , \;\; a_1 a_j \; , \;\; a_1 a_j^{-1} \;\; a_2 a_j \; , \;\; a_2 a_j^{-1}$$

are relators in the presentation of S for $3 \le j \le n$. Put

$$A_1 \;=\; \{a_1, a_2\} \qquad \text{and} \qquad A_2 \;=\; \{a_3, \ldots, a_n\} \; .$$

Then A_1 is left orthogonal and $A_1(A_2 \cup A_2^{-1}) = \{0\}$ in S. By part (1) and Theorem 2.8, $\text{Inv}(A_1)$ has polynomial growth. By part (2), $\text{Inv}(A_2)$ is finite. Hence, by Lemma 2.5, S has polynomial growth. This completes the proof of sufficiency. □

Theorem 7.7. *Let $S = \langle A \mid c_1, \ldots, c_L = 0\rangle$ be a semigroup with polynomial growth given by a two-standard presentation using $n \geq 2$ generators and $L = n^2 - 1$ relators. Suppose that exactly $n-1$ generators are nilpotent. Then S is isomorphic to*

$$\langle\, a_1, \ldots, a_n \mid a_2^2 = \ldots = a_n^2 = a_i a_j = a_i a_j^{-1} = 0\,,\ 1 \leq i < j \leq n\,\rangle\ .$$

Proof. Assume that $A = \{a_1, \ldots, a_n\}$. By part (i)$'$ of Scholium 5.15, we may suppose that $c_i = a_{i+1}^2$ for $1 \leq i \leq n-1$, and that $R_{i,j} = 2$ for $1 \leq i < j \leq n$. For $j \geq 2$, we have that $S_{1,j}$ has polynomial growth and uses three relators, so, by taking $C = a^2$ in Theorem 2.8, we deduce that

$$S_{1,j} \ \cong\ \langle a, b \mid ab = a^{-1}b = a^2 = 0\rangle \ \cong\ \langle a, b \mid b^2 = ab = ab^{-1} = 0\rangle\ . \tag{29}$$

The second isomorphism in (29) determines uniquely one letter of the alphabet that is not nilpotent and dominates a second letter from the left, and the second letter is determined up to inversion. In particular, we may rename generators and relators so that

$$S_{1,2} \ =\ \langle a_1, a_2 \mid a_2^2 = a_1 a_2 = a_1 a_2^{-1} = 0\rangle\ . \tag{30}$$

This proves the theorem in the case that $n = 2$ (following also directly from Theorem 2.8, for the case that exactly one generator is nilpotent). Henceforth we may suppose that $n \geq 3$.

It follows from the above that, for $j \geq 3$, we may rename generators and relators so that, for some $\varepsilon_j \in \{\pm 1\}$, we have

$$S_{1,j} \ =\ \langle a_1, a_j \mid a_j^2 = a_1^{\varepsilon_j} a_j = a_1^{\varepsilon_j} a_j^{-1} = 0\rangle\ . \tag{31}$$

We show that $\varepsilon_j = 1$ for each $j \geq 3$. Suppose, to the contrary that $\varepsilon_j = -1$ for some $j \geq 3$. Then, from (30) and (31), we have that a_1 is not nilpotent, a_1 dominates a_2 from the left, and a_1^{-1} dominates a_j from the left, so that a_1 dominates a_j from the right. But $R_{2,j} = 2 \leq 3$, so $S_{1,2,j}$ has exponential growth, by Lemma 5.7, which is impossible. Hence $\varepsilon_j = 1$ for each $j \geq 3$, so that (31) becomes

$$S_{1,j} \ =\ \langle a_1, a_j \mid a_j^2 = a_1 a_j = a_1 a_j^{-1} = 0\rangle\ . \tag{32}$$

It follows from Lemma 2.6, taking $A_1 = \{a_1\}$, $A_2 = \{a_2, \ldots, a_n\}$ and $a = a_1$, that $\mathrm{Inv}(a_2, \ldots, a_n)$ is finite. But the presentation for $S_{2,\ldots,n}$ uses $n-1$ generators and

$$L - 2(n-1) \ =\ n^2 - 1 - 2n + 2 \ =\ (n-1)^2$$

relators. Hence, by Corollary 6.5, $A_2 \cup A_2^{-1}$ contains a subset of size $n-1$ that is totally ordered by domination from the left. We may therefore rewrite A_2, so that, without loss of generality,

$$\rho_{i,j} \ =\ \{a_i a_j\,,\ a_i a_j^{-1}\}\ . \tag{33}$$

for $2 \leq i < j \leq n$. Note that this rewriting of A_2 does not disturb (32) up to $\mathcal{J}$-equivalence. By (32) and (33), the presentation for S in the statement of the theorem is proved, up to isomorphism. □

Theorem 7.8. *Let $S = \langle A \mid c_1, \ldots, c_L = 0\rangle$ be given by an irredundant presentation using $n \geq 2$ generators and $L = n^2 - 1$ relators. Suppose that exactly $n-1$ generators are nilpotent. Then S has polynomial growth if and only if the generators may be reordered and the relators rewritten, up to $\mathcal{J}$-equivalence, such that the following hold:*

(1) $\mathrm{Inv}(a_1)$ *is free monogenic (using no relators);*
(2) $\mathrm{Inv}(a_2, \ldots, a_n)$ *is finite using* $(n-1)^2$ *relators:*
(3) $\{a_1\}\mathrm{Inv}(a_2, \ldots, a_n) = \{0\}$, *using* $2n-2$ *relators* $a_1 a_j = a_1 a_j^{-1} = 0$ *for* $2 \le j \le n$.

Proof. To prove necessity, suppose that S has polynomial growth. We may assume $A = \{a_1, \ldots a_n\}$ and that a_1 is the unique generator in A that is not nilpotent. Hence $\mathrm{Inv}(a_1)$ is free monogenic, and part (1) holds. We may assume that

$$c_1 \;=\; a_2^{p_2}\,,\;\ldots\;,\; c_{n-1} \;=\; a_n^{p_n}$$

for some integers $p_2, \ldots, p_n \ge 2$. Let S' be any homomorphic image of S that has a two-standard presentation obtained by choosing relators that are reduced words of length two that divide $c_1, \ldots, c_L$ respectively. By Lemma 7.1,

$$S' \;=\; \langle A \mid c_1', \ldots, c_L' = 0 \rangle$$

where we may assume c_i' divides c_i for $i = 1, \ldots, L$. In particular, we may assume

$$c_1' = a_2^2\,,\;\ldots\;,\; c_{n-1}' = a_n^2\;.$$

Part (ii) of Lemma 7.1 guarantees that a_i^2 does not divide c_j for $1 \le i \le n$ and $j > n$. By Theorem 7.7, S' is isomorphic to

$$T_n \;=\; \langle\, A \,|\, a_2^2 = \ldots = a_n^2 = a_i a_j = a_i a_j^{-1} = 0\,,\; 1 \le i < j \le n \,\rangle\,,$$

and there is no loss in generality in supposing $S' = T_n$. For $1 \le i < j \le n$, denote by $c_{i,j}$ and $\overline{c_{i,j}}$ the relators in the presentation of S for which the relators $a_i a_j$ and $a_i a_j^{-1}$, respectively, were chosen as divisors in forming the presentation $S' = T_n$. Thus the presentation for S may be rewritten as follows:

$$S \;=\; \langle\, A \mid a_2^{p_2} = \ldots = a_n^{p_n} = c_{i,j} = \overline{c_{i,j}} = 0\;,\; 1 \le i < j \le n \,\rangle\;.$$

Consider $1 < j \le n$. By Lemma 7.2, we have

$$\mathrm{content}(c_{1,j}) = \mathrm{content}(\overline{c_{1,j}}) = \{a_1, a_j\}\;.$$

Further,

$$S_{1,j} \;=\; \langle\; a_1, a_j \mid a_j^{p_j} = c_{1,j} = \overline{c_{1,j}} = 0 \;\rangle\;.$$

From Theorem 2.8, it follows that

$$S_{1,j} \;\cong\; \langle\, a, b \mid ab = a^{-1}b = a^{\gamma} = 0 \,\rangle \;\cong\; \langle\, a, b \mid ab = ab^{-1} = b^{\gamma} = 0 \,\rangle$$

for some $\gamma \ge 2$, depending on j. The isomorphisms are determined by the unique letter, in each presentation, that dominates the other nilpotent letter and its inverse from the left. It follows that

$$\{c_{1,j}\;,\;\overline{c_{1,j}}\} \;=_{\mathcal{J}}\; \{a_1 a_j\;,\; a_1 a_j^{-1}\}\;.$$

Part (3) now follows immediately, noting that $2n-2$ relators are employed. By Lemma 2.6, as before, taking $A_1 = \{a_1\}$, $A_2 = \{a_2, \ldots, a_n\}$ and $a = a_1$, it follows that $\mathrm{Inv}(a_2, \ldots, a_n)$ is finite. As before, the number of relators in the presentation for $S_{2,\ldots,n}$ is $(n-1)^2$, which completes the proof of part (2). This completes the proof of necessity.

To prove sufficiency, suppose that we have a semigroup S given by the presentation

$$S \;=\; \langle A \mid c_1, \ldots, c_L = 0 \rangle$$

such that (1), (2) and (3) hold. We prove that S has polynomial growth. We may suppose that

$$a_1 a_j\,,\ \ a_1 a_j^{-1}$$

are relators in the presentation of S for $1 < j \le n$. Put

$$A_1 \;=\; \{a_1\} \qquad \text{and} \qquad A_2 \;=\; \{a_2, \ldots, a_n\}\,.$$

Then A_1 is (trivially) left orthogonal and $A_1(A_2 \cup A_2^{-1}) = \{0\}$ in S. By part (1), certainly $\mathrm{Inv}(A_1)$ has polynomial growth. By part (2), $\mathrm{Inv}(A_2)$ is finite. Hence, by Lemma 2.5, S has polynomial growth. This completes the proof of sufficiency. □

Theorem 7.9. *Let $S = \langle A \mid c_1, \ldots, c_L = 0\rangle$ be a semigroup with polynomial growth given by a two-standard presentation using $n \ge 2$ generators and $L = n^2 - 1$ relators. Suppose that exactly $n-2$ generators are nilpotent. If $n = 2$ then*

$$S \;\cong\; \langle\, a_1, a_2 \mid a_2 a_1 = a_1 a_2 = a_1 a_2^{-1} = 0\,\rangle.$$

If $n \ge 3$ then S is isomorphic to

$$\langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_2^{\varepsilon} a_1 = a_i a_j = a_i a_j^{-1} = 0\,,\ 1 \le i < j \le n\,\rangle\,,$$

for some $\varepsilon \in \{\pm 1\}$.

Proof. Again assume that $A = \{a_1, \ldots, a_n\}$. Suppose first that $n = 2$ and put

$$U \;=\; \langle\, a_1, a_2 \mid a_2 a_1 = a_1 a_2 = a_1 a_2^{-1} = 0\,\rangle.$$

It follows from Theorem 2.8, since neither a_1 nor a_2 is nilpotent, that S is isomorphic to

$$U_1 \;=\; \langle\, a, b \mid ab = a^{-1}b = ab^{-1} = 0\,\rangle \quad \text{or} \quad U_2 \;=\; \langle\, a, b \mid ab = a^{-1}b = ba = 0\,\rangle\,.$$

The mapping of generators that takes $a \mapsto a_2$ and $b \mapsto a_1^{-1}$ induces an isomorphism between U_1 and U, whilst the mapping $a \mapsto a_1^{-1}$ and $b \mapsto a_2^{-1}$ induces an isomorphism between U_2 and U. This proves $S \cong U$, completing the proof of the theorem in the case $n = 2$.

Suppose now that $n \ge 3$. We show that S is isomorphic to either

$$T_1 \;=\; \langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_2 a_1 = a_i a_j = a_i a_j^{-1} = 0\,,\ 1 \le i < j \le n\,\rangle$$

(when $\varepsilon = 1$), or

$$T_2 \;=\; \langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_2^{-1} a_1 = a_i a_j = a_i a_j^{-1} = 0\,,\ 1 \le i < j \le n\,\rangle$$

(when $\varepsilon = -1$). By part (ii)$'$ of Scholium 5.15, we may assume $c_i = a_{i+2}^2$ for $1 \le i \le n-2$, and that

$$R_{1,2} = 3\,, \quad R_{1,j} = 2 \quad \text{and} \quad R_{i,j} = 2$$

for $2 \le i < j \le n$. Hence the number of relators appearing in the presentation for $S_{2,\ldots,n}$ is

$$n^2 - 1 - (2(n-1) + 1) \;=\; n^2 - 2n \;=\; (n-1)^2 - 1\,.$$

But $S_{2,\ldots,n}$ has polynomial growth, so Theorem 7.7 applies (in the case of $n - 1 \ge 2$ generators). After renaming generators and relators, if necessary, we may assume that

$$S_{2,\ldots,n} \;=\; \langle\, a_2, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_i a_j = a_i a_j^{-1} = 0\,,\ 2 \le i < j \le n\,\rangle\,. \tag{34}$$

Since a_1 and a_2 are not nilpotent and $R_{1,2} = 3$, there are four possibilities for $S_{1,2}$, after renaming relators up to $\mathcal{J}$-equivalence:

(i) $S_{1,2} = \langle a_1, a_2 \mid a_2a_1 = a_1a_2 = a_1a_2^{-1} = 0\rangle$;
(ii) $S_{1,2} = \langle a_1, a_2 \mid a_2^{-1}a_1 = a_1a_2 = a_1a_2^{-1} = 0\rangle$;
(iii) $S_{1,2} = \langle a_1, a_2 \mid a_2a_1 = a_1a_2 = a_1^{-1}a_2 = 0\rangle$;
(iv) $S_{1,2} = \langle a_1, a_2 \mid a_2a_1 = a_1^{-1}a_2 = a_1a_2^{-1} = 0\rangle$.

Suppose first that case (i) holds. By Lemma 5.1, since $S_{1,3}$ has polynomial growth, $\rho_{1,3}$ is odd and a_1 dominates a_3 from the left or right. Suppose first that a_1 dominates a_3 from the left. Let j be any integer such that $3 < j \leq n$. Note that, by (34), a_3 dominates a_j from the left. By Lemma 5.2, $\rho_{1,j}$ is odd and a_1 dominates a_j from the left. Hence a_1 dominates a_j from the left for all j such that $3 \leq j \leq n$. Putting this together with the relators we have in this case, and with the relators in (34), we may assume, after renaming relators up to $\mathcal{J}$-equivalence, that

$$S = S_{1,\ldots,n} = T_1 . \tag{35}$$

Suppose now that a_1 dominates a_3 from the right. Then a_1^{-1} dominates a_3 from the left. By the same argument as before, a_1^{-1} dominates a_j from the left for all j such that $3 \leq j \leq n$. Putting this together with all of the relators we have in this case, and with the relators from (34), we may assume, after renaming relators up to $\mathcal{J}$-equivalence, that

$$\begin{aligned} S = S_{1,\ldots,n} = \langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_2a_1 = a_1a_2 = a_1a_2^{-1} = \\ a_1^{-1}a_j = a_1^{-1}a_j^{-1} = a_ia_j = a_ia_j^{-1} = 0\,,\ 2 \leq i < j \leq n \,\rangle\ . \end{aligned}$$

First interchanging the roles of a_1 and a_1^{-1}, then some slight reordering of relators and replacement of a relator up to $\mathcal{J}$-equivalence, and then finally interchanging the roles of a_1 and a_2 we get the following isomorphisms:

$$\begin{aligned} S &\cong \langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_2a_1^{-1} = a_1^{-1}a_2 = a_1^{-1}a_2^{-1} = \\ &\qquad a_1a_j = a_1a_j^{-1} = a_ia_j = a_ia_j^{-1} = 0\,,\ 2 \leq i < j \leq n \,\rangle \\ &= \langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_1^{-1}a_2 = a_2a_1 = a_2a_1^{-1} = a_1a_3 = a_1a_3^{-1} = \\ &\qquad a_1a_j = a_1a_j^{-1} = a_2a_j = a_2a_j^{-1} = a_ia_j = a_ia_j^{-1} = 0\,,\ 3 \leq i < j \leq n \,\rangle \\ &\cong \langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_2^{-1}a_1 = a_1a_2 = a_1a_2^{-1} = a_2a_3 = a_2a_3^{-1} = \\ &\qquad a_2a_j = a_2a_j^{-1} = a_1a_j = a_1a_j^{-1} = a_ia_j = a_ia_j^{-1} = 0\,,\ 3 \leq i < j \leq n \,\rangle \\ &= \langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_2^{-1}a_1 = a_ia_j = a_ia_j^{-1} = 0\,,\ 1 \leq i < j \leq n \,\rangle\ . \end{aligned}$$

Thus, after renaming generators and relators, we may assume

$$S = S_{1,\ldots,n} = T_2 . \tag{36}$$

Suppose next that case (ii) holds. As in the previous case, $\rho_{1,3}$ is odd and a_1 dominates a_3 on the left or right. If a_1 dominates a_3 on the right, then, by inspection, a_2 and $a_1a_3a_2$ are nonzero reduced words in S that label different loops at the vertex a_2 in Γ_S, so that S has exponential growth, which is impossible. Hence a_1 dominates a_3 on the left. Exactly as before a_1 dominates a_j on the left for all $j \geq 3$. Now, putting all relators together, and renaming relators up to $\mathcal{J}$-equivalence, we may assume (36) holds.

Suppose next that case (iii) holds. Interchanging the roles of a_1 and a_1^{-1}, we may assume, up to isomorphism of S, that

$$S_{1,2} \;=\; \langle a_1, a_2 \mid a_2 a_1^{-1} = a_1^{-1} a_2 = a_1 a_2 = 0 \rangle \;=\; \langle a_1, a_2 \mid a_2^{-1} a_1 = a_1 a_2 = a_1 a_2^{-1} = 0 \rangle \ ,$$

and we are back in case (ii). Similarly, case (iv) reduces to case (i), by interchanging the roles of a_1 and a_1^{-1}.

Thus, in all cases, up to isomorphism of S, we may assume (35) or (36) holds. This proves that S is isomorphic to T_1 or T_2. □

Remark 7.10. It is not difficult to prove that T_1 and T_2, defined in the previous proof in the case $n \geq 3$, are not isomorphic. One can see this by considering uniqueness of nilpotent generators, up to inversion, and the uniqueness of letters a_1 and a_2 dominating nilpotent generators from the left: neither of the mappings $a_1 \mapsto a_1, a_2 \mapsto a_2$ nor $a_1 \mapsto a_2, a_2 \mapsto a_1$ induce isomorphisms between T_1 and T_2.

Theorem 7.11. *Let $S = \langle A \mid c_1, \dots, c_L = 0 \rangle$ be a semigroup given by an irredundant presentation using $n \geq 2$ generators and $L = n^2 - 1$ relators. Suppose that exactly $n-2$ generators are nilpotent. If $n = 2$ then S has polynomial growth if and only if there exist nonnegative integers γ_1 and γ_2 such that $\gamma_1 + \gamma_2 > 0$ and S is isomorphic to*

$$\langle\, a, b \mid ab = ab^{-1} = b^{\gamma_1} a a^{-1} b^{\gamma_2} = 0 \,\rangle \ .$$

If $n \geq 3$ then S has polynomial growth if and only if the generators may be reordered and the relators rewritten, up to $\mathcal{J}$-equivalence, such that the following hold:

1. $\mathrm{Inv}(a_1, a_2) = \langle a_1, a_2 \mid a_1 a_2 = a_1 a_2^{-1} = a_2^{\gamma} a_1 a_1^{-1} a_2^{\delta} = 0 \rangle$ *for some $\gamma, \delta \geq 0$ such that $\gamma + \delta > 0$;*
2. $\mathrm{Inv}(a_3, \dots, a_n)$ *is finite given by $(n-2)^2$ relators;*
3. $\{a_1, a_2\}\mathrm{Inv}(a_3, \dots, a_n) = \{0\}$*, using $4n - 8$ relators $a_i a_j = a_i a_j^{-1} = 0$ for $i = 1, 2$ and $3 \leq j \leq n$.*

Proof. Suppose first that $n = 2$ and S has polynomial growth. By Theorem 2.8,

$$S \;\cong\; \langle a, b \mid ab = a^{-1} b = C = 0 \rangle$$

where C divides $a^{\gamma} b^{-1} b a^{\gamma}$ for some integer $\gamma \geq 1$. Interchanging the roles of a and b^{-1} we get

$$S \;\cong\; \langle a, b \mid b^{-1} a^{-1} = ab^{-1} = C' = 0 \rangle \;=\; \langle a, b \mid ab = ab^{-1} = C' = 0 \rangle$$

where C' divides $b^{-\gamma} a a^{-1} b^{-\gamma} \;\mathcal{J}\; b^{\gamma} a a^{-1} b^{\gamma}$. But b is not nilpotent. It follows quickly that $C' \;\mathcal{J}\; b^{\gamma_1} a a^{-1} b^{\gamma_2}$ for some nonnegative integers γ_1 and γ_2 such that $\gamma_1 + \gamma_2 > 0$. Hence

$$S \;\cong\; \langle\, a, b \mid ab = ab^{-1} = b^{\gamma_1} a a^{-1} b^{\gamma_2} = 0 \,\rangle \ . \tag{37}$$

Conversely, any semigroup described by a presentation of the form (37) has polynomial growth, by Theorem 2.8, completing the proof of the statement of the theorem for $n = 2$.

Hence we may suppose that $n \geq 3$. To prove necessity, suppose that S has polynomial growth. We may assume that $A = \{a_1, \dots, a_n\}$ and

$$c_1 \;=\; a_3^{p_3} \ , \ \dots \ , \ c_{n-2} \;=\; a_n^{p_n}$$

for some integers $p_3, \ldots, p_n \geq 2$. Let S' be any homomorphic image of S that has a two-standard presentation obtained by choosing relators that are reduced words of length two that divide $c_1, \ldots, c_L$ respectively. By Lemma 7.1,

$$S' \;=\; \langle A \mid c_1', \ldots, c_L' = 0 \rangle$$

where we may assume c_i' divides c_i for $i = 1, \ldots, L$. In particular, we may assume

$$c_1' = a_3^2 \,,\; \ldots \,,\; c_{n-2}' = a_n^2 \;.$$

Part (ii) of Lemma 7.1 guarantees that a_i^2 does not divide c_j for $3 \leq i \leq n$ and $j > n-2$. By Theorem 7.9, S' is isomorphic to

$$U_n \;=\; \langle\, a_1, \ldots, a_n \mid a_3^2 = \ldots = a_n^2 = a_2^{\varepsilon} a_1 = a_i a_j = a_i a_j^{-1} = 0 \,,\; 1 \leq i < j \leq n \,\rangle$$

for some $\varepsilon \in \{\pm 1\}$. There is no loss in generality in supposing

$$S' \;=\; U_n \;.$$

For $1 \leq i < j \leq n$, denote by c, $c_{i,j}$ and $\overline{c_{i,j}}$ the relators in the presentation of S for which the relators $a_2^{\varepsilon} a_1$, $a_i a_j$ and $a_i a_j^{-1}$, respectively, were chosen as divisors in forming the presentation $S' = U_n$. Thus the presentation for S may be rewritten as follows:

$$S \;=\; \langle\, A \mid a_3^{p_3} = \ldots = a_n^{p_n} = c = c_{i,j} = \overline{c_{i,j}} = 0 \;,\; 1 \leq i < j \leq n \,\rangle \;.$$

By Lemma 7.2,

$$\mathrm{content}(c) = \{a_1, a_2\} \qquad \text{and} \qquad \mathrm{content}(c_{i,j}) = \mathrm{content}(\overline{c_{i,j}}) = \{a_i, a_j\} \;,$$

for $1 \leq i < j \leq n$. We have

$$S_{1,2} \;=\; \langle\, a_1, a_2 \mid c = c_{1,2} = \overline{c_{1,2}} = 0 \,\rangle \;\cong\; \langle\, a, b \mid ab = ab^{-1} = b^{\gamma_1} a a^{-1} b^{\gamma_2} = 0 \,\rangle$$

for some nonnegative integers γ_1 and γ_2 such that $\gamma_1 + \gamma_2 > 0$. We may therefore reorder the generators and rewrite the relators, up to $\mathcal{J}$-equivalence, so that

$$\mathrm{Inv}(a_1, a_2) \;=\; S_{1,2} \;=\; \langle a_1, a_2 \mid a_1 a_2 = a_1 a_2^{-1} = a_2^{\gamma_1} a_1 a_1^{-1} a_2^{\gamma_2} = 0 \rangle \;,$$

completing the proof of part (1) of the theorem.

Suppose that $3 \leq j \leq n$. Now, we have, by Theorem 7.8 (in the case of two generators),

$$\mathrm{Inv}(a_1, a_j) \;=\; S_{1,j} \;=\; \langle\, a_1, a_j \mid a_j^{p_j} = c_{1,j} = \overline{c_{1,j}} = 0 \,\rangle \;\cong\; \langle\, a, b \mid ab = ab^{-1} = b^{\gamma} \,\rangle$$

for some integer $\gamma \geq 2$. The isomorphism is determined by the unique letter, in each presentation, that dominates the other nilpotent letter and its inverse from the left. It follows that

$$\{c_{1,j} \,,\, \overline{c_{1,j}}\} \;=_{\mathcal{J}}\; \{a_1 a_j \,,\, a_1 a_j^{-1}\} \;.$$

By a similar argument,

$$\{c_{2,j} \,,\, \overline{c_{2,j}}\} \;=_{\mathcal{J}}\; \{a_2 a_j \,,\, a_2 a_j^{-1}\} \;.$$

Part (3) of the theorem now follows quickly. The proof of part (2) follows exactly as in the proof of part (2) of Theorem 7.6. This completes the proof of necessity.

To prove sufficiency, suppose that we have a semigroup S given by the presentation

$$S \;=\; \langle A \mid c_1, \ldots, c_L = 0 \rangle$$

such that (1), (2) and (3) hold. In particular, we may suppose that

$$a_1a_2^{-1}\,,\ \ a_1a_j\,,\ \ a_1a_j^{-1}\,,\ \ a_2a_j\,,\ \ a_2a_j^{-1}$$

are relators in the presentation of S for $3 \le j \le n$. We prove that S has polynomial growth. Put

$$A_1 \;=\; \{a_1, a_2\} \qquad \text{and} \qquad A_2 \;=\; \{a_3, \dots, a_n\}\ .$$

Then A_1 is left orthogonal and $A_1(A_2 \cup A_2^{-1}) = \{0\}$ in S. By part (1) and Theorem 2.8, $\mathrm{Inv}(A_1)$ has polynomial growth. By part (2), $\mathrm{Inv}(A_2)$ is finite. Hence, by Lemma 2.5, S has polynomial growth. This completes the proof of sufficiency. □

Remark 7.12. Consider the case $n = 2$ in Theorem 7.11. If $\gamma_2 = 0$ then

$$b^{\gamma_1}aa^{-1}b^{\gamma_2} = b^{\gamma_1}aa^{-1}\ \mathcal{J}\ b^{\gamma_1}a\ ,$$

so that $S \cong \langle\ a, b \mid ab = ab^{-1} = b^{\gamma}a\ \rangle$, where $\gamma = \gamma_1$ is a positive integer. On the other hand, if $\gamma_1 = 0$ then

$$b^{\gamma_1}aa^{-1}b^{\gamma_2} = aa^{-1}b^{\gamma_2}\ \mathcal{J}\ b^{-\gamma_2}a\ ,$$

so that $S \cong \langle\ a, b \mid ab = ab^{-1} = b^{\gamma}a\ \rangle$, where now $\gamma = -\gamma_2$ is a negative integer. Thus, if $\gamma_1 = 0$ or $\gamma_2 = 0$, then we get the simpler description

$$S\ \cong\ \langle a, b \mid ab = ab^{-1} = b^{\gamma}a\ \rangle\ ,$$

for some nonzero integer γ. Similarly if $n > 2$ and $p_1 = 0$ or $p_2 = 0$ then

$$S\ \cong\ \langle\ a_1, \dots, a_n \mid a_3^{p_3} = \dots = a_n^{p_n} = a_2^{p}a_1 = a_ia_j = a_ia_j^{-1} = 0\,,\ 1 \le i < j \le n\ \rangle\ ,$$

for some nonzero integer p. If, further $p_3 = \dots = p_n = 2$ then

$$S\ \cong\ \begin{cases} T_1 & \text{if } p = 1,\\ T_2 & \text{if } p = -1,\end{cases}$$

recovering the two-standard presentations given in the proof of Theorem 7.9.

8. An application

We recall from [10] the operator $\mathcal{Z}$ that takes a general inverse semigroup presentation (not necessarily with zero)

$$\Pi\ =\ \mathrm{Inv}\langle A \mid C_i = D_i \quad \text{for} \quad i = 1, \dots, k\rangle\ ,$$

where A is our usual alphabet and all C_i, D_i are words over $B = A \cup A^{-1}$, and produces the homomorphic image

$$\mathcal{Z}(\Pi)\ =\ \langle A \mid C_i = 0\,,\ D_i = 0 \quad \text{for} \quad i = 1, \dots, k\rangle$$

in the class $\mathfrak{M}_{FI}$. Clearly, if $\mathcal{Z}(\Pi)$ has exponential growth then so does Π, and if $\mathcal{Z}(\Pi)$ contains a non-monogenic free subsemigroup then so does Π.

The first theorem in this section is an application of our results to deduce connections between growth of arbitrary finitely presented inverse semigroup and the number of relations.

Theorem 8.1. *Consider the inverse semigroup*

$$\Pi \;=\; \mathrm{Inv}\langle\, A \mid C_i = D_i \;\text{ for }\; i = 1, \ldots, k \,\rangle$$

defined by k relations where A is an alphabet of size $n \geq 2$ and C_i, D_i are words over B neither of which is $\mathcal{J}$-related to a single letter for each i. If $k < \frac{n^2-1}{2}$ then Π contains a noncyclic free subsemigroup, so has exponential growth. If Π has polynomial growth and no C_i divides or is divided by D_j, for all i and j, then $k \geq \frac{n^2-1}{2}$.

Proof. Observe that the number of relators in the presentation for $\mathcal{Z}(\Pi)$ is $L = 2k$. Suppose first that $k < \frac{n^2-1}{2}$, so $L < n^2 - 1$. If condition (v) of irredundancy fails, then $\mathcal{Z}(\Pi)$ contains a noncyclic free subsemigroup for a trivial reason, and hence Π does also. Suppose then that condition (v) of irredundancy holds. If condition (iv) of irredundancy fails, then we may delete relators that $\mathcal{J}$-divide other relators, yielding an irredundant presentation for $\mathcal{Z}(\Pi)$ using $L' < L$ relators. If condition (iv) holds then we put $L' = L$. In either case, $L' \leq L < n^2 - 1$, so that $\mathcal{Z}(\Pi)$ does not have polynomial growth, by Theorem 5.13, so contains a noncyclic free subsemigroup, by part (b) of Theorem 2.2. Hence Π also contains a noncyclic free subsemigroup.

Suppose now that Π has polynomial growth and no C_i divides or is divided by D_j, for all i and j. Certainly $\mathcal{Z}(\Pi)$ has polynomial growth. Condition (v) of irredundancy must therefore hold automatically with respect to its presentation; condition (iv) of irredundancy holds by supposition. Hence $L \geq n^2 - 1$, again by Theorem 5.13, so that $k \geq \frac{n^2-1}{2}$. □

The final theorem is an application of our results to give a sufficient condition that guarantees certain finitely generated inverse semigroups (which need not be finitely presented) possess noncyclic free subsemigroups, so have exponential growth.

Theorem 8.2. *Consider the inverse semigroup*

$$\Pi \;=\; \mathrm{Inv}\langle\, A \mid C_i = D_i \;\text{ for }\; i \in I \,\rangle$$

where A is an alphabet of size $n \geq 2$, I is a nonempty indexing set (not necessarily finite), and C_i, D_i are words over B neither of which is $\mathcal{J}$-related to a single letter for each $i \in I$. Then there exists a finite set M of smallest size consisting of reduced words over $A \cup A^{-1}$ of length 2 such that the ideal generated by M in FI_A contains all C_i and D_i for $i \in I$. If $|M| < n^2 - 1$ then Π contains a noncyclic free subsemigroup, so has exponential growth.

Proof. Clearly M exists (as in the proof of Corollary 5.14). Suppose that $|M| < n^2 - 1$. By Corollary 5.14, $\mathcal{Z}(\Pi)$ contains a noncyclic free subsemigroup, and the theorem follows. □

We finish with a simple application of our techniques to analyse growth of a novel class of two-generated inverse semigroups all of which have exponential growth.

Example 8.3. Consider the following hyperword that is infinite to the right:

$$w \;=\; bab^{-2}b^2a^{-1}b^{-3}b^3ab^{-4}b^4a^{-1}b^{-5}\ldots\; b^{2k-1}ab^{-2k}b^{2k}a^{-1}b^{-2k-1}\ldots$$

Observe that every finite subword of w is the label of a traversal of some subtree of the word tree $T(b^nab^{-n})$ for some sufficiently large positive integer n. Consider an inverse semigroup Π given by the presentation

$$\Pi \;=\; \mathrm{Inv}\langle\, a, b \mid C_i = D_i \;\text{ for }\; i \in I \,\rangle$$

where I is some (possibly infinite) indexing set and C_i and D_i are finite subwords of w whose Munn trees have at least two edges, for each $i \in I$. Let M denote the set of divisors of C_i and D_i that are reduced words of length 2. By inspection of w, we have

$$M \subseteq \{b^2, b^{-2}, ba, a^{-1}b^{-1}, ab^{-1}, ba^{-1}\} .$$

If the set of $\mathcal{J}$-classes of elements of M has cardinality $< 3 = n^2 - 1$, when $n = 2$, then the previous theorem applies, so that Π contains a noncyclic free subsemigroup. Suppose then that the set of $\mathcal{J}$-classes of elements of M has cardinality 3 (so that the previous theorem does not apply). Then

$$S = \langle a, b \mid b^2 = ba = ba^{-1} = 0 \rangle$$

is a homomorphic image of Π. But S does not have polynomial growth by the characterisation given in Theorem 2.8 (or more directly by observing that (a, b) and (a^{-1}, b) are adjacent pairs in Γ_S and then applying part (e)(ii) of Theorem 2.2). It follows now that S, and hence also Π, contains a noncyclic free subsemigroup. Hence Π has exponential growth in all possible cases.

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