FUNCTIONAL CALCULUS VIA THE EXTENSION TECHNIQUE: A FIRST HITTING TIME APPROACH

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Abstract. In this article, we present a solution to the problem:

Which type of linear operators can be realized by the Dirichlet-to-Neumann operator associated with the operator $-\Delta - a(z)\frac{\partial^2}{\partial z^2}$ on an extension problem?

which was raised in the pioneering work [Comm. Par.Diff. Equ. 32 (2007)] by Caffarelli and Silvestre. In fact, we even go a step further by replacing the negative Laplace operator $-\Delta$ on $\mathbb{R}^d$ by an m-accretive operator A on a general Banach space X and the Dirichlet-to-Neumann operator by the Dirichlet-to-Wentzell operator. We establish uniqueness of solutions to the extension problem in this general framework, which seems to be new in the literature and of independent interest. Our aim of this paper is to provide a new Phillips-Bochner type functional calculus which uses probabilistic tools from excursion theory. With our method, we are able to characterize all linear operators $\psi(A)$ among the class $\mathcal{CBF}$ of complete Bernstein functions ψ, resulting in a new characterization of the famous *Phillips' subordination theorem* within this class $\mathcal{CBF}$.

Contents

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1. Introduction and main results

1.1. **Introduction.** In [11], Caffarelli and Silvestre showed that the fractional Laplace operator $\psi(-\Delta) = (-\Delta)^{\sigma}$ on $\mathbb{R}^d$, $d \geq 1$, $\psi(s) = s^{\sigma}$, $0 < \sigma < 1$, defined as the singular integral operator

$$(-\Delta)^{\sigma} f(x) = C_{\sigma,d}\,\mathrm{C.V.}\int_{\mathbb{R}^d} \frac{f(x) - f(\xi)}{|x-\xi|^{d+2\sigma}}\mathrm{d}\xi,$$

$f \in C_c^{\infty}(\mathbb{R}^d)$, can be characterized by the *Dirichlet-to-Neumann operator*

$$f_{|\mathbb{R}^d} \mapsto \Lambda_{\sigma} f := \lim_{y\to 0+} -y^{1-2\sigma} u_y(.,y)_{|\mathbb{R}^d} \tag{1.1}$$

corresponding to the *incomplete Dirichlet/extension problem*

$$\begin{cases} -\Delta u - \frac{1-2\sigma}{y} u_y - u_{yy} = 0 & \text{on } \mathbb{R}^{d+1}_+, \\ \qquad\qquad\qquad\quad u = f & \text{on } \partial\mathbb{R}^{d+1}_+ = \mathbb{R}^d, \end{cases} \tag{1.2}$$

for the negative *Bessel operator* $\mathcal{A} := -\left(\Delta + \frac{1-2\sigma}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right)$ on the half space $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0,\infty)$. In particular, it was observed that by applying the change of variable $z = (y/2\sigma)^{2\sigma}$ to the solution $u(y) := u(\cdot, y)$ of the *Bessel equation*

$$-\Delta u - \frac{1-2\sigma}{y} u_y - u_{yy} = 0 \qquad \text{in } \mathbb{R}^{d+1}_+, \tag{1.3}$$

then $u(z)$ becomes a solution of the elliptic *extension equation*

$$-\Delta u - a(z) u_{zz} = 0 \qquad \text{in } \mathbb{R}^{d+1}_+, \tag{1.4}$$

for the coefficient $a(z) = z^{-\frac{1-2\sigma}{\sigma}}$. Moreover, one has that

$$y^{1-2\sigma} u_y(y) = (2\sigma)^{1-2\sigma} u_z(z)$$

and hence, the Dirichlet-to-Neumann operator Λ_{σ} given by (1.1) reduces to

$$\Lambda_{\sigma} f = -(2\sigma)^{1-2\sigma} u_z(0).$$

Subsequently, the following problem was posed in [11, Section 7.]:

Problem 1. *What other linear operators $\psi(-\Delta)$ can be obtained from the Dirichlet-to-Neumann operator associated with the operator $-\Delta - a(z)\frac{\partial^2}{\partial z^2}$?*

The aim of this paper, is to provide an answer to this problem, and even to go beyond. We replace the negative Laplace operator $-\Delta$ (with vanishing conditions at infinity) on $\mathbb{R}^d$ by a general closed linear m-accretive operator A defined on a Banach space X. Then A admits the property that $-A$ generates a C_0-semigroup $\{e^{-tA}\}_{t\geq 0}$ of contractions (see Section 3.6 for further details). Further, if for a given *string m on $\mathbb{R}$ of infinite length* (see Definition 1.1), ψ_m is the associated *complete Bernstein function* with Lévy triple $(0, m(0+), \nu_m)$ (see Definition 1.11 and (1.21)), and $\psi_m(A)$ is the operator defined by (1.25) via the Lévy triple $(0, m(0+), \nu_m)$, then the main result of this article (Theorem 1.13) states that the characterization

$$\psi_m(A) = \Lambda_m \tag{1.5}$$

holds, where Λ_m is the *Dirichlet-to-Wentzell operator* given by

$$\Lambda_m f := m(0+)Au(0) - \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}z_+}(0) \tag{1.6}$$

for every $f \in D(A)$ with corresponding unique bounded *weak solution* u of the *(incomplete) Dirichlet problem* (see Definition 1.5 below)

$$\begin{cases} \mathcal{A}_m u(z) = 0 & \text{for } z \in (0,\infty), \\ u(0) = f, \end{cases} \tag{1.7}$$

for the *extension operator*

$$\mathcal{A}_m := A + B_m \tag{1.8}$$

on the *extended space* $\mathcal{X}_+ = X \times (0,\infty)$. We also refer to Dirichlet problem (1.7) as the *extension problem* since the operator $\mathcal{A}_m$ extends the operator A acting on X by the 2nd-order differential operator

$$B_m := -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z} \qquad \text{acting on } (0,\infty),$$

where we denote by

$$\frac{\mathrm{d}u}{\mathrm{d}m}(z) := \lim_{h\to 0}\frac{u(z+h)-u(z)}{m(z+h)-m(z)} \tag{1.9}$$

(if the limit exits in X) the *m-derivative* at $z \in (0,\infty)$ of a function $u : [0,\infty) \to X$. We refer to Definition 1.3 below for a formulation of the this notion in the *sense of distributions* and also to Appendix B for further discussions.

For the proof of the characterization (1.5), we employ an intermezzo of probabilistic tools and functional analysis. We begin by proving existence and uniqueness of bounded weak solutions of the Dirichlet problem (1.7). Motivated from the interpretation that problem (1.7) can be considered as a classic elliptic Dirichlet problem on the "extended region" $\mathcal{X}_+$, our existence proof relies on the classical approach of stopping a related stochastic process $\{(X_t, Z_t\}_{t\geq 0}$ in $\mathcal{X}_+$ at $z = 0$. To be more precise, for a given string m on $\mathbb{R}$, the operator $-B_m$ generates a stochastic process $\{Z_t\}_{t\geq 0}$ on $[0,\infty)$, which we call *generalized diffusion associated with m* (see Definition 3.3). Thus, one can define the *first hitting time* τ of zero by $\{Z_t\}_{t\geq 0}$ starting from $z \in (0,\infty)$. If one assumes that the Banach space $X = X(\Sigma)$ is a function space with domain Σ and $-A$ generates a Markov process $\{X_t\}_{t\geq 0}$ in the state space Σ, then it is clear that for every $f \in X(\Sigma)$, the unique solution u of Dirichlet problem (1.7) is given by

$$u(x,z) = \mathbb{E}_{(x,z)}\big(f\big(X_\tau\big)\big) \tag{1.10}$$

for every $(x,z) \in \Sigma \times [0,\infty)$. Since the first hitting time τ admits a *density* (see Section 3.1 for details)

$$\omega_\tau(t,z) := \frac{\mathbb{P}_z(\tau \in \mathrm{d}t)}{\mathrm{d}t}$$

and since the process $\{X_t\}_{t\geq 0}$ in Σ is one-to-one related to the transition semigroup $\{e^{-tA}\}_{t\geq 0}$ on $X(\Sigma)$. Thus the representation (1.10) of a weak solution u of (1.7) is equivalent to the *Poisson formula*

$$u(z) = \int_0^\infty (e^{-tA}f)\,\omega_\tau(t,z)\,\mathrm{d}t \tag{1.11}$$

for every $z \in [0,\infty)$. One crucial advantage of formula (1.11) is that the process $\{X_t\}_{t\geq 0}$ is not directly involved anymore to describe a weak solution u of (1.7). Hence, formula (1.11) provides a strong candidate for being a solution of Dirichlet problem (1.7) even though $-A$ may not necessarily generate a stochastic process, but still generate a C_0-semigroup of contractions on X. By using Kent's theorem [22] on the spectral decomposition of the first hitting time density, we are able to derive a link between the first hitting time density ω_τ in (1.11) and the Lévy measure density of the inverse local time $\{\tilde{L}_t^{-1}\}_{t\geq 0}$ at zero of the generalized diffusion $\{Z_t\}_{t\geq 0}$. This connections allow us to easily calculate the Dirichlet-to-Wentzell map Λ_m given by (1.6), to show that the characterization (1.5) holds, and to establish a Phillips-Bochner type functional calculus.

1.2. **Main results.** In order to state the main theorems of this paper, we need first to introduce some classic definitions and helpful notations. Throughout this section, X denotes a Banach space and A a closed, linear operator on X with dense domain $D(A)$, and $\{e^{-tA}\}_{t\geq 0}$ the C_0-semigroup of contractions on X generated by $-A$.

Definition 1.1. A non-decreasing, right-continuous function $m : \mathbb{R} \to [0,\infty)$ is called a *string on* $\mathbb{R}$ *of infinite length* provided m has the following properties

(i) $m(x) = 0$ for all $x < 0$ and $m(-0) = 0$,
(ii) $m(x_0) < \infty$ for some $x_0 \geq 0$, and
(iii) $m(x) > 0$ for all $x > 0$.

We denote the family of all strings of infinite length by $\mathfrak{m}_\infty$.

In the following, if nothing else is said, we always refer to m as a string on $\mathbb{R}$ of infinite length. The case of strings of *finite length* shall be studied in a forthcoming work. Then, there is a unique Radon measure $\mu_m : \mathcal{B}(\mathbb{R}) \to [0,+\infty]$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ such that

$$\mu_m((a,b]) = m(b+) - m(a+) \qquad \text{for every } a,\, b \in \mathbb{R} \text{ with } a < b \tag{1.12}$$

(cf, [28, Theorem 6.7] and see also [28, Remark 6.11]). Since $m(x) = 0$ for all $x < 0$, (1.12) yields that the $\mu_m((a, b]) = 0$ for all $a < b < 0$. Further, we denote by $E_m = \mathrm{supp}(\mu_m)$ the *support* of μ_m.

For a better understanding of the functional calculus, we develop here, but also, in order to illustrate that our results generalize previous ones obtained for the fractional power case (cf , for example, [11, 37, 16, 5]), we provide the following example.

Example 1.2. Let $A = -\Delta$ be the negative Laplace operator on $X = L^2(\mathbb{R}^d)$ equipped with vanishing conditions at infinity. Then, in the case of the fractional power $\psi_m(-\Delta) = (-\Delta)^\sigma$, $\sigma \in (0, 1)$ in $X = L^2(\mathbb{R}^d)$, in the extension equation (1.4) the coefficient $a(z)$ is given by

$$a_\sigma(z) = \tfrac{1}{2m'_\sigma(z)} \qquad \text{and} \qquad a_\sigma(z)u_{zz} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m_\sigma}\frac{\mathrm{d}u}{\mathrm{d}z}(z),\ z \in (0, \infty), \tag{1.13}$$

for the string $m_\sigma \in \mathfrak{m}_\infty$ on $\mathbb{R}$ given by

$$m_\sigma(z) := \begin{cases} \frac{1}{2}\frac{\sigma}{1-\sigma} z^{\frac{1-\sigma}{\sigma}} & \text{if } z > 0, \\ 0 & \text{if } z \le 0, \end{cases} \tag{1.14}$$

for every $z \in \mathbb{R}$. For convenience, we write in (1.13) $m'_\sigma(z)$ to denote $\frac{\mathrm{d}m_\sigma}{\mathrm{d}z}$.Thus, equation (1.4) can be rewritten in $X = L^2(\mathbb{R}^d)$ by

$$\mathcal{A}_{m_\sigma} u(z) = 0 \qquad \text{for } z > 0. \tag{1.15}$$

It is worth mentioning that the generalized diffusion $\{Z_t\}_{t\ge 0}$ generated by $-B_{m_\sigma}$, in this case, coincides up to a multiple constant with the scaled Bessel process $\{Y_t^{2\sigma}\}_{t\ge 0}$. We continue discussing the fractional power case A^σ in the Sections 2.3 & 2.4, and in Example 3.20 of Section 3.4. For the convenience of the reader, we provide in Appendix A of this paper a brief review of the Bessel process and related properties, which are relevant here.

For $1 \le q < \infty$, let $L^q_{\mu_m}(0, \infty; X)$ (respectively, $L^q_{loc,\mu_m}((0, \infty); X)$) denote the weighted Lebesgue spaces of all μ_m-a.e. rest-classes of measurable functions $u : [0, \infty) \to X$ with finite integral $\int_{[0,\infty)} \|u\|^q_X \mathrm{d}\mu_m$ (respectively, finite $\int_K \|u\|^q_X \mathrm{d}\mu_m$ for every compact subset $K \subseteq (0, \infty)$).

Definition 1.3. For a given $u \in L^1_{loc}((0, \infty); X)$, one calls a function $g \in L^1_{loc,\mu_m}((0, \infty); X)$ a *weak m-derivative of f* provided u and g satisfy

$$\int_0^\infty g(z)\,\xi(z)\,\mathrm{d}\mu_m(z) = -\int_0^\infty u(z)\,\frac{\mathrm{d}\xi}{\mathrm{d}z}(z)\,\mathrm{d}z \tag{1.16}$$

for every $\xi \in C_c^\infty((0, \infty))$. Due to Lemma B.3 (in the appendix of this paper), a function g is uniquely defined through (1.16). Thus, we can call g *the weak m-derivative of u* and set $\frac{\mathrm{d}u}{\mathrm{d}m} = g$.

For $1 \le q < \infty$, we write $W^{1,q}_{loc,\mu_m}((0, \infty); X)$ to denote the *mixed* 1^{st}*-Sobolev space* of all functions $u \in L^q_{loc}((0, \infty); X)$ with weak m-derivative $\frac{\mathrm{d}u}{\mathrm{d}m} \in L^q_{loc,\mu_m}((0, \infty); X)$ and by $W^{1,q}_{\mu_m}((0, \infty); X)$ the Sobolev space of all functions $u \in L^q(0, \infty; X)$ with $\frac{\mathrm{d}u}{\mathrm{d}m} \in L^q_{\mu_m}(0, \infty; X)$. If $\mu_m = \mu_{Leb}$ is the Lebesgue measure on $(0, \infty)$, then we use the standard notation $W^{1,q}_{loc}((0, \infty); X)$ instead of $W^{1,q}_{loc,\mu_{Leb}}((0, \infty); X)$ and $W^{1,q}(0, \infty; X)$ instead of $W^{1,q}_{\mu_{Leb}}((0, \infty); X)$.

Remark 1.4. On the other hand, for given $u \in L^1_{loc}((0,\infty);X)$ and $g \in L^1_{loc,\mu_m}((0,\infty);X)$, (1.16) means that the *regular* (vector-valued) distribution $[u]$ given by

$$\langle [u], \xi\rangle := \int_0^\infty u(z)\,\xi(z)\,\mathrm{d}z, \quad \xi \in C_c^\infty((0,\infty)),$$

has the *(vector-valued) measure* $g\,\mu_m$ as its *distributional derivative*

$$\langle [u]', \xi\rangle := -\int_0^\infty u(z)\,\frac{\mathrm{d}\xi}{\mathrm{d}z}(z)\,\mathrm{d}z = \int_0^\infty \xi(z)\,g(z)\,\mathrm{d}\mu_m(z)$$

for every $\xi \in C_c^\infty((0,\infty))$. Thus, the weak m-derivative $\frac{\mathrm{d}u}{\mathrm{d}m}$ of u can be characterized by

$$[u]' = \frac{\mathrm{d}u}{\mathrm{d}m}\,\mu_m \qquad \text{in } \mathcal{D}'((0,\infty);X).$$

It is worth noting that Revuz and Yor [31] employed the notation $g\,\mu_m$ instead of $\frac{\mathrm{d}u}{\mathrm{d}m}$ to study the *generalized Sturm-Liouville problem* (1.20) below for given Radon measure μ_m.

Now, we are ready to introduce the notion of a *weak solution* of the incomplete Dirichlet problem (1.7) associated with the extension operator $\mathcal{A}_m$ on $\mathcal{X}_+$.

Definition 1.5 (*Weak solution of the incomplete Dirichlet problem)***.** A function $u : (0,\infty) \to X$ is called a *weak solution* of the extension equation

$$\mathcal{A}_m u = 0 \qquad \text{in } (0,\infty), \tag{1.17}$$

if $u \in W^{1,1}_{loc}((0,\infty);X)$ with $\frac{\mathrm{d}u}{\mathrm{d}z} \in W^{1,1}_{loc,\mu_m}((0,\infty);X)$ satisfying

$$u(z) \in D(A) \quad \text{ and } \quad B_m u(z) = A(u(z)) \quad \text{ for } \mu_m\text{-a.e. } z \in (0,\infty).$$

Further, for given $f \in X$, we define a function $u \in C([0,\infty);X)$ to be a *weak solution* of Dirichlet problem (1.7) for the extension operator $\mathcal{A}_m$ defined by (1.8) if $u(0) = f$ in X, $u \in W^{1,1}_{loc}([0,\infty);X)$ with $\frac{\mathrm{d}u}{\mathrm{d}z} \in W^{1,1}_{loc,\mu_m}([0,\infty);X)$, and u is a weak solution of (1.17).

For our next definition, it is worth noting that for a function

$$u \in W^{1,1}_{loc}([0,\infty);X) \quad \text{ with } \quad \tfrac{\mathrm{d}u}{\mathrm{d}z} \in W^{1,1}_{loc,\mu_m}([0,\infty);X),$$

the right hand-side derivative $\frac{\mathrm{d}u}{\mathrm{d}z+}(0)$ exists. We refer to Remark 4.1 for more details on this.

Definition 1.6 (The Dirichlet-to-Wentzell operator associated with $\mathcal{A}_m$)**.** Let $D(\Lambda_m)$ be the set of all $f \in D(A)$ such that there exists a unique weak solution $u \in C([0,\infty);X)$ of Dirichlet problem (1.7) with boundary value $u(0) = f$. Then, we call the linear operator $\Lambda_m : D(\Lambda_m) \to X$ defined by (1.6), where u is the weak solution of (4.2) with boundary value $u(0) = f$, the *Dirichlet-to-Wentzell operator associated with* $\mathcal{A}_m$.

Remark 1.7. For strings $m \in \mathfrak{m}_\infty$ with right hand-side limit $m(0+) = 0$, the Dirichlet-to-Wentzell operator Λ_m reduces to the classical *Dirichlet-to-Neumann operator* associated with $\mathcal{A}_m$.

It is convenient to apply the preceding two definitions at a fundamental example.

Example 1.8. Let the string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$ be the *Heaviside step function*

$$m(x) := \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{1.18}$$

The associated measure $\mu_m = \delta_0$ is the *Dirac measure* at $x = 0$. Hence, according to the two Definitions 1.3 and 1.5, a function $u \in C([0,\infty);X) \cap W^{1,1}_{loc}((0,\infty);X)$ is a weak solution of extension equation (1.17) provided $\frac{\mathrm{d}u}{\mathrm{d}z} \in W^{1,1}_{loc,\mu_m}((0,\infty);X)$, $u(z) \in D(A)$ for every $z > 0$, and

$$-\frac{1}{2}\int_0^\infty \frac{\mathrm{d}u}{\mathrm{d}z}\frac{\mathrm{d}\xi}{\mathrm{d}z}\,\mathrm{d}z = 0 \qquad \text{for every } \xi \in C^\infty_c((0,\infty)). \tag{1.19}$$

By Lemma B.6 and (1.19) implies that u is constant on $[0,\infty)$. Thus, for given $f \in X$, the unique bounded weak solution u of Dirichlet problem (1.7) is given by $u(z) \equiv f$. This implies that the associated Dirichlet-to-Wentzell operator Λ_m associated with $\mathcal{A}_m$ reduces to

$$\Lambda_m f = m(0+)Au(0) = Af \quad \text{for all } f \in D(A).$$

In other words, the Dirichlet-to-Wentzell operator Λ_m associated with $\mathcal{A}_m$ coincides with the operator A if m is the Heaviside step function. This can be understood as a time change with the trivial subordinator $\mathrm{id}_{\mathbb{R}_+}(s) := s$, $s \in \mathbb{R}_+ := [0,\infty)$. We refer the interested reader to Subsection 3.2 for further details regarding the notion of *inverse local times*. In Example 1.14 we provide an alternative proof of this case by using one of our main results in this paper.

Our first theorem provides sufficient conditions to ensure the existence, uniqueness, and a Poisson formula of bounded weak solutions of the Dirichlet problem (1.7) for the extension operator $\mathcal{A}_m$ defined by (1.8) on $\mathcal{X}_+$.

Theorem 1.9. *Let $\{e^{-tA}\}_{t\geq 0}$ be a C_0-semigroup of contractions on X and $-A$ its infinitesimal generator on X. Further, let $m \in \mathfrak{m}_\infty$ be a string on $\mathbb{R}$, ω_τ and ψ_m be the first hitting time density and the complete Bernstein function associated with the string m. Then, for every $f \in D(A)$, the Dirichlet problem (1.7) admits a unique bounded weak solution u, and this solution u is given by the Poisson formula (1.11).*

We outline the proof of Theorem 1.9 in Section 4, in which we combine arguments from stochastic analysis with tools from nonlinear functional analysis in a refined way. We begin in Section 4.2 by establishing uniqueness of bounded weak solutions of Dirichlet problem (1.7); see Theorem 4.3.

To the best of our knowledge, uniqueness results for Dirichlet problem (1.7) are only known in the case when the string m is given by (1.14), which corresponds to the fractional power case A^σ; either A being a sectorial operator (see [5] and [29]), or A being an m-accretive (possibly, nonlinear and multi-valued) operator on a Hilbert space (see [18]). Thus Theorem 4.3 essentially improves this result within the class of accretive operators A on a Banach space X.

Our proof of this result uses arguments from nonlinear functional analysis and exploits the fact that A is accretive. In Section 4.4, we establish existence of bounded weak solutions of Dirichlet problem (1.7) by simply verifying that if u is given by the integral (1.11) then u satisfies another integral representation (see Theorem 4.4), which implies that u satisfies all conditions of being a weak solutions of Dirichlet problem (1.7). Then, by our uniqueness result (Theorem 4.3), every weak solution of (1.7) can be represented by the Poisson formula (1.11). Since the weight $\omega_\tau(t,z)$ in the integral (1.11) is the density function of the first hitting time τ by a given generalized diffusion, this method provides a *first hitting time approach*.

Remark 1.10. In order to keep this paper well organized, we separated our research results obtained in the interesting Hilbert case $X = H$ (see [19]) from the one presented here. By focusing on sectorial operators A on a Hilbert space H which are defined by a continuous, coercive form $\mathcal{E} : V \times V \to \mathbb{C}$, where V is another Hilbert space continuously and densely embedded into H, we obtain existence of a weak (variational) solution u of Dirichlet problem (1.7), admitting stronger regularity properties. In addition, in the case H is separable and A admits a compact resolvent, applying the Fourier series yields that finding a weak solution u of Dirichlet problem (1.7) becomes equivalent to determining for every eigenvalue $\lambda \geq 0$ of A the unique bounded weak solution $\phi \in C([0,\infty);\mathbb{R})$ of the *generalized Sturm-Liouville problem*

$$\begin{cases} -\frac{1}{2}\phi'' + \lambda\,\phi\,\mu_m = 0 & \text{in } (0,\infty),\\ \phi(0) = 1. \end{cases} \tag{1.20}$$

We note that similar ideas as given in [19] can be applied to pseudo-differential operators A on $L^2(\mathbb{R}^d)$ with a strictly positive symbol. By using the Fourier transform, one reduces the Dirichlet problem (1.7) to a Sturm-Liouville problem. In fact, this was essentially done in [11, Section 7].

Next, we intend characterizing the operator $\psi_m(A)$ among the class of complete Bernstein functions ψ_m in terms of the Dirichlet-to-Wentzell operator Λ_m. Before doing this, we briefly recall from [36] the following definition.

Definition 1.11. A function $\psi : (0,\infty) \to \mathbb{R}$ is called a *Bernstein function with Lévy triple* (a,b,ν) provided ψ is given by

$$\psi(\lambda) = a + b\lambda + \int_0^\infty \left(1 - e^{-\lambda r}\right) \mathrm{d}\nu(r) \qquad \text{for all } \lambda > 0,$$

for some a, $b \geq 0$ and a Lévy measure ν on $(0,\infty)$ with finite integral $\int_0^\infty (r \wedge \mathbb{1}_{(0,\infty)}(r))\,\mathrm{d}\nu(r)$. The triple (a,b,ν) is referred to as a Lévy triple. Now, a Bernstein function ψ is called *complete* if it has the Lévy triple $(0,b,\nu)$ and admits the following two properties

(i) the Lévy measure ν is absolutely continuous with respect to the Lebesgue measure;
(ii) the density $h := \frac{\mathrm{d}\nu}{\mathrm{d}r}$ of the Lévy measure ν is a *completely monotone* function; that is, h is smooth and its derivatives satisfy

$$(-1)^n\,h^{(n)}(r) \geq 0 \qquad \text{for all } n \in \mathbb{N} \cup \{0\}, \text{ and } r > 0.$$

We denote the set of complete Bernstein functions by $\mathcal{CBF}$.

Due to Kreĭn's theorem [25] (see also [24, Theorem 1.1]), for every string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$, there is a unique complete Bernstein function

$$\psi_m(\lambda) := m(0+)\lambda + \int_0^\infty \left(1 - e^{-\lambda r}\right) \mathrm{d}\nu_m(r), \qquad \lambda > 0. \tag{1.21}$$

Moreover, the mapping $\Psi : \mathfrak{m}_\infty \to \mathcal{CBF}$ $m \mapsto \Psi(m) = \psi_m$ with ψ_m defined by (1.21) is bijective, and called the *Kreĭn's correspondence*.

Notation 1.12. For a given string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$, we refer to the function $\psi_m : [0,\infty) \to \mathbb{R}$ given in (1.21) as the *complete Bernstein function associated with* m, ν_m the *Lévy measure associated with* m, $h_m = \frac{\mathrm{d}\nu_m}{\mathrm{d}r}$ the density of the Lévy measure ν_m, and by $(0, m(0+), \nu_m)$ the *Lévy triple* associated with ψ_m.

According to [36, Theorem 5.2], ψ is a Bernstein function if and only if there exists a unique *vaguely continuous convolution semigroup* $\{\gamma_t\}_{t\ge 0}$ *of sub-probability measures* γ_t on $[0,\infty)$ (see Definition 3.41) such that the Laplace transform of γ_t

$$\int_0^\infty e^{-\lambda s}\,\mathrm{d}\gamma_t(s) = e^{-t\psi(\lambda)} \qquad \text{for all } \lambda > 0,\ t \ge 0. \tag{1.22}$$

In addition, by Knight's theorem (see Theorem 3.8 in Section 3.2) for a given string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$, if $\{Z_t\}_{t\ge 0}$ denotes the generalized diffusion associated with m, $\{\tilde{L}_t^{-1}\}_{t\ge 0}$ the *local inverse time at zero* of $\{Z_t\}_{t\ge 0}$ (see Section 3.2 for the construction of this notion), and if ψ_m is the complete Bernstein function associated with m, then the Laplace transform determines uniquely (see (3.11)) that the convolution semigroup $\{\gamma_t\}_{t\ge 0}$ of sub-probability measures γ_t on $[0,\infty)$ associated with ψ_m has to be given by the push-forward measure

$$\gamma_t((a,b]) = \mathbb{P}(\tilde{L}_t^{-1} \in (a,b]) \qquad \text{for all } a, b \in [0,\infty), \text{ and } t \ge 0. \tag{1.23}$$

Now, for a given C_0-semigroup $\{e^{-tA}\}_{t\ge 0}$ of contractions $e^{-tA} \in \mathcal{L}(X)$ with infinitesimal generator $-A$, and a vaguely continuous convolution semigroup $\{\gamma_t\}_{t\ge 0}$ of sub-probability measures on $[0,\infty)$, the family $\{e^{-t\psi(A)}\}_{t\ge 0}$ of operators $e^{-t\psi(A)}$ on X defined by the Bochner integral

$$e^{-t\psi(A)}f := \int_{[0,\infty)} e^{-sA}f\,\mathrm{d}\gamma_t(s) \qquad \text{for every } t \ge 0 \text{ and } f \in X \tag{1.24}$$

defines a C_0-semigroup of contractions on X (see [36, Theorem 13.1]). Thanks to Phillips' subordination theorem [2] (cf, Theorem 3.45 in Section 3.7), the abstractly defined infinitesimal generator $-\psi(A)$ of the semigroup $\{e^{-t\psi(A)}\}_{t\ge 0}$ can be expressed by

$$\psi(A)f = af + bAf + \int_0^\infty (f - e^{-tA}f)\,\mathrm{d}\nu(t) \tag{1.25}$$

for all $f \in D(A)$, where the integral (1.25) is to be understood in the Bochner sense, ψ is the unique Bernstein function associated with the given vaguely continuous convolution semigroup $\{\gamma_t\}_{t\ge 0}$ through (1.22), and (a, b, ν) the corresponding Lévy triple.

Now, for a given string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$ with associated Bernstein function ψ_m, our second main result provides an alternative characterization of the operator $\psi_m(A)$ in terms of the Dirichlet-to-Wentzell operator Λ_m.

Theorem 1.13. *Let A be an m-accretive operator on a Banach space X. Given a string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$, let ψ_m be the corresponding complete Bernstein function with Lévy triple $(0, m(0+), \nu_m)$, and $\psi_m(A)$ the operator given* *by (1.25) for this triple. Then, the operator $\psi_m(A)$ coincides with the Dirichlet-to-Wentzell operator Λ_m given by (1.6). Moreover, the semigroup $\{e^{-t\psi_m(A)}\}_{t\geq 0}$ generated by $-\psi_m(A)$ is given by*

$$e^{-t\psi_m(A)}f = \mathbb{E}\left(e^{-\tilde{L}_t^{-1}A}f\right) = \int_{[0,\infty)} e^{-sA}f\,\mathrm{d}\gamma_t(s) \tag{1.26}$$

for every $t \geq 0$, $f \in X$, where the convolution semigroup $\{\gamma_t\}_{t\geq 0}$ is given by (1.23) involving the local inverse time $\{\tilde{L}_t^{-1}\}_{t\geq 0}$ at zero of the generalized diffusion process $\{Z_t\}_{t\geq 0}$ associated with m.

Example 1.14 (Example 1.8 revisited)**.** Thanks to (1.26) in Theorem 1.13, we can now give an alternative proof of the fact that the the Dirichlet-to-Wentzell operator Λ_m for the Heaviside step function m given by (1.18) coincides with A. Namely, for the semigroup $\{e^{-t\Lambda_m}\}_{t\geq 0}$ generated by $-\Lambda_m$ on X, one has then that

$$\begin{aligned} e^{-t\Lambda_m}f &= \mathbb{E}\Big(e^{-\frac{1}{2}\int_{[0,\infty)} L_{L_t^{-1}}(z)\,\mathrm{d}\delta_0(z)A}f\Big) \\ &= \mathbb{E}\Big(e^{-\frac{t}{2}A}f\Big) \\ &= e^{-\frac{t}{2}A}f \end{aligned}$$

for every $f \in X$ and $t \geq 0$, yielding that $\Lambda_m = \frac{1}{2}A$.

Due to Kreĭn's correspondence Ψ and by Theorem 1.13, we obtain the following new characterization of Phillips's subordination theorem characterizing $\psi(A)$ for any ψ of the class $\mathcal{CBF}$.

Corollary 1.15. *Let A be an m-accretive operator on a Banach space X. If ψ is a complete Bernstein function with Lévy triple $(0, b, \nu)$ and $m \in \mathfrak{m}_\infty$ the unique string on $\mathbb{R}$ given by the Kreĭn's correspondence $\Psi(m) = \psi$, then the two operators $\psi(A)$ given by (1.25) and the Dirichlet-to-Wentzell operator Λ_m given by (1.6) coincide.*

Remark 1.16 (*Problem 1 & the Dirichlet-to-'Wentzell-Robin' operator.*)*.* Since for the string m introduced by (1.14), the Dirichlet-to-Wentzell operator (1.6) reduces to the Dirichlet-to-Neumann operator (1.1), Theorem 1.13 provides an answer to **Problem 1**. Moreover, given a string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$ and m-accretive operator A on a Banach space X, translating the associated Dirichlet-to-Wentzell operator Λ_m by $\alpha \in \mathbb{R}$ leads to the *Dirichlet-to-'Wentzell-Robin' operator* $W_{\alpha,m}$ (cf, [4]) given by

$$W_{\alpha,m}f = (\alpha\,\mathrm{id}_X + \Lambda_m)f \qquad \text{for every } f \in D(\Lambda_m). \tag{1.27}$$

If ψ_m is the complete Bernstein function associated with m, having Lévy triple $(0, m(0+), \nu_m)$, then $\psi_{\alpha,m}(\lambda) := \alpha + \psi_m(\lambda)$, $\lambda \geq 0$, is a complete Bernstein function with Lévy triple $(\alpha, m(0+), \nu_m)$. Thus, according to Theorem 1.13, the operator $\psi_{\alpha,m}(A)$ defined by (1.25) can be characterized by

$$\psi_{\alpha,m}(A)f = (\alpha\,\mathrm{id}_X + \Lambda_m)f \qquad \text{for every } f \in D(A).$$

The main results of this paper and and the preceding Remark 1.16 lead naturally to the following open problem.

Open Problem. *Given a general Bernstein function ψ on $\mathbb{R}$ and an accretive operators A on a Banach space X, can the operator $\psi(A)$ still be characterized as an operator similar to the Dirichlet-to-'Wentzell-Robin' operator $W_{\alpha,m}$, which is associated with an extension problem similar to (1.7)?*

We conclude this paper with a stability result of the operator $\psi_m(A)$ by varying the string m.

Theorem 1.17. *For a given sequence $\{m_n\}_{n\ge1} \subseteq \mathfrak{m}_\infty$ of strings m_n on $\mathbb{R}$ of infinite length, let ψ_{m_n} be the corresponding complete Bernstein functions with Lévy triple $(0, m_n(0+), \nu_{m_n})$. Further, for a given m-accretive operator A on X, let $\psi_{m_n}(A)$ be the operator given by (1.25) for these triples, and $\{e^{-t\psi_{m_n}(A)}\}_{t\ge0}$ the semigroups generated by $-\psi_{m_n}(A)$. If there is a string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$ of infinite length such that*

$$\lim_{n\to\infty} m_n(z) = m(z) \qquad \textit{pointwise for every continuity point } z \textit{ of } m, \tag{1.28}$$

then there is a complete Bernstein function ψ_m with Lévy triple $(0, m(0+), \nu_m)$ such that for every $f \in D(A)$, there exists an $f_n \in D(\psi_{m_n}(A))$ such that

$$\lim_{n\to\infty} x_n = x \quad \textit{in } X \quad \textit{and} \quad \lim_{n\to\infty} \psi_{m_n}(A) f_n = \psi_m(A) f \quad \textit{in } X. \tag{1.29}$$

The next remark is a reminder for later reference.

Remark 1.18. For a better understanding of the type of convergence obtained in 1.29 of Theorem 1.17, it worth recalling Trotter-Kato's first approximation theorem (cf, [14, Theorem 1.8 in Chapter IV.]). Accordingly to this theorem, the following statements are equivalent:

(1) $\psi_{m_n}(A)$ converges to $\psi_m(A)$ in the *graph sense*, that is, for every $f \in D(A)$, there exists an $f_n \in D(\psi_{m_n}(A))$ such that (1.29) holds;

(2) $\psi_{m_n}(A)$ converges to $\psi_m(A)$ *strongly in the resolvent sense*, that is, for every $f \in X$, and some or all $\lambda > 0$,
$$\lim_{n\to\infty} R(\lambda, \psi_{m_n}(A)) f = R(\lambda, \psi_m(A)) f \qquad \text{in } X,$$
where for every $\lambda > 0$ and $n \ge 1$, $R(\lambda, \psi_{m_n}(A)) := (\lambda + \psi_{m_n}(A))^{-1}$ denotes the *resolvent* operator of $\psi_{m_n}(A)$ and $R(\lambda, \psi_m(A))$ the resolvent operator of $\psi_m(A)$;

(3) For every $f \in X$, $e^{-t\psi_{m_n}(A)} f \to e^{-t\psi_m(A)} f$ in X uniformly for t in compact subintervals of $[0, \infty)$.

We outline the proof of Theorem 1.17 in Section 4.5.

1.3. **Organization of this paper.** The structure of this paper is as follows. In the subsequent Section 2, we provide a historical development of the so-called *extension technique*, mention important related contributions, and discuss various approaches to establish this technique. Section 3 is dedicated to collect various notions and intermediate results, which are necessary to prove our main results in Section 4. In particular, we recall the construction of *generalized diffusion processes* (Section 3.1), and we briefly review the notion of an *inverse local time* and it relation to complete Bernstein functions (Section 3.2). Since in [11], the idea to derive the extension equation (1.4) relies on applying the *change of variable* $z = (y/2\sigma)^{2\sigma}$ to the solution u of the Bessel equation (1.3), we provide in the two Sections 3.3 & 3.4 a stochastic point of view of the impact of such a change of variable on the extension equation (1.8) and illustrate its usefulness later in Section 5 on two applications. Section 3.5 is dedicated to the hitting time τ of generalized diffusion $\{Z_t\}_{t\geq 0}$ associated with a string m on $\mathbb{R}$, the probability density ω_τ of τ, the transition density $\hat{p}$ of the killed process $\{\hat{Z}_t\}_{t\geq 0}$ of $\{Z_t\}_{t\geq 0}$ and derive the spectral representations of the hitting time density ω_τ. The intermediate results gathered in this subsection are crucial to prove our main theorems in this paper. In the sections 3.6 & 3.7, we briefly review the notions of C_0-semigroups, m-accretive operators on Banach spaces, and subordination of semigroups. As mentioned above, Section 4 is dedicated to the proof of our two main results Theorem 1.9 and Theorem 1.13. In Section 5, we apply our main theorems for providing a short proof of the classic limit $A^\sigma \to A$ in the *graph sense* as $\sigma \to 1-$. We also discuss briefly the case $A^\sigma \to \mathrm{id}_X$ in the *graph sense* as $\sigma \to 0+$.

For the reader, who is not familiar with the mixed framework of stochastic analysis and PDEs, we provide in the appendix of this paper a short primer on Bessel processes (Appendix A). In addition, in Appendix B, we provide important properties of the m-derivative (1.9), which are not available in the literature, but are necessary for the proofs in this paper.

2. Historical development of the extension technique

Throughout this section, let Σ be an open subset of $\mathbb{R}^d$, $d\geq 1$, $\mathrm{d}\eta$ a positive measure defined on Σ, and A a second order partial differential operator such that A is positive, densely defined, and self-adjoint in $L^2(\Sigma,\eta)$. Let Σ_+ denote the half-open cylinder $\Sigma\times(0,\infty)$ with pairs $(x,y)\in\Sigma_+$. Then, the main object of this section are the Dirichlet problem

$$\begin{cases} Au - y^{2\sigma-1}\{y^{1-2\sigma}u_y\}_y &= 0 \qquad \text{in } \Sigma_+,\\ u(x,0) &= f \qquad \text{on } \Sigma, \end{cases} \tag{2.1}$$

and the associated Dirichlet-to-Neumann operator

$$f \mapsto \Lambda_\sigma f := \frac{(-1)}{2\sigma}\lim_{y\to 0^+} y^{1-2\sigma}u_y(\cdot,y). \tag{2.2}$$

We note that the above framework is the same as in Stinga and Torrea [37], who extended some of the results in [11] on fractional powers A^σ to this more general framework.

2.1. A trace process approach.

To the best of our knowledge, Sato and Ueno [34, Theorem 9.1] were the first, who studied the *trace process* $\{X_t^*\}_{t\geq 0}$ on a smooth boundary $\partial\Sigma$ of a bounded domain Σ in a sufficiently smooth N-dimensional manifold, whose generator is the negative Dirichlet-to-Neumann operator $-\Lambda$ given by

$$f_{|\partial\Sigma} \mapsto \Lambda f := \tfrac{\partial u_f}{\partial\nu}_{|\partial\Sigma} \tag{2.3}$$

associated with a second-order uniformly elliptic differential operator A with symmetric coefficients on Σ. Their original aim was to analyze second-order uniformly elliptic differential operator operators equipped with Wentzell boundary conditions and their corresponding stochastic processes. They showed that

$\{X_t^*\}_{t\ge 0}$ is the Markov process obtained from a reflecting diffusion $\{X_t\}_{t\ge 0}$ through the *time change*

$$X_t^* := X_{L^{-1}(t)}, \qquad (t \ge 0), \tag{2.4}$$

by the local time $\{L_t\}_{t\ge 0}$ of $\{X_t\}_{t\ge 0}$ on the boundary $\partial\Sigma$ with

$$L_t^{-1} := \sup\{s \ge 0 \,|\, L_s \le t\} \qquad \text{for all } t \ge 0. \tag{2.5}$$

In the mid 80s, Hsu [20] constructed a Skorokhod-type lemma (see [39, Lemma 2.1] for the original result by Skorokhod which applies to the half-space) for bounded domains Σ in $\mathbb{R}^N$ with a C^2-boundary $\partial\Sigma$. By applying this lemma to a Brownian motion $\{B_t\}_{t\ge 0}$ inside Σ, Hsu obtained a reflecting Brownian motion $\{X_t\}_{t\ge 0}$ inside of Σ. Then by using Itô's lemma, Hsu showed that the Dirichlet-to-Neumann operator (2.3) associated with the (scaled) Laplacian $\frac{1}{2}\Delta$ is the infinitesimal generator of the trace process $\{X_t^*\}_{t\ge 0}$ again obtained from $\{X_t\}_{t\ge 0}$ via (2.4) through the time change (2.5) (see [20, Proposition 4.1]).

To see the two-step construction of the Dirichlet-to-Neumann operator via stochastic analysis, it is worth recalling that a reflecting Brownian motion $\{X_t\}_{t\ge 0}$ inside a domain Σ is a classical Brownian motion before it hits the first time the boundary $\partial\Sigma$. But Brownian motion killed at the boundary $\partial\Sigma$ corresponds to the classical Dirichlet problem for the (scaled) Laplacian $\frac{1}{2}\Delta$ on Σ, and reflecting Brownian motion corresponds to the Neumann problem involving the (scaled) Laplacian $\frac{1}{2}\Delta$.

The trace process $\{X_t^*\}_{t\ge 0}$ on the bottom Σ of the cylinder Σ_+ generated by the Dirichlet-to-Neumann operator Λ_σ given in (2.2) is obtained by starting with the *joint process*

$$\{\mathbb{X}_t\}_{t\ge 0} := \{(X_t, Y_t)\}_{t\ge 0} \qquad \text{in } \Sigma_+$$

for given processes $\{X_t\}_{t\ge 0}$ generated by the operator A in Σ and $\{Y_t\}_{t\ge 0}$ generated by the Bessel-operator $\mathcal{B}_{1-2\sigma} = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1-2\sigma}{y}\right)$ on the half-line $(0, +\infty)$.

To the best of our knowledge, the joint process $\{\mathbb{X}_t\}_{t\ge 0}$ occurred the first time in the short paper [30] by Molčanov and Ostrovskiĭ. In the case Σ is the Euclidean space $\mathbb{R}^N$, they proved that the trace process $\{X_t^*\}_{t\ge 0}$ on $\Sigma = \mathbb{R}^N$ obtained from (2.4) through the time change (2.5) of the local time $\{L_t\}_{t\ge 0}$ of $\{Y_t\}_{t\ge 0}$ at $y = 0$ is generated by the fractional Laplace operator $(-\Delta)^\sigma$ on $\mathbb{R}^N$.

In the literature, one finds claims that instead of the highly-cited paper [11] by Caffarelli and Silvestre, the approach by Molčanov and Ostrovskiĭ was the first (stochastic) proof to the extension property of $(-\Delta)^\sigma$ (i.e., Theorem 1.13 applied to $A = -\Delta$ on Σ). But, the final step, the identification of the infinitesimal generator of the trace process with the Dirichlet-to-Neumann map Λ_s, was not a considered aim in [30]. We agree with the sentiment in [6] that the proof of this identification is far from obvious. In fact, if one intends to prove the result in [11] by using the trace process of the joint process given by $\{(B_t, Y_t)\}_{t\ge 0}$ in $\mathbb{R}^{d+1}_+$, where $\{B_t\}_{t\ge 0}$ is a Brownian motion, then Itô's lemma cannot be applied immediately since the Bessel process $\{Y_t\}_{t\ge 0}$ is no longer a semi-martingale when $\delta \in (0, 1)$ (see Appendix A for more details).

But a potentially more fruitful approach would instead be to consider the trace process of the joint process $\{(B_t, Y_t^{2\sigma})\}_{t\ge 0}$ in $\mathbb{R}^{d+1}_+$ since according to Theorem A.1, the process $\{Y_t^{2\sigma}\}_{t\ge 0}$ is a submartingale. Nevertheless, in this setting one still needs to overcome certain technical details, in order to apply the ideas in [20]. We direct the reader to [6] for details. The scaling of the process $\{Y_t\}_{t\ge 0}$ to $\{Y_t^{2\sigma}\}_{t\ge 0}$ is a common technique in stochastic analysis outlined in Section 3.3 and Section 3.4 and was also used in [11].

2.2. **A Fourier approach.** A first approach to use the Fourier transform to prove Theorem 1.13 in the case of the negative Laplace operator $A = -\Delta$ on $\Sigma = \mathbb{R}^N$ was provided by Caffarelli and Silvestre [11]. Under the additional assumption that the operator A has a discrete spectrum $\sigma(A) = \{\lambda_k\}_{k\ge 0}$, Stinga and Torrera [38] outlined the following Fourier-series approach to the existence and uniqueness of solutions u to the Dirichlet problem (2.1). Under this hypothesis, there is an orthonormal basis $\{\phi_k\}_{k\ge 0}$ in $L^2(\Sigma,\mu)$ such that $A\phi_k = \lambda_k\phi_k$ for all $k \ge 0$. Then, for given $f \in L^2(\Sigma,\mu)$, the Fourier-series

$$u(\cdot,y) = \sum_{k\ge 0} c_k(y)\phi_k \qquad \text{converging in } L^2(\Sigma,\eta) \text{ for every } y > 0, \tag{2.6}$$

is the unique solution of Dirichlet problem (2.1) if and only if for every $k \ge 0$, c_k is the unique solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}_{\sigma,k}c_k &= 0, \qquad\qquad \text{on } (0,\infty),\\ c_k(0) &= \langle f,\phi_k\rangle_{L^2(\Sigma,\eta)},\\ \lim\limits_{y\to+\infty} c_k(y) &= 0. \end{cases} \tag{2.7}$$

for the *Sturm-Liouville operator*

$$\mathcal{L}_{\sigma,k}c(y) := -\frac{1}{y^{1-2\sigma}}\{y^{1-2\sigma}c'\}' + \lambda_k\, c.$$

If K_σ denotes the *modified Bessel function of the third kind*, then the unique solution c_k to Dirichlet problem (2.7) is given by

$$c_k(y) = y^\sigma \frac{2^{1-\sigma}}{\Gamma(\sigma)}\lambda_k^{\sigma/2}\langle f,\phi_k\rangle_{L^2(\Sigma,\eta)}K_\sigma(\lambda_k^{1/2}y).$$

By using the asymptotic of K_σ as $y \to 0+$ and $y \to +\infty$, one sees that the series (2.6) is a solution of Dirichlet problem (2.1).

Motivated by these ideas, we provide in this paper a stochastic proof to our main result (Theorem 1.13). This approach simplifies essentially the recently appeared proofs related to Theorem 1.13 by Assing and Herman [6], and Kwaśnicki and Mucha [27].

2.3. **A stochastic representation formula.** From a stochastic analytical point of view, it is worth mentioning that the *Sturm-Liouville equation*

$$-\, y^{2\sigma-1}\{y^{1-2\sigma}c'\}' + \lambda_k\, c = 0 \qquad \text{on } (0,\infty) \tag{2.8}$$

related to Dirichlet problem (2.7) is studied heavily in relation with the *first hitting time* τ at zero of the $2(1-\sigma)$-Bessel Process $\{Y_t\}$ starting at $y >$

0. To simplify notation, we suppress the y-dependence of τ and we refer to Appendix A. In fact, this is quite natural, since in the *Poisson formula*

$$u(x,y) = \int_0^\infty (e^{-tA}f)(x)\,\omega_\sigma(t,y)\,dt \qquad \text{on } \Sigma_+, \tag{2.9}$$

of the weak solution u to the Dirichlet problem (2.1), the density

$$\omega_\sigma(t,y) = \frac{y^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)} e^{-\frac{y^2}{4t}} \frac{1}{t^{1+\sigma}} \qquad \text{for every } t,\, y > 0, \tag{2.10}$$

coincides exactly with the *first hitting time density* τ (cf., [17], and [10, No. 43 & 44, p.75]). We note that the Poisson formula (2.9) was obtained in [37] by Stinga and Torrea. In other words, for (2.9), we have

$$\begin{aligned}
u(x,y) &= \frac{y^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)} \int_0^\infty (e^{-tA}f)(x)\, e^{-\frac{y^2}{4t}} \frac{\mathrm{d}t}{t^{\sigma+1}}\\
&= \frac{y^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)} \int_0^\infty \mathbb{E}^x\left(f(X_t)\right)\, e^{-\frac{y^2}{4t}} \frac{\mathrm{d}t}{t^{\sigma+1}}\\
&= \int_0^\infty \mathbb{E}_x\left(f(X_t)\right)\, \mathrm{d}\mathbb{P}_{\tau_{y/\sqrt{2}}}(t)\\
&= \int_0^\infty \mathbb{E}_x(f(X_t)\,|\tau_y \in dt)\,\mathbb{P}(\tau_y \in dt)\\
&= \int_0^\infty \mathbb{E}_x(f(X_{\tau_y})\,|\tau \in dt)\,\mathbb{P}(\tau \in dt)\\
&= \mathbb{E}_{(x,y)}\left(\mathbb{E}_{(x,y)}\left(f(X_\tau)\,\middle|\,\tau\right)\right) = \mathbb{E}_{(x,y)}\left(f(X_\tau)\right),
\end{aligned}$$

where we used that $\{e^{-tA}\}_{t\geq 0}$ with $e^{-tA}f(x) := \mathbb{E}_x\left(f(X_t)\right)$ is the transition semigroup of the process $\{X_t\}_{t\geq 0}$, and the independence of $\{X_t\}_{t\geq 0}$ and τ. Thus, the Poisson formula (2.9) is nothing less than a stochastic representation formula to Dirichlet problem (2.1). In this paper, we outline how to derive from Sturm-Liouville equation (2.8) the stochastic representation formula (1.11) to the more general Dirichlet problem (1.7).

2.4. **The Lévy measure of the inverse local time.** The following observation is critical for the complete Bernstein case. Let $s(y) = y^{2\sigma}$, which is also the *scale function* of the Bessel process $\{Y_t\}_{t\geq 0}$ (see Section 3.3 for more details). Then, for solutions u to the extension problem (2.1), Stinga and Torrera [38] showed that

$$\lim_{y\to 0+} \frac{1}{s'(y)} \frac{\partial u}{\partial y}(x,y) = \lim_{y\to 0+} \frac{1}{s(y)}(u(x,y) - u(x,0)).$$

Inserting (2.9) into $u(x,y)$ and $u(x,0)$, and by using that

$$\mathrm{d}\nu(t) = \frac{\mathbb{1}_{\{t>0\}}}{2^{2\sigma}\Gamma(\sigma)} \frac{\mathrm{d}t}{t^{\sigma+1}} \tag{2.11}$$

is a Lévy measure, dominated convergence and Phillips' subordination theorem yield that

$$\lim_{y\to 0+} \frac{1}{s'(y)} \frac{\partial u}{\partial y}(x,y) = \lim_{y\to 0+} \int_0^\infty (e^{-tA}f(x) - f(x)) \left(\frac{1}{s(y)}\omega_\sigma(t,y)\right) dt$$

$$= \int_0^\infty (e^{-tA} f(x) - f(x)) \frac{1}{2^{2\sigma}\Gamma(\sigma)} \frac{1}{t^{1+\sigma}}\, dt$$
$$= -\frac{\Gamma(1-\sigma)}{2^{2\sigma}\Gamma(1+\sigma)} A^\sigma f(x).$$

Here, the important observation is that for every $t > 0$,

$$\lim_{y\to 0+} \frac{1}{s(y)} \omega_\sigma(t,y) = \frac{1}{2^{2\sigma}\Gamma(\sigma)} \frac{1}{t^{1+\sigma}} = \lim_{y\to 0+} \frac{1}{s'(y)} \frac{\partial \omega_\sigma}{\partial y}(t,y),$$

where $\frac{1}{2^{2\sigma}\Gamma(\sigma)}\frac{1}{t^{1+\sigma}}$ is the density of the measure ν given in (2.11). However, it is well-known for quite a while (cf [22] or [12]) that the measure ν is the Lévy measure of the inverse local time process of the $2\sigma^{\text{th}}$-powered process $\{Y_t^{2\sigma}\}_{\ge 0}$ of the $2(1-\sigma)$-Bessel process $\{Y_t\}_{\ge 0}$ (cf, Theorem A.1).

The goal of this paper is to exploit the idea of the first hitting time density τ for proving the extension technique for operators $\psi(A)$ provide ψ is a complete Bernstein function.

3. Preliminaries

In this section, we introduce several important notions and collect related results, which are required to proof our main theorems stated in Section 1.2.

3.1. **A primer on generalized diffusion processes.** In this section, we briefly collect some basic notions from the theory of generalized diffusion processes and strings needed throughout this paper. For an excellent review of the subject, we refer the interested reader to [36, Chapter 14] and for a more in depth resource see [13, Chapter 5 & 6].

After providing the definition of a string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$ (Definition 1.1), and the associated measure μ_m, we still need to introduce to the following notion.

Definition 3.1. For every string $m \in \mathfrak{m}_\infty$ we denote by

$$E_m := \operatorname{supp}(\mu_m)$$

the *support of* μ_m.

Remark 3.2. We note that in the language of Feller's theory of diffusion processes (cf., [31, Chapter III and VII]), the measure μ_m plays the role of a *speed measure.*

With this preliminary in mind, we can start introducing generalized diffusion processes. Fix a string $m \in \mathfrak{m}_\infty$, and for given $z \in [0,\infty)$, let $\{B_t^+\}_{t\ge 0}$ be a reflecting Brownian motion on $[0,\infty)$ starting at z. We refer to $\{B_t^+\}_{t\ge 0}$ as the *reflecting Brownian motion.*

It is well known that the reflecting Brownian motion $\{B_t^+\}_{t\ge 0}$ is a Hunt process, which is symmetric with respect to the Lebesgue measure and hence, the process $\{B_t^+\}_{t\ge 0}$ has an associated Dirichlet form $\mathcal{E}$ on $L^2((0,\infty),\mathrm{d}x)$. We refer to [36, Appendix A.2] and the classical textbook [15] concerning Dirichlet forms and symmetric Hunt processes. To the Dirichlet form $\mathcal{E}$, one can define the *capacity* Cap by

$$\operatorname{Cap}(O) := \inf\Big\{\mathcal{E}(\phi,\phi) + \|\phi\|^2_{L^2((0,\infty),\mathrm{d}x)} \,|\, \phi \ge 1 \text{ a.e.}\Big\}$$

for every open subset O of $[0,\infty)$. Now, it is worth mentioning that for the string m, the associated measure m is *smooth* in the sense that it does not charge any set of zero capacity and if there is an increasing sequence $(A_n)_{n\geq 1}$ of closed subsets $A_n \subseteq [0,\infty)$ satisfying $m(A_n) < \infty$ for all $n \geq 1$ and $\lim_{n\to\infty} \mathrm{Cap}(K \setminus A_n) = 0$ for all compact subsets K of $[0,\infty)$.

For every $z \in (0,\infty)$, let $\{L_t(z)\}_{t\geq 0}$ be the local time process at level z of the reflecting Brownian motion $\{B_t^+\}_{t\geq 0}$. Then, the family $\{L_t(z)\}_{t\geq 0, 0\leq z<\infty}$ is jointly continuous and (after normalization) the following *occupation time formula*

$$\int_0^t g(B_r^+)\,\mathrm{d}r = \int_0^\infty g(z)\,L_t(z)\,\mathrm{d}z \qquad \text{holds for all } t \geq 0, \tag{3.1}$$

and all Borel functions $g : [0,\infty) \to [0,\infty)$. Since for the string m, the associated measure m is *smooth* and for the reflecting Browning motion $\{B_t^+\}_{t\geq 0}$ only the empty set has capacity zero, we can define a positive continuous additive process $\{A_t\}_{t\geq 0}$ by setting

$$A_t = \int_{[0,\infty)} L_t(z)\,\mathrm{d}\mu_m(z) \qquad \text{for all } t \geq 0. \tag{3.2}$$

It follows from our construction (cf., [15, Chapter 5]) that the *Revuz measure* of the process $\{A_t\}_{t\geq 0}$ is the measure μ_m. Now, let $\{Z_t\}_{t\geq 0}$ be the process defined by the *time change* $\{A_t^{-1}\}_{t\geq 0}$, that is,

$$Z_t := B^+_{A_t^{-1}} \qquad \text{for every } t \geq 0, \tag{3.3}$$

where $\{A_t^{-1}\}_{t\geq 0}$ is the right-continuous inverse of $\{A_t\}_{t\geq 0}$ given by

$$A_t^{-1} = \inf\{s > 0 \,|\, A_s > t\}, \qquad (t \geq 0).$$

Then, $\{Z_t\}_{t\geq 0}$ is an m-symmetric, continuous Hunt process in $[0,\infty)$ with infinitesimal generator (see, for example, [15, Theorem 6.2.1])

$$-B_m = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z} \qquad \text{on } L^2(E_m,\mu_m). \tag{3.4}$$

Definition 3.3. For given $m \in \mathfrak{m}_\infty$, the process $\{Z_t\}_{t\geq 0}$ defined by (3.3) is called a *generalized diffusion process associated with m.*

Here, the operator B_m given in (3.4) also occurs in the next definition (cf, [36, p. 271]).

Definition 3.4. Let E be a non-empty, connected subset of $\mathbb{R}$ with $-\infty \leq l := \inf E < r := \sup E \leq \infty$, m be a Radon measure on E, and $s : E \to \mathbb{R}$ a continuous, strictly increasing function. Then, let $W^{2,1}_{loc,ds,\mu}(E)$ be the set of functions $u : \mathbb{R} \to \mathbb{C}$ of the form

$$u(z) = \alpha + \beta z + \int_l^z \int_{(l,y]} g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}s(y), \qquad (z \in \mathbb{R}),$$

for some α, $\beta \in \mathbb{C}$ and locally m-integrable $g : \mathbb{R} \to \mathbb{C}$. Note, the second integral w.r.t. $\mathrm{d}s(y)$ is a Lebesgue-Stieltjes integral. If $-\infty < l$ then for every $z \leq l$, one interprets the interval $[l,y] = \emptyset$ for $y < z$ and hence, $u(z) = \alpha + \beta z$ for $z < l$. Then, the 2nd-order differential operator

$$-B_{m,s} = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}$$

is defined by $-B_{m,s}u = g$ for every $u \in W^{2,1}_{loc,ds,\mu_m}(E)$.

Moreover, the following properties hold.

Proposition 3.5. *Let $m \in \mathfrak{m}_\infty$ be a string, $s = id_{\mathbb{R}}$ the identity on $\mathbb{R}$, and B_m be the operator given by (3.4) with domain $D(B_m)$. Then, for every $u \in D(B_m)$, one has that*

(1) u is locally absolutely continuous on $\mathbb{R}$,
(2) u is linear on $\mathbb{R} \setminus E_m$,
(3) the right derivative $\frac{\mathrm{d}u}{\mathrm{d}z+}$ and left derivative $\frac{\mathrm{d}u}{\mathrm{d}z-}$ exist on $\mathbb{R}$, and

$$\frac{\mathrm{d}u}{\mathrm{d}z+}(x) - \frac{\mathrm{d}u}{\mathrm{d}z-}(x) = \mu_m\{x\}B_m u(x) \qquad \textit{for every } 0 \le x \le \sup E_m.$$

We omit the elementary proofs to the above proposition.

For a generalized diffusion process $\{Z_t\}_{t\ge 0}$, we can associate for every $z \in E_m$, a local time process $\{\tilde{L}_t(z)\}_{t\ge 0}$ at level z, which can be realized as a time-change of the local time process $\{L_t(z)\}_{t\ge 0}$ of the reflecting Brownian motion $\{B_t^+\}_{t\ge 0}$. Namely, we have that

$$\tilde{L}_t(z) = L_{A_t^{-1}}(z) \qquad \text{for every } t \ge 0,\ z \in [0,\infty), \tag{3.5}$$

and that the following *occupation times formula*

$$\int_0^t g(Z_r)\,\mathrm{d}r = \int_{[0,\infty)} g(z)\tilde{L}_t(z)\,\mathrm{d}\mu_m(z) \qquad \text{for all } t \ge 0 \tag{3.6}$$

holds for all essentially bounded Borel functions $g : [0,\infty) \to [0,\infty)$.

3.2. **Inverse local times and complete Bernstein functions.** For a generalized diffusion process $\{Z_t\}_{t\ge 0}$, let $\{\tilde{L}_t\}_{t\ge 0}$ be the local time $\{\tilde{L}_t(0)\}_{t\ge 0}$ of $\{Z_t\}_{t\ge 0}$ at the level $z = 0$, and $\{\tilde{L}_t^{-1}\}_{t\ge 0}$ be the inverse local time give by

$$\tilde{L}_t^{-1} = \inf\left\{r > 0 \,\middle|\, \tilde{L}_r > t\right\} \qquad \text{for every } t \ge 0.$$

Then, by (3.2) and (3.5), we have that

$$\tilde{L}_t^{-1} = A_{L_t^{-1}} = \int_{[0,\infty)} L_{L_t^{-1}}(z)\,\mathrm{d}\mu_m(z) \qquad \text{for every } t \ge 0. \tag{3.7}$$

It is worth noting that the inverse local time $\{\tilde{L}_t^{-1}\}_{t\ge 0}$ is a *subordinator* (cf., [9, p. 114]), that is, a one-dimensional Lévy process that is non-decreasing (a.s.). Hence (see [2, Theorem 1.3.15]), there is a Lévy measure ν_m on $\mathcal{B}(\mathbb{R} \setminus \{0\})$ satisfying $\nu_m(-\infty, 0) = 0$,

$$\int_0^\infty \left(r \wedge \mathbb{1}_{(0,\infty)}(r)\right)\mathrm{d}\nu_m(r) < \infty, \tag{3.8}$$

and there is some $b \ge 0$ such that the *Lévy symbol* η of $\{\tilde{L}_t^{-1}\}_{t\ge 0}$ is given by

$$\eta_m(s) = ibs + \int_0^\infty \left(e^{isr} - 1\right)\mathrm{d}\nu_m(r) \qquad \text{for all } s \in \mathbb{R}.$$

From this, one finds that the *Laplace exponent* ψ_m of $\{\tilde{L}_t^{-1}\}_{t\ge 0}$ is given by

$$\psi_m(\lambda) = m(0+)\lambda + \int_0^\infty \left(1 - e^{-\lambda r}\right)\mathrm{d}\nu_m(r) \qquad \text{for all } \lambda \ge 0. \tag{3.9}$$

The right hand-side in (3.9) of the Laplace exponent ψ_m provides an important example of a *Bernstein function* (cf, Definition 1.11 and, cf., [36, Theorem 3.2]).

Remark 3.6. Due to the Bernstein's theorem of monotone functions [36, Theorem 4.8], one has that for every completely monotone function h, there is an associated measure $\hat{\Delta}$ satisfying

$$h(r) = \int_0^\infty e^{-r\,s}\,\mathrm{d}\hat{\Delta}(s) \qquad \text{for all } r > 0. \tag{3.10}$$

Definition 3.7. For given $m \in \mathfrak{m}_\infty$, let $\{\hat{Z}_t\}_{t\geq 0}$ be the generalized process $\{Z_t\}_{t\geq 0}$ associated with m killed at level 0, and $h_m = \frac{\mathrm{d}\nu_m}{\mathrm{d}r}$ be the density of the Lévy measure ν_m associated with m. Then, we call the measure $\hat{\Delta}_m$ satisfying (3.10) for h_m the *principal measure* of the process $\{\hat{Z}_t\}_{t\geq 0}$.

The Laplace exponent of the *inverse local time processes* of a given generalized diffusion process can be described in terms of Bernstein functions thanks to the famous result [23] by Knight.

Theorem 3.8 (Knight's theorem [23, Theorem 3.1 and Theorem 1.2])**.** *For given string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$, let $\{Z_t\}_{t\geq 0}$ be a generalized diffusion associated with m and $\{\tilde{L}_t^{-1}\}_{t\geq 0}$ be the corresponding inverse local time at 0. Then the Laplace transform of $\tilde{L}_t^{-1}$ is given by*

$$\mathbb{E}\Big(e^{-\lambda \tilde{L}_t^{-1}}\Big) = e^{-t\psi_m(\lambda)} \qquad \textit{for all } \lambda > 0, \tag{3.11}$$

where $\psi_m : (0,\infty) \to \mathbb{R}$ is a complete Bernstein function given by

$$\psi_m(\lambda) = m(0+)\lambda + \int_0^\infty \Big(1 - e^{-\lambda r}\Big)\,\mathrm{d}\nu_m(r) \qquad \textit{for all } \lambda > 0. \tag{3.12}$$

Moreover, the mapping $m \mapsto \psi_m$ with ψ_m defined by (3.12) is a bijection $\Psi : \mathfrak{m}_\infty \to \mathcal{CBF}$, $\Psi(m) := \psi_m$ for every $m \in \mathfrak{m}_\infty$ known as the Kreĭn's correspondence.

We shall discuss properties of the principal measure $\hat{\Delta}_m$ in Section 3.5.

3.3. **Substitution and Scale functions.** The aim of this section is to outline the connection between the substitution used in [11] to simplify the extension problem (1.2) and the scaling of associated stochastic processes and to give an overview of the impact on the corresponding Dirichlet-to-Wentzell operator (see Section 3.4).

It is clear that the substitution

$$z = \Big(\frac{y}{2\sigma}\Big)^{2\sigma} \qquad \text{for } y > 0 \text{ and fixed } 0 < s < 1,$$

transforms the differential equation in the Dirichlet problem (2.1) to the non-divergence form equation

$$Au - \frac{1}{z^{-\frac{2\sigma-1}{\sigma}}}u_{zz} = 0 \qquad \text{in } \Sigma \times (0,\infty);$$

see [11], or alternatively for more details, see [18, Lemma 3.4]. Moreover, Stinga and Torrea [37] proved that for the scale function

$$s(y) = y^{2\sigma} \qquad \text{for } y > 0 \text{ and fixed } 0 < \sigma < 1, \tag{3.13}$$

the co-normal derivative (2.2) can be rewritten as

$$\frac{\mathrm{d}u}{\mathrm{d}s}(\cdot,0) := \lim_{y\to 0+}\frac{u(\cdot,y)-u(\cdot,0)}{s(y)} = \frac{(-1)}{2\sigma}\lim_{y\to 0+} y^{1-2\sigma}u_y(\cdot,y).$$

On the other hand, in stochastic analysis the function s given by (3.13) is used to *scale* the $2(1-\sigma)$-Bessel process $\{Y_t\}_{t\ge 0}$ in $[0,\infty)$ starting at $y=0$ in order to derive stronger properties of the process (see Theorem A.1 in the appendix). In order to give more details t this, we briefly need to review the necessary definition and results on *scale functions* from [31, Chapter 7].

For given $-\infty \le l < r \le \infty$, let $E_{l,r}$ be either a closed, open or semi-closed interval in $\mathbb{R}$, and let $\{Y_t\}_{t\ge 0}$ be a continuous, regular, strong Markov process on the state space $E_{l,r}$ with a killing time ζ. Further, suppose, $\{Y_t\}_{t\ge 0}$ can only be killed at the end-points l and r of $E_{l,r}$, provided they do not belong to $E_{l,r}$. For given $c\in E_{l,r}$, we define the *first time* $\{Y_t\}_{t\ge 0}$ hits $y=c$ by

$$\tau_Y^c := \inf\Big\{t>0\,\Big|\, Y_t = c\Big\}. \tag{3.14}$$

In this framework the following important existence result of a *scale function* holds.

Proposition 3.9 ([31, Chapter VII, Proposition 3.2])**.** *Under the hypothesis of this subsection, there exists a continuous, strictly increasing function $s: E_{l,r}\to\mathbb{R}$ with the property that for every a, b, $y\in E_{l,r}$ satisfying $l\le a<y<b\le r$, one has that*

$$\mathbb{P}_y(\tau_Y^b < \tau_Y^a) = \frac{s(y)-s(a)}{s(b)-s(a)}, \tag{3.15}$$

where $\mathbb{P}_y(A)$ denotes the conditional probability of the event A under the condition that the process $\{Y_t\}_{t\ge 0}$ starts at y.

With this in mind, we can now give the definition of a scale function.

Definition 3.10. A continuous, strictly increasing function $s: E_{l,r}\to\mathbb{R}$ is called a *scale function* of a continuous, regular, strong Markov process $\{Y_t\}_{t\ge 0}$ on the state space $E_{l,r}$ if for every a, b, $y\in E_{l,r}$ satisfying $l\le a<y<b\le r$, one has that (3.15) holds.

We note that if $s: E_{l,r}\to\mathbb{R}$ is a scale function of the process $\{Y_t\}_{t\ge 0}$, then it is not hard to see that for every α, $\beta\in\mathbb{R}$, the function $\tilde{s}: E_{l,r}\to\mathbb{R}$ given by the *affine transformation*

$$\tilde{s}(y) = \alpha s(y)+\beta \qquad \text{for every } y\in E_{l,r}$$

is another scale function of the process $\{Y_t\}_{t\ge 0}$. Thus, if there is a scale function s of the process $\{Y_t\}_{t\ge 0}$, then there are infinitely many scale functions of $\{Y_t\}_{t\ge 0}$, and moreover, there is no loss of generality to assume that $s(0)=0$.

Now, if there is a scale function s of the process $\{Y_t\}_{t\ge 0}$ on $E_{l,r}$, then we can define the *scaled process* $\{Z_t\}_{t\ge 0}$ on the state space $E_{s(l),s(r)}$ by setting

$$Z_t := s(Y_t) \qquad \text{for every } t\ge 0,$$

where $s(l)$ and $s(r)$ are the possibly improper limits

$$s(l) := \lim_{y\to l+} s(y) \quad\text{ and }\quad s(r) := \lim_{y\to r-} s(y).$$

Now, for every a, b, $z \in E_{s(l),s(r)}$ satisfying $l \leq a < y < b \leq r$, one has that

$$\mathbb{P}_z(\tau_Z^b < \tau_Z^a) = \frac{z-a}{b-a},$$

showing that the identity $s(z) = z$ is a scale function of the scaled process $\{Z_t\}_{t\geq 0}$.

Definition 3.11. One says that a process $\{X_t\}_{t\geq 0}$ on the state space $E_{l,r}$ is of *natural scale* if the identity $s(x) = x$, $(x \in E_{l,r})$, is a scale function of the the process $\{X_t\}_{t\geq 0}$.

Processes of natural scale have the following characterization.

Proposition 3.12 ([31, Chapter VII, Proposition 3.5])**.** *A continuous, regular, strong Markov process process $\{X_t\}_{t\geq 0}$ on state space $E_{l,r}$ is of natural scale if and only if $\{X_t\}_{0\leq t\leq \tau_X^l\wedge\tau_X^r}$ is a local martingale.*

From this proposition, we can conclude the following characterization of a scale function s.

Corollary 3.13. *A continuous, strictly increasing function $s : E_{l,r} \to \mathbb{R}$ is called a* scale function *of a continuous, regular, strong Markov process $\{Y_t\}_{t\geq 0}$ on the state space $E_{l,r}$ if and only if for the scaled process $\{Z_t\}_{t\geq 0}$,*

$$\{Z_t\}_{0\leq t\leq \tau_Z^{s(l)}\wedge\tau_Z^{s(r)}} \qquad \textit{is a local martingale.} \tag{3.16}$$

With this preliminary, we can now come back to the example of the Bessel process.

Example 3.14. For $\sigma \in (0,1)$, the $2(1-\sigma)$-Bessel process $\{Y_t\}_{t\geq 0}$ starting at $y = 0$ for is a continuous, regular, strong Markov process has state space $E_{0,\infty} = [0,\infty)$. By Theorem A.1, the $2\sigma^{\text{th}}$-powered process $\{Y_t^{2\sigma}\}_{t\geq 0}$ is submartingale with the Doob-Meyer decomposition $Y_t^{2\sigma} = M_t + L_t$, consisting of a continuous martingale $\{M_t\}_{t\geq 0}$ and a continuous, non-decreasing process $\{L_t\}_{t\geq 0}$ carried by the zeros of $\{Y_t\}_{t\geq 0}$. Thus, for the function $s(y) = y^{2\sigma}$, the scaled process $\{Z_t\}_{t\geq 0}$ give by $Z_t = Y_t^{2\sigma}$, $(t \geq 0)$, satisfies (3.16) and hence by Corollary 3.13, $\{Z_t\}_{t\geq 0}$ is of natural scale; or in other words, s is a scale function of the $2(1-\sigma)$-Bessel process $\{Y_t\}_{t\geq 0}$.

3.4. **Speed measures.** In this section, we intend to outline the impact of scale function s on speed measures $\hat{m}$ and the associated generalized diffusion $\{Z_t\}_{t\geq 0}$ associated with $m \in \mathfrak{m}_\infty$.

We built upon the scenario from Section 3.3. For given $-\infty \leq l < r \leq \infty$, let $E_{l,r}$ be either a closed, open or semi-closed interval in $\mathbb{R}$, and $\{Y_t\}_{t\geq 0}$ be a continuous, regular, strong Markov process on the state space $E_{l,r}$ with a killing time ζ. Further, suppose, $\{Y_t\}_{t\geq 0}$ can only be killed at the end-points l and r of $E_{l,r}$, provided they do not belong to $E_{l,r}$.

Let $I = (a,b)$ be an open interval such that the closure $[a,b] \subset E_{l,r}$, and let $\sigma_I := \inf\{t > 0 \,|\, Y_t \notin I\}$ be the *first time* $\{Y_t\}_{t\geq 0}$ *exiting* the interval I. Then, for the starting point y of $\{Y_t\}_{t\geq 0}$, one has that $y \in I$ yields that the first exit time $\sigma_I = \tau_Y^a \wedge \tau_X^b$ almost surely, and $y \notin I$ implies that the first exit time $\sigma_I = 0$ almost surely. Further, let

$$m_I(y) := \mathbb{E}_y(\sigma_I) \qquad \text{for every } y \in \mathbb{R}.$$

Now, let us take $J = (c, d)$ to be an open subinterval of I. Then by the strong Markov property of $\{Y_t\}_{t\geq 0}$ and due to property (3.15), one obtains that for every scale function $s : E_{l,r} \to \mathbb{R}$ of $\{Y_t\}_{t\geq 0}$, one has that

$$\begin{aligned} m_I(y) &= m_J(y) + \mathbb{E}_y(\mathbb{E}_{Y_{\sigma_J}}(\sigma_I)) \\ &= m_J(y) + \frac{s(d)-s(y)}{s(d)-s(c)} m_I(c) + \frac{s(y)-s(c)}{s(d)-s(c)} m_I(d) \end{aligned}$$

for all $a < c < x < d < b$.

Given a scale function $s : E_{l,r} \to \mathbb{R}$ of $\{Y_t\}_{t\geq 0}$ and an open interval $I = (a, b)$ with closure $[a, b]$ in $E_{l,r}$, let

$$G_{s,I}(x, y) := \begin{cases} \dfrac{(s(x)-s(a))(s(b)-s(y))}{s(b)-s(a)} & \text{if } a \le x \le y \le b, \\ \dfrac{(s(y)-s(a))(s(b)-s(x))}{s(b)-s(a)} & \text{if } a \le x \le y \le b, \\ 0 & \text{otherwise.} \end{cases}$$

With this preliminary, we can now recall the following existence theorem of the *speed measure* $\hat{m}$.

Theorem 3.15 ([31, Chapter VII, Proposition 3.5])**.** *Let $\{Y_t\}_{t\geq 0}$ be a continuous, regular, strong Markov process on the state space $E_{l,r}$. Then, there exists a unique Radon measure $\mu_{\hat{m}}$ on the interior $\mathring{E}_{l,r}$ of $E_{l,r}$ and a scale function $s : E_{l,r} \to \mathbb{R}$ of $\{Y_t\}_{t\geq 0}$ such that for every open sub-interval $I = (a, b)$ of E, one has*

$$m_I(y) = \int_I G_{s,I}(y, r)\, \mathrm{d}\mu_{\hat{m}}(r) \qquad \textit{for every } y \in I. \tag{3.17}$$

The above existence theorem leads to the following definition.

Definition 3.16. For a continuous regular, strong Markov process $\{Y_t\}_{t\geq 0}$ on the state space $E_{l,r}$, one calls the Radon measure $\hat{m}$ satisfying (3.17) the *speed measure* of the process $\{Y_t\}_{t\geq 0}$.

For our next theorem, we recall the following elementary definition.

Definition 3.17. For a strictly increasing continuous function $s : (a, b) \to \mathbb{R}$ defined on an open interval (a, b) with range $\mathrm{Rg}(s) \subseteq \mathbb{R}$, a function $f : \mathrm{Rg}(s) \to \mathbb{R}$ is called *s-differentiable at $y \in \mathrm{Rg}(s)$* if the limit

$$\lim_{\hat{y}\to y} \frac{f(\hat{y}) - f(y)}{s(\hat{y}) - s(y)}$$

exists and then, we denote by

$$\frac{\mathrm{d}f}{\mathrm{d}s}(y) := \lim_{\hat{y}\to y} \frac{f(\hat{y}) - f(y)}{s(\hat{y}) - s(y)}$$

the *s-derivative of f at y*.

Remark 3.18. Recall, every continuous, strictly increasing function $s : (a, b) \to \mathbb{R}$ is a.e. differentiable in (a, b). Thus, if f is differentiable at $y \in \mathrm{Rg}(s)$ and $s'(y) > 0$ exists, then the s-derivative of f

$$\frac{\mathrm{d}f}{\mathrm{d}s}(y) = \frac{\mathrm{d}f}{\mathrm{d}y}(y) \frac{1}{s'(y)}. \tag{3.18}$$

Now, our next theorem outlines the relation between the speed measure $\hat{m}$ and *scale function* $s : E_{l,r} \to \mathbb{R}$ in sense of infinitesimal generator of the process $\{Y_t\}_{t\geq 0}$. Here, we denote by A_Y the infinitesimal generator of $\{Y_t\}_{t\geq 0}$ and by $D(A_Y)$ its domain.

Theorem 3.19 ([31, Chapter VII, Theorem 3.12 and Proposition 3.13])**.** *Let $\{Y_t\}_{t\geq 0}$ be a continuous, regular, strong Markov process on the state space $E_{0,r}$, $(0 < r \leq \infty)$, with infinitesimal generator B. Then, there is a scale function $s : E_{0,r} \to \mathbb{R}$ of $\{Y_t\}_{t\geq 0}$ such that*

$$Bf = \frac{\mathrm{d}}{\mathrm{d}\hat{m}} \frac{\mathrm{d}f}{\mathrm{d}s} \qquad \textit{for every } f \in D(B),$$

where $\hat{m}$ is the speed measure of $\{Y_t\}_{t\geq 0}$. Moreover, if $E_{0,r} = [0, r)$, then one has that

$$\hat{m}(0+)Bf(0) - \frac{\mathrm{d}f}{\mathrm{d}s_+}(0) = 0 \qquad \textit{for every } f \in D(B).$$

From this theorem, one see that the pair $(s, \hat{m})$ of the scale function s and the speed measure $\hat{m}$ given by Theorem 3.19 characterizes the infinitesimal generator B of the continuous, regular, strong Markov process $\{Y_t\}_{t\geq 0}$.

Let us consider the $2(1-\sigma)$-Bessel process $\{Y_t\}_{t\geq 0}$ as an example.

Example 3.20. We recall (cf., [12, Sect. 1.2]) that for $\sigma \in (0, 1)$, the infinitesimal generator B_Y of the $2(1-\sigma)$-Bessel process $\{Y_t\}_{t\geq 0}$ on $[0, \infty)$ is given by

$$\begin{aligned} B_Y u &= \frac{1}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \frac{1-2\sigma}{2y}\frac{\mathrm{d}u}{\mathrm{d}y} \\ &= \frac{1}{2y^{1-2\sigma}}\frac{\mathrm{d}}{\mathrm{d}y}\Big(y^{1-2\sigma}\frac{\mathrm{d}u}{\mathrm{d}y}\Big) \\ &= \frac{1}{2}\Big(\frac{y}{2\sigma}\Big)^{2\sigma-1}\frac{\mathrm{d}}{\mathrm{d}y}\Big(\Big(\frac{y}{2\sigma}\Big)^{1-2\sigma}\frac{\mathrm{d}u}{\mathrm{d}y}\Big). \end{aligned}$$

From the above computation, it is natural that the scale function s and speed measure $\mu_{\hat{m}_Y}$ are given by

$$s(y) = \Big(\frac{y}{2\sigma}\Big)^{2\sigma} \qquad \text{and} \qquad \mathrm{d}\mu_{\hat{m}_Y}(y) = 2\Big(\frac{y}{2\sigma}\Big)^{1-2\sigma}\mathrm{d}y,$$

and so, by (3.18), one sees that

$$B_Y u = \frac{1}{2}\Big(\frac{y}{2\sigma}\Big)^{2\sigma-1}\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}\hat{m}_Y}\frac{\mathrm{d}u}{\mathrm{d}s}.$$

Next, if we apply the substitution

$$z = s(y) = \Big(\frac{y}{2\sigma}\Big)^{2\sigma}$$

to the process $\{Y_t\}_{t\geq 0}$; that is, one sets $Z_t = s(Y_t)$, then the process $\{Z_t\}_{t\geq 0}$ is the (up to a multiple constant) $2\sigma^{\text{th}}$-power process (cf., Theorem A.1), which is of natural scale (cf., Example 3.14) with speed measure $\mu_{\hat{m}_Z}$ given by

$$\mathrm{d}\mu_{\hat{m}_Z}(z) = 2z^{-\frac{2\sigma-1}{\sigma}}\mathrm{d}z.$$

Hence, the infinitesimal generator B_Z of $\{Z_t\}_{t\geq 0}$ is given by

$$B_Z = \frac{1}{2} z^{\frac{2\sigma-1}{\sigma}} \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = \frac{1}{2z^{\frac{1-2\sigma}{\sigma}}} \frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}\hat{m}_Z}\frac{\mathrm{d}u}{\mathrm{d}z}.$$

On the other hand, the process $\{Z_t\}_{t\geq 0}$ is a generalized diffusion associated with the string m given by

$$m(z) = \begin{cases} \displaystyle\int_0^z \mathrm{d}\mu_{\hat{m}_Z}(s) = \frac{2\sigma}{1-\sigma} z^{\frac{1-\sigma}{\sigma}} & \text{if } z \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the context of local times and generalized diffusions, it is natural to ask what the effect is for a given continuous, regular, strong Markov process $\{Z_t\}_{t\geq 0}$ associated with a given string $m \in \mathfrak{m}_\infty$, if one switches from *natural scale* to the *original scale*, that is, from $\{Z_t\}_{t\geq 0}$ to $\{Y_t\}_{t\geq 0}$ by setting

$$Y_t = s^{-1}(Z_t) \qquad \text{for all } t \geq 0, \tag{3.19}$$

for a scale function $s : E_{l,r} \to \mathbb{R}$ of $\{Y_t\}_{t\geq 0}$. To see this, we employ the occupation times formula (3.6) involving the local time process $\{\tilde{L}_t(z)\}_{t\geq 0}$ of $\{Z_t\}_{t\geq 0}$ at level z. Then,

$$\int_0^t g(s^{-1}(Z_r))\,\mathrm{d}r = \int_{E_m} g(s^{-1}(z))\tilde{L}_t(z)\,\mathrm{d}\mu_m(z)$$

for every $g \in L^\infty(E_m)$. Applying the change of variable $z = s(y)$ to the integral on the right hand-side of the last equation, yields that

$$\int_0^t g(Y_r)\,\mathrm{d}r = \int_{s^{-1}(E_m)} g(y)\tilde{L}_t(s(y))\,\mathrm{d}s_{\#}^{-1}\mu_m(y), \tag{3.20}$$

where $s_{\#}^{-1}\mu_m$ is the push-forward measure of m by s^{-1}. Since $s(0) = 0$, the rescaling (3.19) with the scale function s does not affect the local time $\tilde{L}_t(0)$ of $\{Z_t\}_{t\geq 0}$ at level 0. Moreover, the process $\{\tilde{L}_t(s(y))\}$ is a local time process of $\{Y_t\}_{t\geq 0}$ at level $s(y)$. Thus, the occupation times formula (3.20) suggests that the speed measure $\hat{m}_Y$ associated with the process $\{Y_t\}_{t\geq 0}$ is the push-forward measure $s_{\#}^{-1}m$, which certainly is the case (cf., [31, see Chapter VII, Exercise 3.18]).

In Table 1, we summarize the relation between a scale function s, the speed measure $\mu_{\hat{m}_Y}$ associated with $\{Y_t\}_{t\geq 0}$, and the *natural scaled* process $\{Z_t\}_{t\geq 0}$ via (3.19), and its impact to the Dirichlet problem and corresponding Dirichlet-to-Wentzell operator (DtW operator).

We note that the generator is slightly different than what is proposed in Theorem 3.19 in the sense that we add a factor of $\frac{1}{2}$. The reason for this is that in Section 3.1 we obtain a *generalized diffusion* by considering a time change of reflecting Brownian motion. The generator of a reflecting Brownian motion is $\frac{1}{2}\frac{d^2}{dx^2}$ however the generator of the generalized diffusion is $\frac{1}{2}\frac{d}{dm}\frac{d}{dx}$ hence for us it is convenient to keep a factor of $\frac{1}{2}$.

We conclude this section with the following remark.

string m	*original scale* s	*natural scale* $\mathrm{id}_{\mathbb{R}}$
speed measure	$\mu_{\hat{m}_Y} = s^{-1}_{\#}\mu_m$	$\mu_{\hat{m}_Z} = \mu_m$
generator	$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\hat{m}_Y}\frac{\mathrm{d}}{\mathrm{d}s}$	$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z}$
Dirichlet prb.	$Au - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\hat{m}_Y}\frac{\mathrm{d}}{\mathrm{d}s}u = 0$	$Au - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z}u = 0$
DtW operator	$u(0) \mapsto m(0+)Au(0) - \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}s_+}(0)$	$u(0) \mapsto m(0+)Au(0) - \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}z_+}(0)$

TABLE 1. Scale functions, speed measures, DtW operator

Remark 3.21. It is worth noting that for every string $m \in \mathfrak{m}_\infty$, the generalized diffusion process $\{Z_t\}_{t\geq 0}$ associated with m, is already in *natural scale* $s = 2\,\mathrm{id}_{\mathbb{R}}$ and the speed measure $\hat{m}_Z$ coincides with the associated measure μ_m of m. This follows immediately from the fact that the infinitesimal generator B of $\{Z_t\}_{t\geq 0}$ is given by (3.4).

3.5. **Hitting times and spectral representations.** Hitting time densities admit the useful property of a spectral decomposition (see, for instance, the papers [22], [26] or [24], the survey paper [33, Section 3], and [17] containing interesting examples). We intend to use the spectral decomposition of hitting time densities to directly calculate the Dirichlet-to-Wentzell operator Λ_m.

For a given string $m \in \mathfrak{m}_\infty$, let $\{Z_t\}_{t\geq 0}$ be a generalized diffusion process associated with m, and

$$\tau := \inf\left\{t > 0 \,\middle|\, Z_t = 0\right\} \tag{3.21}$$

be the *first hitting time* of $z = 0$ by $\{Z_t\}_{t\geq 0}$. Then through τ, we can define the *killed process* $\{\hat{Z}_t\}_{t\geq 0}$ by setting

$$\hat{Z}_t := \begin{cases} Z_t & t \leq \tau, \\ \partial & t > \tau, \end{cases}$$

where ∂ denotes the *coffin state*.

Remark 3.22. We note that the killed process $\{\hat{Z}_t\}_{t\geq 0}$ has the same speed measure m and the same scale function $s = 2\,\mathrm{id}_{\mathbb{R}}$ as $\{Z_t\}_{t\geq 0}$ (cf., Remark 3.21). Moreover, the *infinitesimal generator* $-\hat{B}_m$ of the killed process $\{\hat{Z}_t\}_{t\geq 0}$ is the 2nd-order differential operator $-B_m$ defined in (3.4) equipped with the boundary conditions $f(0) = 0$.

Further, the killed process $\{\hat{Z}_t\}_{t\geq 0}$ has a *transition density*

$$\hat{p}(t, z, y) := \frac{\mathbb{P}_z(\hat{Z}_t \in \mathrm{d}y)}{\mathrm{d}\mu_m} = \frac{\mathbb{P}_z(Z_t \in \mathrm{d}y \,|\, t < \tau)}{\mathrm{d}\mu_m}$$

for every $t \geq 0$, z, $y \in [0, \infty)$. In the next proposition, we recall some important properties from [21, Section 4.11] of the transition density $\hat{p}(t, z, y)$.

Proposition 3.23 ([21, Section 4.11] & [26]). *For a given string $m \in \mathfrak{m}_\infty$, let $\{Z_t\}_{t\ge 0}$ be the generalized diffusion process with generator $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z}$. Then the transition density $\hat{p}(t,z,y)$ of the killed process $\{\hat{Z}_t\}_{t\ge 0}$ is a jointly continuous function $\hat{p} : [0,\infty)\times[0,\infty)\times[0,\infty)\to[0,\infty)$ with the following properties.*

1. *$\hat{p}(t,z,0) = \hat{p}(t,0,y) = 0$ for every $t \ge 0$, z, $y \in [0,\infty)$.*
2. *$\hat{p}$ is symmetric; that is,*

$$\hat{p}(t,z,y) = \hat{p}(t,y,z) \qquad \text{for all } t\in[0,\infty) \text{ and } y,\, z \in (0,\infty).$$

The next definition contains the core density of our *Poisson formula* (1.11).

Definition 3.24. For a given string $m \in \mathfrak{m}_\infty$, let τ be the first hitting time of $z = 0$ of a generalized diffusion process $\{Z_t\}_{t\ge 0}$ associated with m. Then τ has the *probability density*

$$\omega_\tau(t,z) := \frac{\mathbb{P}_z(\tau\in \mathrm{d}t)}{\mathrm{d}t} \qquad \text{for every } z\in(0,\infty) \text{ and } t>0.$$

To derive the spectral representation of the first hitting time density ω_τ, we begin by recalling from [22, Section 5] the so-called *eigenfunction expansion* $\{C(\cdot,\gamma)\}_{\gamma>0}$ of the operator $\hat{B}_m$, which is directly related to the killed process $\{\hat{Z}_t\}_{t\ge0}$ (cf. Remark 3.22). To do so, for given $\gamma>0$, let $C(\cdot,\gamma)$ be the unique solution of

$$\begin{cases} -\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}m}\dfrac{\mathrm{d}}{\mathrm{d}z}C(z,\gamma) = \gamma\, C(z,\gamma) & \text{for } z\in(0,\infty),\\ C(0;\gamma)=0, \quad \lim_{z\to 0+}\frac{C(z;\gamma)}{2z} = 1. \end{cases} \tag{3.22}$$

Alternatively, for every $\gamma>0$, $C(\cdot,\gamma)$ satisfies

$$C(z;\gamma) = 2z - 2\gamma\int_0^z\int_{(0,x]} C(r;\gamma)\,\mathrm{d}\mu_m(r)\,\mathrm{d}x \tag{3.23}$$

for every $z\in[0,\infty)$. Next, let $\{C_n\}_{n\ge0}$ be a sequence of functions $C_n : [0,\infty)\to\mathbb{R}$ recursively defined by $C_0(z) := 2z$, and for every $n\ge1$,

$$C_n(z) := 2\int_0^z\int_{(0,x]} C_{n-1}(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}x, \qquad (z\in[0,\infty)). \tag{3.24}$$

For this sequence $\{C_n\}_{n\ge0}$ (cf., [13, Section 5.4] for a similar construction), the function $C(\cdot,\gamma)$ has the following *series representation*

$$C(z,\gamma) = \sum_{n=0}^{\infty}(-\gamma)^n C_n(z) \tag{3.25}$$

for every $z\in[0,\infty)$ and $\gamma>0$. Note, the series on the right hand-side in (3.25) converges locally uniformly on $[0,\infty)$. This follows from the Weierstrass M-test, and the estimates (3.26) in the subsequent lemma. The next statements will be useful later in the proofs of our theorems to verify the hypothesis of dominated convergence.

Lemma 3.25 ([33, Lemma 3.1 and 3.2]). *For every $n\ge0$, the functions C_n defined in (3.24) are positive and increasing along $[0,\infty)$, and satisfy*

$$C_n(z) \le \frac{2z}{n!}\Big(2\int_0^z M(x)\,\mathrm{d}x\Big)^n. \tag{3.26}$$

where $M(x) := m(x) - m(0)$ for $x > 0$. Moreover, for the unique solution $C(\cdot, \gamma)$ of (3.12),one has that

$$|C(z,\gamma)| \leq 2z\, e^{\gamma 2 \int_0^z M(x)\mathrm{d}x} \tag{3.27}$$

for every $\gamma > 0$ and $z \in [0, \infty)$.

Proof. It follows from (3.24) and since $C_0(z) = 2z$ that each C_n is positive and increasing along $[0, r_m)$. Next, we show that inequality (3.26) by an induction over $n \geq 0$. Obviously, (3.26) is satisfies by $C_0(z) = 2z$ and hence, the case $n = 0$ holds. Now, suppose that (3.26) holds for an integer $n \geq 1$. Then, it remains to show that (3.26) holds for $n+1$. But by induction hypothesis, one sees that

$$\begin{aligned}
C_{n+1}(z) &= 2 \int_0^z \int_{(0,x]} C_n(r)\, \mathrm{d}\mu_m(r)\, \mathrm{d}x \\
&\leq 2 \int_0^z \int_{(0,x]} \frac{r}{n!}\Big(2\int_0^r M(u)\,\mathrm{d}u\Big)^n \mathrm{d}\mu_m(r)\,\mathrm{d}x \\
&\leq \frac{2z}{n!} \int_0^z \int_{(0,x]} \Big(2\int_0^r M(u)\,\mathrm{d}u\Big)^n \mathrm{d}\mu_m(r)\,\mathrm{d}x \\
&\leq \frac{2z}{n!} \int_0^z \int_{(0,x]} \Big(2\int_0^x M(u)\,\mathrm{d}u\Big)^n \mathrm{d}\mu_m(r)\,dx \\
&\leq \frac{2z}{n!} \int_0^z \Big(2\int_0^x M(u)\,\mathrm{d}u\Big)^n M(x)\,\mathrm{d}x \\
&= \frac{2z}{(n+1)!}\Big(2\int_0^z M(x)\mathrm{d}x\Big)^{n+1}.
\end{aligned}$$

This proves (3.26) for $n+1$ and hence, this inequality holds for all integers $n \geq 0$. One sees that estimate (3.27) holds after applying (3.26) to the series representation (3.25). This completes the proof of this lemma. ☐

By [26, Theorem 2.3], for the killed process $\{\hat{Z}_t\}_{t\geq 0}$, there exists a σ-finite measure $\hat{\Delta}_m$, called the *principal measure* of $\{\hat{Z}_t\}_{t\geq 0}$, for which the following holds

$$\int_0^\infty \frac{1}{\gamma(\gamma+1)}\,\mathrm{d}\hat{\Delta}_m(\gamma) < \infty \tag{3.28}$$

and

$$\int_0^\infty \frac{1}{\gamma}\,\mathrm{d}\hat{\Delta}_m(\gamma) = \infty. \tag{3.29}$$

From this, one can deduce the following *integral representation*

$$\hat{p}(t,z,y) = \int_0^\infty e^{-\gamma t} C(z,\gamma)C(y,\gamma)\,\mathrm{d}\hat{\Delta}_m(\gamma), \quad (t \geq 0,\ z,\ y \in [0,\infty)), \tag{3.30}$$

of the transition density $\hat{p}$ of $\{\hat{Z}_t\}_{t\geq 0}$. Further, the following *spectral representation* of the hitting time density ω_τ and *representation of the density h_m* of the Lévy measure ν_m associated with m (cf, Definition 1.11) is available.

Theorem 3.26 ([26, Theorem 3.1 and 3.2])**.** *For a given string $m \in \mathfrak{m}_\infty$, let τ be the first hitting time of $z = 0$ by a generalized diffusion process $\{Z_t\}_{t\geq 0}$ associated with m, and ω_τ the probability density of τ. Further, for given $\gamma >$*

0, let $C(\cdot,\gamma)$ be the unique solution of (3.12). Then, the following statements hold.

(1) The probability density ω_τ has the spectral representation

$$\omega_\tau(t,z) = \int_0^\infty e^{-\gamma t} C(z,\gamma)\,\mathrm{d}\hat{\Delta}_m(\gamma) \tag{3.31}$$

for every $t > 0$, $z \in (0,\infty)$.

(2) One has that

$$\omega_\tau(t,z) = \lim_{y\to 0+} \frac{\hat{p}(t,z,y)}{2y} \qquad \text{for every } z \in (0,\infty). \tag{3.32}$$

(3) The density $h_m = \frac{d\nu_m}{dt}$ of the Lévy measure ν_m associated with m, one has that

$$h_m(t) = \lim_{z\to 0+} \frac{\omega_\tau(t,z)}{2z} = \int_0^\infty e^{-\gamma t}\,\mathrm{d}\hat{\Delta}_m(\gamma) \qquad \text{for every } t > 0. \tag{3.33}$$

Integrating formula (3.31) of the first hitting density ω_τ of τ over (t,∞) for $t > 0$, then by (3.27) and Fubini's theorem yields the following.

Corollary 3.27 ([33, Proposition 3.7])**.** *For a given string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$, let τ be the first hitting time (3.21) of $z = 0$ by a generalized diffusion $\{Z_t\}_{t\ge 0}$ associated with m, and for $\gamma > 0$, let $C(\cdot,\gamma)$ be the unique solution of (3.12). Then for every $z \in (0,\infty)$, one has that*

$$\mathbb{P}_z(\tau > t) = \int_0^\infty \frac{e^{-\gamma t}}{\gamma} C(z,\gamma)\,\mathrm{d}\hat{\Delta}_m(\gamma) \qquad \text{for all } t \in [0,\infty).$$

In our next proposition, we collect some important properties of an anti-derivative β_m of the density h_m of ν_m, which we require later in Section 4.3 to derive an *integration by parts* argument (see Lemma 4.6).

Proposition 3.28. *For a given string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$, let $\hat{\Delta}_m$ be the principal measure associated with m, and set*

$$\beta_m(t) := \int_0^\infty \frac{1}{\gamma} e^{-\gamma t}\,\mathrm{d}\hat{\Delta}_m(\gamma) \qquad \text{for every } t > 0. \tag{3.34}$$

Then, the following properties hold.

(1) The function $\beta_m : (0,\infty) \to [0,\infty)$ is differentiable with derivative

$$\frac{\mathrm{d}\beta_m}{\mathrm{d}t}(t) = h_m(t) \qquad \text{for every } t > 0. \tag{3.35}$$

(2) One has that

$$\lim_{t\to 0+} t\beta_m(t) = \lim_{t\to\infty} \beta_m(t) = 0. \tag{3.36}$$

Proof. By applying the elementary inequality

$$(\gamma+1)e^{-\gamma t} \le \frac{1}{t} e^t \qquad \text{holding for all } t > 0, \tag{3.37}$$

to the integrand in (3.34) shows that

$$\beta_m(t) \le \frac{1}{t} e^t \int_0^\infty \frac{1}{\gamma(1+\gamma)}\,\mathrm{d}\hat{\Delta}_m(\gamma).$$

Since the integral on right hand-side is finite (see (3.28)), β_m is a finite positive function on $(0,\infty)$. To see that β_m is differentiable at $t>0$, note that the integrand $f:(0,\infty)\times(0,\infty)\to(0,\infty)$ given by

$$f(\gamma,t)=\frac{1}{\gamma}e^{-\gamma t}\qquad\text{for every }(\gamma,t)\in(0,\infty)\times(0,\infty),$$

is continuously differentiable in t with partial derivative $\frac{\partial f}{\partial t}(\gamma,t)=-e^{-\gamma t}$. Moreover, since $e^{-\gamma t}\le e^{-\gamma\varepsilon}$ for any $0<\varepsilon<t$ and since

$$0\le e^{-\gamma\varepsilon}\lesssim\frac{1}{\gamma(\gamma+1)}\qquad\text{for every }\gamma\in(0,\infty),$$

it follows from (3.28) and Lebesgue's dominated convergence theorem that the parameter integral

$$\beta_m(t)=\int_0^\infty f(\gamma,t)\,\mathrm{d}\hat{\Delta}_m(\gamma)$$

is differentiable and $\frac{\mathrm{d}\beta_m}{\mathrm{d}t}(t)=h_m(t)$ with h_m given by (3.35). Further, since for every $\gamma>0$, $\lim_{t\to\infty}f(\gamma,t)=0$, and since

$$0\le f(\gamma,t)\le f(\gamma,M)\qquad\text{for every }t>M>0,$$

we can conclude again from Lebesgue's dominated convergence theorem that $\lim_{t\to\infty}\beta_m(t)=0$. Finally, by using again (3.37), one sees that

$$0\le t\,f(\gamma,t)\le\frac{e^t}{\gamma(\gamma+1)}\qquad\text{for every }(\gamma,t)\in(0,\infty)\times(0,\infty),$$

and since $\lim_{t\to0+}t\,f(\gamma,t)=0$, it follows from (3.28) and Lebesgue's dominated convergence theorem that $\lim_{t\to0+}t\beta_m(t)=0$. This completes the proof of (3.36) and thereby the proof of this proposition. ☐

The following proposition provides a crucial identity to prove existence of a weak solution to the extension problem (1.7) (see Section 4.3).

Proposition 3.29. *For a given string $m\in\mathfrak{m}_\infty$ on $\mathbb{R}$, let ω_τ be the probability density (3.32) of the first hitting time τ of $z=0$ by a generalized diffusion process $\{Z_t\}_{t\ge0}$ associated with m, and β_m given by (3.34). Then, one has that*

$$2\int_0^z\int_{(0,x]}\omega_\tau(t,v)\,\mathrm{d}\mu_m(v)\mathrm{d}x=2\,z\,\beta_m(t)-\mathbb{P}_z(\tau>t)\tag{3.38}$$

for every $z\in(0,\infty)$ and $t>0$.

Proof. Let $t>0$ and $z\in(0,\infty)$. According to the spectral representation (3.31) of the first hitting time density ω_τ (Theorem 3.26), one sees that

$$\begin{aligned}\int_0^z\int_{(0,x]}&\omega_\tau(t,v)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x\\&=\int_0^z\int_{(0,x]}\int_0^\infty e^{-\gamma t}C(v,\gamma)\,\mathrm{d}\hat{\Delta}_m(\gamma)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x\\&=\int_{\Omega_z}\int_0^\infty e^{-\gamma t}C(v,\gamma)\,\mathrm{d}\hat{\Delta}_m(\gamma)\,\mathrm{d}(\mu_{\mathrm{Leb}}\otimes\mu_m)(x,v).\end{aligned}\tag{3.39}$$

where we set $\Omega_z=\{(x,v)\in[0,\infty)^2:0\le v\le x\le z\}$ and $\mu_{\mathrm{Leb}}\otimes\mu_m$ is the product measure of the one-dimensional Lebesgue measure μ_{Leb} and

μ_m. Further, if $\hat{p}$ is the transition density of the killed process $\{\hat{Z}_t\}_{t\ge 0}$ of the generalized diffusion $\{Z_t\}_{t\ge 0}$ and h_m the density h_m of the Lévy measure ν_m associated with m, then by Cauchy-Schwarz's inequality, by (3.30), and by (3.33), one sees that

$$\begin{aligned}
&\int_0^\infty e^{-\gamma t}|C(v,\gamma)|\,\mathrm{d}\hat{\Delta}_m(\gamma)\\
&\quad\le \left(\int_0^\infty e^{-\gamma t}\,\mathrm{d}\hat{\Delta}_m(\gamma)\right)^{\frac{1}{2}}\left(\int_0^\infty e^{-\gamma t}|C(v,\gamma)|^2\,\mathrm{d}\hat{\Delta}_m(\gamma)\right)^{\frac{1}{2}}\\
&\quad= \sqrt{h_m(t)}\,\sqrt{\hat{p}(t,v,v)}
\end{aligned}$$

and so,

$$\begin{aligned}
&\int_{\Omega_z}\int_0^\infty e^{-\gamma t}|C(v,\gamma)|\,\mathrm{d}\hat{\Delta}_m(\gamma)\,\mathrm{d}(\mu_{\mathrm{Leb}}\otimes\mu_m)(x,v)\\
&\quad\le \int_{\Omega_z}\sqrt{h_m(t)}\,\sqrt{\hat{p}(t,v,v)}\,\mathrm{d}(\mu_{\mathrm{Leb}}\otimes\mu_m)(x,v)\\
&\quad= \sqrt{h_m(t)}\int_{\Omega_z}\sqrt{\hat{p}(t,v,v)}\,\mathrm{d}(\mu_{\mathrm{Leb}}\otimes\mu_m)(x,v)\\
&\quad= \sqrt{h_m(t)}\int_0^z\int_{(0,x]}\sqrt{\hat{p}(t,v,v)}\,\mathrm{d}\mu_m(v)\mathrm{d}x\\
&\quad\le \sqrt{h_m(t)}\,z\,\|\sqrt{\hat{p}(t,\cdot,\cdot)}\|_{L^\infty([0,z]^2)}\,\mu_m((0,z]).
\end{aligned}$$

Note, the right hand-side of the last estimate above is finite since $\hat{p}(t,\cdot,\cdot)$ is continuous on $[0,\infty)^2$ by Proposition 3.23. Therefore, we have thereby shown that the function

$$g(\gamma,v) := e^{-\gamma t}C(v,\gamma) \quad \text{belongs to } L^1((0,\infty)\times\Omega_z;\hat{\Delta}_m\otimes(\mu_{\mathrm{Leb}}\otimes\mu_m)).$$

Hence, by Fubini's theorem and subsequently by applying the identity (3.23) for $C(v,\gamma)$, (3.34), and by Corollary 3.27, one sees that

$$\begin{aligned}
&2\int_{\Omega_z}\int_0^\infty e^{-\gamma t}C(v,\gamma)\,\mathrm{d}\hat{\Delta}_m(\gamma)\,\mathrm{d}(\mu_{\mathrm{Leb}}\otimes\mu_m)(x,v)\\
&\quad= \int_0^\infty\left(2\int_{\Omega_z}e^{-\gamma t}C(v,\gamma)\,\mathrm{d}(\mu_{\mathrm{Leb}}\otimes\mu_m)(y,v)\right)\mathrm{d}\hat{\Delta}_m(\gamma)\\
&\quad= \int_0^\infty\left(2\int_0^z\int_{(0,y]}e^{-\gamma t}C(v,\gamma)\,\mathrm{d}\mu_m)(v)\,\mathrm{d}y\right)\mathrm{d}\hat{\Delta}_m(\gamma)\\
&\quad= 2z\int_0^\infty\frac{1}{\gamma}e^{-\gamma t}\mathrm{d}\hat{\Delta}_m(\gamma) - \int_0^\infty\frac{1}{\gamma}e^{-\gamma t}C(z,\gamma)\mathrm{d}\hat{\Delta}_m(\gamma)\\
&\quad= 2z\,\beta_m(t) - \mathbb{P}_z(\tau > t).
\end{aligned}$$

Combining this with (3.39), one sees that (3.38) holds. □

3.6. **Semigroups and a bit of convex analysis.** This section is dedicated to recall the definition of a C_0-*semigroup of contractions*, Hille-Yosida-Phillips' characterization of the infinitesimal generator $-A$ of such a semigroup, and to recall some important definitions and notions from convex analysis used throughout this paper. Here, let X denote a Banach space equipped with norm $\|\cdot\|_X$, X' its dual space and $\langle\cdot,\cdot\rangle_{X',X}$ the duality pairing.

Definition 3.30 (C_0-semigroup of contractions)**.** A family $\{T_t\}_{t\geq 0}$ of linear operators $T_t \in \mathcal{L}(X)$ is called a C_0*-semigroup of contractions* on X provided

(i) $T_{t+s} = T_t \circ T_s$ for every t, $s \geq 0$;
(ii) $T_0 = \mathrm{id}_X$;
(iii) for every $x \in X$, the function $t \mapsto T_t x$ belongs to $C([0,\infty); X)$;
(iv) $\|T_t\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$.

Definition 3.31 (Infinitesimal generator)**.** For a given C_0-semigroup $\{T_t\}_{t\geq 0}$ on X, one can define an associated *infinitesimal generator* $-A$ on X by setting

$$D(A) := \Big\{ x \in X \,\Big|\, \lim_{h\to 0+} \frac{T_h x - x}{h} \text{ exists in } X \Big\}$$

and

$$-Ax := \lim_{h\to 0+} \frac{T_h x - x}{h} \qquad \text{for every } x \in D(A).$$

Notation 3.32. To emphasize that for a given operator A and a C_0-semigroup $\{T_t\}_{t\geq 0}$, $-A$ is the infinitesimal generator of $\{T_t\}_{t\geq 0}$, we write $\{e^{-tA}\}_{t\geq 0}$ instead of $\{T_t\}_{t\geq 0}$.

In order to characterize operators A, for which $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, we require to recall some notions from convex analysis.

Definition 3.33. For a given convex, proper functional $\varphi : X \to (-\infty, \infty]$ with *effective domain* $D(\varphi) := \{u \in X \,|\, \phi(u) < \infty\}$, the *sub-differential* $\partial\varphi : X \to 2^X$ is a possibly multi-valued mapping given by

$$\partial\varphi(u) = \Big\{ x' \in X' \,\Big|\, \langle x', v - u\rangle_{X',X} \leq \varphi(v) - \varphi(u) \;\forall\, v \in X \Big\}$$

for every $u \in D(\varphi)$.

Definition 3.34. Let $\xi \in C([0,\infty))$ be a continuous, monotone, and surjective function satisfying $\omega(0) = 0$. Then the (multi-valued) operator $J_\xi : X \to 2^{X'}$ given by

$$J_\xi(x) = \Big\{ x' \in X' \,\Big|\, \langle x', x\rangle_{X',X} = \|x'\|_{X'} \|x\|_X,\; \|x'\|_X = \xi(\|x\|_X) \Big\}$$

for every $x \in X$, is called the *duality map with gauge function* ξ. In the case, the gauge function $\xi(r) = r$, $r \in [0,\infty)$, we simply write J instead of J_ξ and call it the *normalized duality map.*

Remark 3.35. Since the duality map J_ξ is the sub-differential operator $\partial\varphi : X \to 2^{X'}$ of the convex functional $\varphi(x) = \zeta(\|x\|_X)$ for $\zeta(r) := \int_0^r \xi(s)\,\mathrm{d}s$, $r \geq 0$, it follows that J_ξ is *monotone*, that is, one has that

$$\langle x_1' - x_2', x_1 - x_2\rangle_{X',X} \geq 0 \qquad \text{for all pairs } (x_1, x_1'),\, (x_2, x_2') \in J_\xi.$$

Moreover, for every $x \in X$, $J_\xi(x)$ is closed, convex and non-empty subset of X'.

Definition 3.36. Let $J : X \to 2^{X'}$ be the *normalized duality map* given by

$$J(x) = \Big\{ x' \in X' \,\Big|\, \langle x', x\rangle_{X',X} = \|x\|_X^2 = \|x'\|_{x'}^2 \Big\} \quad \text{for every } x \in X.$$

Then, a linear operator $A : D(A) \to X$ with domain $D(A) \subseteq X$ is called *accretive* if for every $u \in D(A)$, there is an $x' \in J(u)$

$$\mathrm{Re}\langle x', Au\rangle_{X',X} \geq 0.$$

Further, A is called *m-accretive* if A is accretive and satisfies the so-called *range condition*; that is, for every $f \in X$ and $\lambda > 0$, there exists a unique $u \in D(A)$ such that $u + \lambda Au = f$.

Theorem 3.37 ([3, Corollary 3.3.5])**.** *Let A be a linear operator on X. Then, $-A$ is the infinitesimal generator of a C_0-semigroup of contractions $\{e^{-tA}\}_{t\geq 0}$ on X if and only if A is m-accretive on X.*

3.7. **Subordination of semigroups.** In this section, we briefly review the definition of $\psi(A)$ for a given Bernstein function ψ and a linear m-accretive operator A on a Banach space X. Here, we follow closely [36, Chapter 13]

We begin by recalling the following definitions.

Definition 3.38. A finite Borel measure ν on $[0,\infty)$ is called a *sub-probability measure* provided $\nu([0,\infty)) \leq 1$.

Definition 3.39. Let (X,τ) be a locally compact topological Hausdorff space. Then a sequence $\{\nu_n\}_{n\geq 1}$ of Radon measures ν_n on X *converges vaguely* to a measure ν if

$$\int_X \varphi(x)\,\mathrm{d}\nu_n(x) \to \int_X \varphi(x)\,\mathrm{d}\nu(x) \qquad \text{as } n\to\infty$$

for all compactly supported, continuous, real-valued functions φ on X (we denote this set of functions by $C_c(X)$.

Definition 3.40. The *convolution $\mu*\nu$ of two sub-probability measures* μ and ν on $[0,\infty)$ is defined by

$$\int_{[0,\infty)} \varphi(t)\,\mathrm{d}(\mu*\nu)(t) = \int_{[0,\infty)}\int_{[0,\infty)} \varphi(t+s)\,\mathrm{d}\mu(t)\,\mathrm{d}\nu(s)$$

for every bounded continuous function φ on $[0,\infty)$ (which we summarize in the set $C_c([0,\infty)))$.

We note that the convolution $\mu*\nu$ of two sub-probability measures μ and ν is again a sub-probability measure on $[0,\infty)$. With the *convolution*-operation, we can defined now the following type of semigroups.

Definition 3.41. A family $\{\gamma_t\}_{t\geq 0}$ of finite Borel measures γ_t on $[0,\infty)$ is called a *vaguely continuous convolution semigroup of sub-probability measures* provided the family $\{\gamma_t\}_{t\geq 0}$ satisfies

(i) $\gamma_t([0,\infty)) \leq 1$ for all $t \geq 0$ (sub-probability condition);
(ii) $\mu_{t+s} = \gamma_t * \mu_s$ for all $t,\, s \geq 0$ (semigroup property);
(iii) $\lim\limits_{t\to 0} \gamma_t = \delta_0$ vaguely (vague continuity).

The next theorem highlights the bijective relation between Bernstein functions ψ and vaguely continuous convolution semigroup of sub-probability measures.

Theorem 3.42 ([36, Theorem 5.2])**.** *Let $\{\gamma_t\}_{t\geq 0}$ be a vaguely continuous convolution semigroup of sub-probability measures on $[0,\infty)$. Then there exists a unique Bernstein function ψ such that the Laplace transform of γ_t*

$$\int_0^\infty e^{-\lambda s}\,\mathrm{d}\gamma_t(s) = e^{-t\psi(\lambda)} \qquad \text{for all } \lambda > 0,\ t \geq 0. \tag{3.40}$$

Conversely, for a given Bernstein function ψ, there exists a unique vaguely continuous convolution semigroup $\{\gamma_t\}_{t\geq 0}$ of sub-probability measures on $[0,\infty)$ satisfying (3.40).

The following proposition provides the existence theorem of the operator $\psi(A)$ via the infinitesimal generator of a semigroup.

Proposition 3.43 ([36, Proposition 13.1])**.** *Let $\{e^{-tA}\}_{t\geq 0}$ be a C_0-semigroup of contractions with infinitesimal generator A on a Banach space X, and $\{\gamma_t\}_{t\geq 0}$ be a vaguely continuous convolution semigroup of sub-probability measures on $[0,\infty)$ with the corresponding Bernstein function ψ. Then the family $\{e^{-t\psi(A)}\}_{t\geq 0}$ defined by the Bochner integral*

$$e^{-t\psi(A)}f := \int_{[0,\infty)} e^{-sA}f\,\mathrm{d}\gamma_t(s) \qquad \text{for every } t \geq 0,\ f \in X, \tag{3.41}$$

defines a C_0-semigroup $\{e^{-t\psi(A)}\}_{t\geq 0}$ of contractions $e^{-t\psi(A)} \in \mathcal{L}(X)$.

Notation 3.44 (The operator $\psi(A)$). For a given m-accretive operator A on a Banach space X, and vaguely continuous convolution semigroup $\{\gamma_t\}_{t\geq 0}$ of sub-probability measures γ_t associated the corresponding Bernstein function ψ, we denote by $\psi(A)$ the *infinitesimal generator* (see Definition 3.31) of the semigroup $\{e^{-t\psi(A)}\}_{t\geq 0}$ given by (3.41).

Theorem 3.45 (Phillips' subordination theorem, [36, Theorem 13.6])**.** *Let $\{e^{-tA}\}_{t\geq 0}$ be a C_0-semigroup of contractions with infinitesimal generator $-A$ on Banach space X, and ψ be a Bernstein function with the Lévy triple (a,b,ν). Consider the semigroup $\{e^{-t\psi(A)}\}_{t\geq 0}$ with the infinitesimal generator $-\psi(A)$. Then, the domain $D(A)$ of A is an operator core of the domain $D(\psi(A))$ of $\psi(A)$ and*

$$\psi(A)f = af + bAf + \int_0^\infty \left(f - e^{-tA}f\right)\mathrm{d}\nu(t)$$

for all $f \in D(A)$, where the integral is to be understood in the Bochner sense.

4. Proofs of the Main Results

Throughout this section, let $m \in \mathfrak{m}_\infty$ and associated Lebesgue-Stieltjes measure μ_m. Further, $\{Z_t\}_{t\geq 0}$ be a generalized diffusion associated with m, τ the first hitting time of $z=0$ by the process $\{Z_t\}_{t\geq 0}$ as defined in (3.21), and ψ_m the associated complete Bernstein function (3.12) from Theorem 3.8.

For the moment, we assume that $A : D(A) \to X$ is merely a closed, linear operator defined on a Banach X with norm $\|\cdot\|_X$, and denote by X' the dual space, and $\langle\cdot,\cdot\rangle_{X',X}$ the corresponding duality brackets.

Then, the main object of this section is the *Dirichlet-to-Wentzell operator*

$$f \mapsto \Lambda_m f := m(0+)Au(0) - \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}z_+}(0) \tag{4.1}$$

associated with the *Dirichlet problem*

$$\begin{cases} Au(z) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z}u(z) &= 0, \qquad \text{for } z \in (0,\infty),\\ \qquad\qquad u(0) &= f, \end{cases} \tag{4.2}$$

for given $f \in D(A)$. In the next subsection, we briefly discuss the notion of a weak solution u to the Dirichlet problem (4.2) introduced in Definitions 1.5.

4.1. **Weak solutions.** We begin by discussing the the notion of weak solutions u to the Dirichlet problem (4.2) as it was introduced in Definition 1.5.

Remark 4.1. For a given string $m \in \mathfrak{m}_\infty$, μ_m the associated measure to m, and E_m the support of μ_m. Then the following comments are worth noting.

(a) We show in the Proposition B.7 (in the appendix) that a function $f \in L^1_{loc}((0,\infty);X)$ has a weak m-derivative $g \in L^1_{loc,\mu_m}((0,\infty);X)$ if and only if f can be represented by

$$f(z_2) = f(z_1) + \int_{z_2}^{z_1} g(r)\,\mathrm{d}\mu_m(r) \qquad \text{for a.e. } z_1, z_2 \in (0,\infty).$$

(b) Due to Remark (a), for given $f \in X$, a *weak solution* u of Dirichlet problem (4.2) is characterized by satisfying $u \in C([0,\infty);X) \cap W^{1,1}_{loc}([0,\infty);X)$, the weak derivative $\frac{\mathrm{d}u}{\mathrm{d}z}$ is weakly m-differentiable, and there is a $g \in L^1_{loc,\mu_m}([0,\infty);X)$ satisfying

$$u(z) = f + \frac{\mathrm{d}u}{\mathrm{d}z_-}(0)\,z + \int_0^z \int_{[0,y]} g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}y \tag{4.3}$$

for every $z \in [0,\infty)$, where $\frac{\mathrm{d}u}{\mathrm{d}z_-}(0)$ denotes the left hand-side derivative of u at $z = 0$, and

$$2Au(z) = g(z) \qquad \text{for } \mu_m\text{-a.e. } z \in (0,\infty).$$

This formulation consistent with the real-valued function case described in [36] provided by Revuz and Yor.

(c) In comparison to Remark 1.4, an alternative characterization of a weak solution u is of the *extension equation*

$$Au(z) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z}u(z) = 0 \qquad \text{in } z \in (0,\infty), \tag{4.4}$$

is given by u belonging to $W^{1,1}_{loc}((0,\infty);X)$, for μ_m-a.e. $z \in (0,\infty)$, one has that $u(z) \in D(A)$, $Au \in L^1_{loc,\mu_m}((0,\infty);X)$, and

$$\frac{1}{2}\frac{\mathrm{d}^2u}{\mathrm{d}z^2} = Au\,\mu_m \qquad \text{in } \mathcal{D}'((0,\infty);X). \tag{4.5}$$

(d) A function u given by (4.3) with weak m-derivative

$$\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}u}{\mathrm{d}z} \in L^1_{loc,\mu_m}([0,\infty);X)$$

is linear on $[0,\infty) \setminus E_m$, the left and right hand-side derivative $\frac{\mathrm{d}u}{\mathrm{d}z_-}(z)$, $\frac{\mathrm{d}u}{\mathrm{d}z_+}(z)$ exist at every $z \in [0,\infty)$ with

$$\frac{\mathrm{d}u}{\mathrm{d}z_+}(z) = \frac{\mathrm{d}u}{\mathrm{d}z_-}(0) + \int_{[0,z]} g(r)\,\mathrm{d}\mu_m(r),$$

$$\frac{\mathrm{d}u}{\mathrm{d}z_-}(z) = \frac{\mathrm{d}u}{\mathrm{d}z_-}(0) + \int_{[0,z)} g(r)\,\mathrm{d}\mu_m(r),$$

and u' is a.e. continuous on $[0,\infty)$.

With these comments in mind, we can now start by establishing uniqueness of bounded solutions of Dirichlet problem (4.2).

4.2. **Uniqueness of weak solutions of the Dirichlet problem.** In this subsection, we outline our uniqueness result of bounded *solutions* of Dirichlet problem (4.2). Our method to prove uniqueness relies essentially on tools and arguments borrowed from the theory of nonlinear evolution theory (see, for instance, [8, 18]).

Throughout this section, we assume that the operator A in the extension problem (4.2) is *accretive* on X as defined in Definition 3.36.

Our first lemma partially generalizes a result by Bénilan [8].

Lemma 4.2. *For a given string $m \in \mathfrak{m}_\infty$, let μ_m be the associated measure to m. Let $\varphi : X \to \mathbb{R}$ be a continuous, convex functional on a Banach space X satisfying $\varphi(0) = 0$, and for some $g \in L^1_{loc,\mu_m}([0,\infty);X)$, x, $\beta \in X$, let $u \in C([0,\infty);X)$ be given by*

$$u(z) = x + \beta\, z + \int_0^z \int_{[0,y]} g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}y \tag{4.6}$$

for every $z \in [0,\infty)$. Then the following statements hold.

(1) The mapping $z \mapsto \varphi(u(z))$ is differentiable from the right at every $z \in [0,\infty)$ with

$$\frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z)) = \max_{w\in\partial\varphi(u(z))} \langle w, \frac{\mathrm{d}u}{\mathrm{d}z_+}(z)\rangle_{X',X}, \tag{4.7}$$

where $\partial\varphi$ denotes the sub-differential of φ (see Definition 3.33).

(2) The mapping $z \mapsto \frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z))$ is m-differentiable μ_m-a.e. on $[0,\infty)$, and locally of bounded variation.

(3) For every $z \in [0,\infty)$ such that $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}u}{\mathrm{d}z_+}(z)$ and $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z))$ exist, one has that

$$\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z)) \ge \langle w, \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}u}{\mathrm{d}z_+}(z)\rangle_{X',X} \tag{4.8}$$

for every $w \in \partial\varphi(u(z))$.

Proof. Let u be given by (4.6). Then u is differentiable from the right at every $z \in [0,\infty)$. Thus, claim (1) follows by the same arguments as in [8, Lemme 1] with $u'(z)$ replaced by $\frac{du}{dz_+}(z)$.

To see that claim (2) holds, let $0 \le z_1 \le z_2 \le T < \infty$ and for $i = 1,2$, let $w_i \in \partial\varphi(u(z_i))$ such that (4.7) holds. Then,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z_2)) - \frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z_1))\\&\qquad = \langle w_2, \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_2) - \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_1)\rangle_{X',X} + \langle w_2 - w_1, \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_1)\rangle_{X',X}.\end{aligned}$$

Since

$$\frac{\mathrm{d}u}{\mathrm{d}z_+}(z) = \beta + \int_{[0,z]} g(r)\,\mathrm{d}\mu_m(r), \qquad z \in [0,\infty),$$

we have that

$$\begin{aligned}\langle w_2, \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_2) - \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_1)\rangle_{X',X} &= \langle w_2, \int_{(z_1,z_2]} g(r)\,\mathrm{d}\mu_m(r)\rangle_{X',X}\\ &\ge -\|w_2\|_{X'} \int_{(z_1,z_2]} \|g(r)\|_X\,\mathrm{d}\mu_m(r).\end{aligned}$$

On the other hand, since

$$\begin{aligned}\frac{u(z_2)-u(z_1)}{z_2-z_1} &= \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_1) - \int_{[0,z_1]} g(r)\,\mathrm{d}\mu_m(r)\\ &\quad + \tfrac{1}{z_2-z_1}\int_{z_1}^{z_2}\int_{[0,z]} g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}z,\end{aligned}$$

one has that

$$\frac{u(z_2)-u(z_1)}{z_2-z_1} = \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_1) - \tfrac{1}{z_2-z_1}\int_{z_1}^{z_2}\int_{(z_1,z]} g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}z. \tag{4.9}$$

Therefore and since $\partial\varphi$ is monotone,

$$\begin{aligned}\langle w_2 - w_1, \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_1)\rangle_{X',X} &\ge \langle w_2 - w_1, \tfrac{1}{z_2-z_1}\int_{z_1}^{z_2}\int_{(z_1,z]} g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}z\rangle_{X',X}\\ &\ge -\|w_2 - w_1\|_{X'}\int_{(z_1,z_2]} \|g(r)\|_X\,\mathrm{d}\mu_m(r).\end{aligned}$$

By hypothesis, the functional φ is continuous on X. Hence, on every bounded subset U of X, there is a constant $M > 0$ such that

$$\|x'\|_{X'} \le M \qquad \text{for every } x' \in \partial\varphi(u) \text{ and } u \in U.$$

Further, by the continuity of $u : [0,T] \to X$, there is a bounded open subset U of X containing $u([0,T])$. Thus, there is an $M > 0$ such that

$$\langle w_2, \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_2) - \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_1)\rangle_{X',X} \ge -M\int_{(z_1,z_2]} \|g(r)\|_X\,\mathrm{d}\mu_m(r).$$

and

$$\langle w_2 - w_1, \frac{\mathrm{d}u}{\mathrm{d}z_+}(z_1)\rangle_{X',X} \ge -2\,M\int_{(z_1,z_2]} \|g(r)\|_X\,\mathrm{d}\mu_m(r)$$

and hence,

$$\frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z_2)) - \frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z_1)) \ge -3\,M\int_{(z_1,z_2]} \|g(r)\|_X\,\mathrm{d}\mu_m(r),$$

or, equivalently, for

$$h(z) := \frac{\mathrm{d}}{\mathrm{d}z_+}\varphi(u(z)) \quad \text{and} \quad H(z) := \int_{[0,z]} \|g(r)\|_X\,\mathrm{d}\mu_m(r),$$

one has that $h(z_2) + H(z_2) \ge h(z_1) + H(z_1)$. Since $0 \le z_1 \le z_2 \le T < \infty$ were arbitrary, we have thereby shown that $z \mapsto h(z) + H(z)$ is monotonically increasing along $[0,T]$, and hence m-differentiable μ_m-a.e. on $[0,\infty)$. Since H is also monotone, it is also m-differentiable μ_m-a.e. on $[0,\infty)$, which implies that $h(z)=\frac{\mathrm{d}}{\mathrm{d}z+}\varphi(u(z))$ is m-differentiable μ_m-a.e. on $[0,\infty)$. This completes the proof of statement (2).

Next, let u be given by (4.6). Then, one has that

$$\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}u}{\mathrm{d}z+}(z)=g(z) \tag{4.10}$$

for μ_m-a.e. $z\in(0,\infty)$. From this, we can infer that

$$\lim_{h\to 0+}\frac{u(z+h)-2u(z)+u(z-h)}{h\,(m(z)-m(z-h))}=\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}u}{\mathrm{d}z+}(z)\qquad\text{in } X \tag{4.11}$$

for μ_m-a.e. $z\in(0,\infty)$. To see that (4.11) holds, we first write

$$\frac{u(z+h)-2u(z)+u(z-h)}{h\,(m(z)-m(z-h))}=\frac{u(z+h)-u(z)}{h\,(m(z)-m(z-h))}-\frac{u(z)-u(z-h)}{h\,(m(z)-m(z-h))}.$$

Then for $h>0$, (4.9) yields that

$$u(z+h)-u(z)=\frac{\mathrm{d}u}{\mathrm{d}z+}(z)h-\int_z^{z+h}\int_{(z,y]}g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}y$$

and

$$u(z)-u(z-h)=\frac{\mathrm{d}u}{\mathrm{d}z+}(z-h)h-\int_{z-h}^{z}\int_{(z-h,y]}g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}y.$$

Thus,

$$\begin{aligned}
&\frac{u(z+h)-u(z)-(u(z)-u(z-h))}{h\,(m(z)-m(z-h))}\\
&\quad=\frac{\frac{\mathrm{d}u}{\mathrm{d}z+}(z)-\frac{\mathrm{d}u}{\mathrm{d}z+}(z-h)}{m(z)-m(z-h)}-\frac{1}{h\,(m(z)-m(z-h))}\int_z^{z+h}\int_{(z,y]}g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}y\\
&\qquad\qquad+\frac{1}{h\,(m(z)-m(z-h))}\int_{z-h}^{z}\int_{(z-h,y]}g(r)\,\mathrm{d}\mu_m(r)\,\mathrm{d}y.
\end{aligned}$$

If (4.10) holds at $z\in(0,\infty)$ and if z a Lebesgue point of μ_m, then the last two terms on the right hand-side of the latter equation tend to zero as $h\to 0+$, showing that (4.11) holds.

Finally, for $w\in\partial\phi(u(z))$, the monotonicity of $\partial\phi$ implies that

$$\begin{aligned}
&\frac{\phi(u(z+h))-2\phi(u(z))+\phi(u(z-h))}{h\,(m(z)-m(z-h))}\\
&\qquad\geq\frac{\langle w,u(z+h)-2u(z)+u(z-h)\rangle_{X',X}}{h\,(m(z)-m(z-h))}\\
&\qquad=\langle w,\frac{u(z+h)-2u(z)+u(z-h)}{h\,(m(z)-m(z-h))}\rangle_{X',X}.
\end{aligned} \tag{4.12}$$

Thus, for every $z\in[0,\infty)$ such that $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}u}{\mathrm{d}z+}(z)$ and $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z+}\varphi(u(z))$ exist, taking the limit as $h\to 0+$ in (4.12) yields that (4.8) holds. $\square$

With the help of Lemma 4.2, we obtain uniqueness of bounded solutions of the Dirichlet problem (4.2) in a Banach spaces X.

Theorem 4.3 (Uniqueness of bounded solutions)**.** *Let $m \in \mathfrak{m}_\infty$ with associated measure μ_m, and A an accretive operator on the Banach space X. Then, for every given $f \in X$, there is at most one solution $u \in L^\infty([0,\infty);X)$ of Dirichlet problem (4.2).*

Proof. Let $u \in L^\infty([0,\infty);X)$ be a solution of Dirichlet problem (4.2) with initial data $f = 0$ and $\varphi(x) := \frac{1}{2}\|x\|_X^2$ for every $x \in X$. Since the operator A is accretive on X, one has that

$$\langle w, \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}u}{\mathrm{d}z_+}(z)\rangle_{X',X} \geq 0 \qquad \text{for every } w \in \partial\varphi(u(z)).$$

Hence by (4.8) of Lemma 4.2, the function $z \mapsto \|u(z)\|_X^2$ is convex on $[0,\infty)$. Since u is bounded, also the function $z \mapsto \|u(z)\|_X^2$ is bounded and hence monotonically decreasing. This implies that

$$\|u(z)\|_X^2 \leq \|u(0)\|_X^2 = 0 \qquad \text{for all } z \in [0,\infty),$$

completing the proof of this theorem. □

4.3. **Existence of weak solutions of the Dirichlet problem.** In this section, we establish existence of bounded solutions u of Dirichlet problem (4.2) under the hypothesis that A is an m-accretive operator on a Banach space X, and for given initial value $f \in D(A)$. Our approach to this combines arguments from stochastic analysis with linear semigroup theory. In particular, it is based on the spectral representation of the *first hitting time* density ω_τ of a given generalized diffusion $\{Z_t\}_{t\geq 0}$ on $[0,\infty)$ (see Theorem 3.26). Thus, we employ the same notation here as introduced in Subsection 3.5.

Throughout this subsection, let $m \in \mathfrak{m}_\infty$ with associated measure μ_m. Further, let A an m-accretive operator on X and $\{e^{-tA}\}_{t\geq 0}$ the semigroup generated by $-A$. Further, let $\{Z_t\}_{t\geq 0}$ be a generalized diffusion on $[0,\infty)$ associated with m, $B_m = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}z}$ the infinitesimal generator of $\{Z_t\}_{t\geq 0}$, and ω_τ the density (3.32) of the first hitting time τ of $z = 0$ by $\{Z_t\}_{t\geq 0}$. Then, for given $f \in X$, we set

$$u(z) := \int_0^\infty (e^{-tA}f)\,\omega_\tau(t,z)\,\mathrm{d}t, \qquad z \in [0,\infty). \tag{4.13}$$

Then, our aim is to show that the function u given by (4.13) is a solution of the Dirichlet problem (4.2).

Since the semigroup $\{e^{-tA}\}_{t\geq 0}$ is contractive, one easily sees that the integral in (4.13) is finite. Moreover, one has that

$$\sup_{z\in[0,\infty)} \|u(z)\|_X \leq \sup_{z\in[0,\infty)} \int_0^\infty \|e^{-tA}f\|_X \omega_\tau(t,z)\,dt \leq \|f\|_X, \tag{4.14}$$

showing that $u \in L^\infty(0,\infty;X)$.

Next, we intend to show that for given $f \in D(A)$, the function u given by (4.13) is a weak solution of the extension equation (4.4) satisfying $u(0) = f$. More specifically, we show that u can be rewritten as follows.

Theorem 4.4. *In addition to the hypotheses of this subsection, let $f \in D(A)$, and u be given by (4.13). Then u belongs to $W^{1,1}_{loc}([0,\infty);X)$ and satisfies*

(1) $u(0) = f$ in X,

(2) $u(z) \in D(A)$ *for every* $z \in [0,\infty)$,
(3) $Au \in L^\infty([0,\infty);X)$, *and*
(4) u *can be rewritten as*

$$u(z) = f - 2z\int_0^\infty \left[f - e^{-tA}f\right]\, h_m(t)\,\mathrm{d}t \qquad + 2\int_0^z \int_{(0,x]} Au(v)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x \tag{4.15}$$

for every $z \in [0,\infty)$.

We note that by the Radon property of the measure μ_m it follows that $L^\infty(0,\infty;X)$ is contained in $L^1_{loc,\mu_m}([0,\infty);X)$. Thus, by Remark 4.1, the representation (4.15) of u implies that u is a *weak solution* of Dirichlet problem (4.2). The statement of Theorem 4.4 establishes our main result Theorem 1.9.

The proof of Theorem 4.4 proceeds in several steps. We begin by showing that $u(z) \in D(A)$ for every $z \in (0,\infty)$ and that statement 3 holds.

Lemma 4.5. *In addition to the hypotheses of this subsection, let* $f \in D(A)$, *and* u *be given by* (4.13). *Then for every* $z \in (0,\infty)$, $u(z) \in D(A)$ *and*

$$Au(z) = \int_0^\infty (Ae^{-tA}f)\,\omega_\tau(t,z)\,\mathrm{d}t. \tag{4.16}$$

Moreover, one has that

$$\sup_{z\in[0,\infty)} \|Au(z)\|_X \le \|Af\|_X. \tag{4.17}$$

Proof. Let $f \in D(A)$, u be given by (4.13), and $z \in (0,\infty)$. Then by $f \in D(A)$, we have that $Ae^{-tA}f = e^{-tA}(Af)$, and since the family $\{e^{-tA}\}_{t\ge 0}$ consists of contractive operators $e^{-tA} \in \mathcal{L}(X)$, one sees that

$$\begin{aligned}\int_0^\infty \|Ae^{-tA}f\|_X\,\omega_\tau(t,z)\,\mathrm{d}t &= \int_0^\infty \|e^{-tA}(Af)\|_X\,\omega_\tau(t,z)\,\mathrm{d}t\\ &\le \|Af\|_X \int_0^\infty \omega_\tau(t,z)\,\mathrm{d}t\\ &\le \|Af\|_X,\end{aligned} \tag{4.18}$$

showing that the function $g(t) := e^{-tA}f\,\omega_\tau(t,z)$ is Bochner integrable. Since A is a closed linear operator on a Banach space X, $g(t) \in D(A)$ for every $t \ge 0$, it follows from [3, see Proposition 1.1.7] that $u(z) = \int_0^\infty g(t)\,\mathrm{d}t$ belongs to $D(A)$ and (4.16) holds. Now, thanks to (4.16), the estimates in (4.18) show that (4.17) holds. □

Our next step is to calculate for the function u given by (4.13) the integral on the left hand-side of (4.19) below. To do this, we employ the integral identity from Lemma 3.29.

Lemma 4.6. *In addition to the hypotheses of this subsection, let* $f \in D(A)$, u *be given by* (4.13), *and* β_m *the antiderivative of the density* h_m *of the Lévy* *measure ν_m is defined as in (3.34). Then one has that*

$$2\int_0^z\int_{(0,x]} Au(v)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x = 2z\int_0^\infty (Ae^{-tA}f)\beta_m(t)\,\mathrm{d}t - \int_0^\infty (Ae^{-tA}f)\,\mathbb{P}_z(\tau>t)\,\mathrm{d}t, \tag{4.19}$$

for every $z\in(0,\infty)$.

Proof. Let $f\in D(A)$ and $z\in(0,\infty)$. Then, by (4.16),

$$\int_0^z\int_{(0,x]} Au(v)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x = \int_0^z\int_{(0,x]}\int_0^\infty (Ae^{-tA}f)\,\omega_\tau(t,v)\,\mathrm{d}t\,\mathrm{d}\mu_m(v)\,\mathrm{d}x.$$

Since

$$\int_0^z\int_{(0,x]}\int_0^\infty \|Ae^{-tA}f\|_X\,\omega_\tau(t,v)\,\mathrm{d}t\,\mathrm{d}\mu_m(v)\,\mathrm{d}x \le z\,\mu_m((0,z])\,\|Af\|_X,$$

it follows from Fubini's theorem for Bochner Integrals (see, for instance, [3, Theorem 1.1.9]) and by Lemma 3.29 that

$$\begin{aligned}
2\int_0^z\int_{(0,x]} & Au(v)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x\\
&= 2\int_0^z\int_{(0,x]}\int_0^\infty (Ae^{-tA}f)\,\omega_\tau(t,v)\,\mathrm{d}t\,\mathrm{d}\mu_m(v)\,\mathrm{d}x\\
&= \int_0^\infty (Ae^{-tA}f)\left(2\int_0^z\int_{(0,x]}\omega_\tau(t,v)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x\right)\mathrm{d}t\\
&= \int_0^\infty (Ae^{-tA}f)\Big(2\,z\,\beta_m(t)-\mathbb{P}_z(\tau>t)\Big)\,\mathrm{d}t.
\end{aligned}$$

From this, one sees that (4.19) holds. □

We are now ready to prove the remaining statements of Theorem (4.4).

Proof of Theorem 4.4. Given $f\in D(A)$ and for $z\in(0,\infty)$, let u be given by (4.13). We focus on proving the characterization (4.15). By Lemma 4.6, and since

$$\tfrac{\mathrm{d}}{\mathrm{d}t}e^{-tA}f + Ae^{-tA}f = 0,$$

it follows that

$$\begin{aligned}
2\int_0^z\int_{(0,x]} & Au(v)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x\\
&= 2z\int_0^\infty (Ae^{-tA}f)\beta_m(t)\,\mathrm{d}t - \int_0^\infty (Ae^{-tA}f)\mathbb{P}_z(\tau>t)\,\mathrm{d}t\\
&= 2z\int_0^\infty (Ae^{-tA}f)\beta_m(t)\,\mathrm{d}t + \int_0^\infty \mathbb{P}_z(\tau>t)\frac{\mathrm{d}}{\mathrm{d}t}e^{-tA}f\,\mathrm{d}t.
\end{aligned} \tag{4.20}$$

To complete the proof of the representation (4.15), it remains to compute the integral terms on the left hand-side of (4.20). Since $\omega_\tau(t,z)$ is the probability density of the first hitting time τ, and since $t\mapsto\omega_\tau(t,z)$ is continuous on $(0,\infty)$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_z(\tau>t) = -\omega_\tau(t,z) \qquad \text{for every } t>0 \text{ and } z\in(0,\infty).$$

By using this together with an integration by parts, one sees that

$$\int_0^\infty \mathbb{P}_z(\tau > t)\frac{\mathrm{d}}{\mathrm{d}t}e^{-tA}f\,\mathrm{d}t = -f + \int_0^\infty e^{-tA}f\,\omega_\tau(t,z)\,\mathrm{d}t = -f + u(z) \tag{4.21}$$

completing the computation of one of the two integral terms on the left hand-side of (4.20). Before we can compute also the second integral term on the left hand-side of (4.20), we require proving that the two limits

$$\lim_{t\to 0+} \beta_m(t)\left(f - e^{-tA}f\right) = \lim_{t\to\infty} \beta_m(t)\left(f - e^{-tA}f\right) = 0 \tag{4.22}$$

exist in X. Since $f \in D(A)$, and since $\{e^{-tA}\}_{t\ge 0}$ is contractive on X, one has that

$$\|f - e^{-tA}f\|_X = \left\|\int_0^t e^{-sA}(-Af)\,\mathrm{d}s\right\|_X \le t\,\|Af\|_X.$$

Thus, and by (3.36) from Proposition 3.36, one sees that

$$\lim_{t\to 0+} |\beta_m(t)|\,\|f - e^{-tA}f\|_X \le \lim_{t\to 0+} \|Af\|_X\,|t\beta_m(t)| = 0.$$

To see that the second limit in (4.22) holds as well, we use again that the semigroup $\{e^{-tA}\}_{t\ge 0}$ is contractive on X and apply (3.36). Then, one easily sees that

$$\lim_{t\to\infty} |\beta_m(t)|\,\|f - e^{-tA}f\|_X \le \lim_{t\to\infty} |\beta_m(t)|\,2\,\|f\|_X = 0.$$

Now, by (3.35) from Proposition 3.36, an integration by parts yields that

$$\begin{aligned}
2\,z\int_0^\infty & \left(Ae^{-tA}f\right)\beta_m(t)\,\mathrm{d}t \\
&= 2\,z\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left(f - e^{-tA}f\right)\beta_m(t)\,\mathrm{d}t \\
&= 2\,z\int_0^\infty \left(f - e^{-tA}f\right)h_m(t)\,\mathrm{d}t - \lim_{t\to\infty} 2\,z\left(f - e^{-tA}f\right)\beta_m(t) \\
&\qquad + \lim_{t\to 0+} 2\,z\left(f - e^{-tA}f\right)\beta_m(t)
\end{aligned}$$

and so, (4.22) implies that

$$2\,z\int_0^\infty \left(Ae^{-tA}f\right)\beta_m(t)\,\mathrm{d}t = 2\,z\int_0^\infty \left(f - e^{-tA}f\right)h_m(t)\,\mathrm{d}t. \tag{4.23}$$

Applying (4.21) and (4.23) to (4.20), one finds the desired representation (4.15) of u. Further, by the integral representation (4.15), it follows that u belongs to $W^{1,1}_{loc}([0,\infty);X)$ and $u(0) = f$ in X. This completes the proof of Theorem 4.4. □

4.4. **Proof of Theorem 1.13: a first hitting time approach.** In this subsection, we outline the proof of Theorem 1.13 based on a first hitting time approach.

Proof of Theorem 1.13. Let $f \in D(A)$, and u be the unique weak solution of Dirichlet problem (4.2). Then, by the continuity of u and by (4.17), the function

$$g(x) := \int_{(0,x]} Au(v)\,\mathrm{d}\mu_m(v) \qquad \text{for every } x \in [0,\infty),$$

is right-continuous at $x = 0$ and $g(0+) = 0$. Therefore, we can conclude from (4.15) that

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}z_+}(0) &= \lim_{h\to 0+}\frac{u(h)-u(0)}{h}\\ &= \lim_{h\to 0+}\Bigg[(-2)\int_0^\infty (f-e^{-tA}f)\,h_m(t)\,\mathrm{d}t\\ &\qquad\qquad + \frac{2}{h}\int_0^h\int_{(0,x]} Au(v)\,\mathrm{d}\mu_m(v)\,\mathrm{d}x\Bigg]\\ &= (-2)\int_0^\infty (f-e^{-tA}f)\,h_m(t)\,\mathrm{d}t\end{aligned}$$

and so,

$$\begin{aligned}\Lambda_m f &= m(0+)Au(0) - \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}z_+}(0)\\ &= m(0+)Af + \int_0^\infty (f-e^{-tA}f)\,h_m(t)\,\mathrm{d}t\\ &= \psi_m(A)f.\end{aligned}$$

According to Phillips' subordination theorem (Theorem 3.45), the domain $\mathcal{D}(A)$ of A is a core of $\mathcal{D}(\psi_m(A))$. Thus, we have thereby shown that the two operators Λ_m and $\psi_m(A)$ coincide.

Further, let $\{\tilde{L}_t^{-1}\}_{t\ge 0}$ be the local inverse time at zero of the generalized diffusion process $\{Z_t\}_{t\ge 0}$ associated with m. Then by Knight's theorem (see Theorem 3.8), the Laplace transform determines uniquely (see (3.11)) that the convolution semigroup $\{\gamma_t\}_{t\ge 0}$ of sub-probability measures γ_t on $[0,\infty)$ associated with ψ_m has to be given by the push-forward measure (1.23). Thus and by (3.41) from Proposition 3.43, one sees that

$$e^{-t\psi_m(A)}f = \int_{[0,\infty)} e^{-sA}f\,\mathrm{d}\gamma_t(s) = \mathbb{E}\left(e^{-\tilde{L}_t^{-1}A}f\right)$$

for every $t \ge 0$, $f \in X$, showing that (1.26) holds. This completes the proof of this theorem. $\square$

4.5. **Stability: Proof of Theorem 1.17.** This section is dedicated to outline the proof of Theorem 1.17.

We begin this subsection with the following lemma.

Lemma 4.7. *Let $m \in \mathfrak{m}_\infty$ and for every $n \in \mathbb{N}$, $m_n \in \mathfrak{m}_\infty$ be strings on $\mathbb{R}$ of infinite length respectively with associated measures μ_m and μ_{m_n}. If $m_n(z) \to m(z)$ as $n \to \infty$ for every continuity point $z \in \mathbb{R}$ of m, then one has that*

$$\lim_{n\to\infty} L_{m_n,t}^{-1} = L_{m,t}^{-1} \qquad \mathbb{P}\text{-a.e. and for all } t \ge 0. \tag{4.24}$$

where denote by $\{L_{m_n,t}^{-1}\}_{t\ge 0}$ and $\{L_{m,t}^{-1}\}_{t\ge 0}$ the inverse local time process at zero of the generalized diffusions $\{Z_{m_n,t}\}_{t\ge 0}$ and $\{Z_{m,t}\}_{t\ge 0}$ associated with m_n and m.

For the proof of this lemma, we use the Portmanteau theorem applied to the special case that the sequence $\{\mu_{m_n}\}_{n\geq 1}$ of measures μ_{m_n} is induced by a string $m_n \in \mathfrak{m}_\infty$.

Theorem 4.8 ([35, Theorem 21.15])**.** *Let m and m_n for every $n \in \mathbb{N}$, be monotone increasing functions on $\mathbb{R}$ and μ_m and μ_{m_n} the associated measures to m and m_n, respectively. Then, the following statements are equivalent.*

(1) One has that $m_n(z) \to m(z)$ as $n \to \infty$ for every continuity point $z \in \mathbb{R}$ of m;
(2) One has that $\mu_{m_n} \to \mu_m$ as $n \to \infty$ vaguely.

With this result in mind, we can now outline the proof of Lemma 4.7.

Proof of Lemma 4.7. Since each m_n and m are strings on $\mathbb{R}$ of infinite length, (3.7) yields that the associated inverse local time process $\{L^{-1}_{m_n,t}\}_{t\geq 0}$ and $\{L^{-1}_{m,t}\}_{t\geq 0}$ at zero of the generalized diffusions $\{Z_{m_n,t}\}_{t\geq 0}$ associated with m_n and $\{Z_{m,t}\}_{t\geq 0}$ associated with m admit the integral representation

$$L^{-1}_{m_n,t} = \int_{[0,\infty)} L_{L^{-1}_t}(z)\, \mathrm{d}\mu_{m_n}(z) \quad \text{and} \quad L^{-1}_{m,t} = \int_{[0,\infty)} L_{L^{-1}_t}(z)\, \mathrm{d}\mu_m(z)$$

$\mathbb{P}$-a.e. and for all $t \geq 0$. Moreover, according to Theorem 4.8, the pointwise convergence of $m_n(z) \to m(z)$ as $n \to \infty$ at every continuity point $z \in \mathbb{R}$ of m yields that $\mu_{m_n} \to \mu_m$ as $n \to \infty$ vaguely. Thus, in order to establish the desired limit (4.24), it is sufficient to show that

$$\lim_{n\to\infty} \int_{[0,\infty)} L_t(z)\, \mathrm{d}\mu_{m_n}(z) = \int_{[0,\infty)} L_t(z)\, \mathrm{d}\mu_m(z) \tag{4.25}$$

$\mathbb{P}$-a.s. for every $t \geq 0$. Note, the limit (4.25) follows from the vaguely convergence $\mu_{m_n} \to \mu_m$ as $n \to \infty$ provided that for every $t \geq 0$, the map L_t is continuous and has a compact support in $[0,\infty)$. To see that the latter holds, recall that the local time process $\{L_t(z)\}_{t\geq 0}$ of $\{B^+_t\}_{t\geq 0}$ is a jointly continuous mapping $L : [0,\infty) \times [0,\infty) \to [0,\infty)$ assigning $(t,z) \mapsto L_t(z)$. Thus, for every $t \geq 0$, $L_t : [0,\infty) \to [0,\infty)$ is continuous. To see that for every $t \geq 0$, the map L_t has a compact support in $[0,\infty)$, set $M := \sup_{s\in[0,t]} B^+_s$. By the continuity of the reflecting Brownian motion $\{B^+_t\}_{t\geq 0}$, the upper bound M is finite and so, by applying the occupation times formula (3.1) to the function $g := \mathbb{1}_{(M,\infty)}$, we can conclude that

$$\int_M^\infty L_t(z)\, \mathrm{d}z = \int_0^t \mathbb{1}_{(M,\infty)}(B^+_s)\, \mathrm{d}s = 0.$$

This shows that for every $t \geq 0$, $L_t(z) = 0$ for every $z > M$, that is, the function $z \mapsto L_t(z)$ has compact support in $[0,\infty)$. This proves (4.25). This completes the proof of this lemma. $\square$

Now, we are ready to give the proof of Theorem 1.17.

Proof of Theorem 1.17. Let A be an m-accretive operator on X, and for given strings $\{m_n\}_{n\geq 1} \subseteq \mathfrak{m}_\infty$ and $m \in \mathfrak{m}_\infty$, let ψ_{m_n} be the associated complete Bernstein function given by (3.12). Further, let $\{e^{-t\psi_{m_n}(A)}\}_{t\geq 0}$ and $\{e^{-t\psi_m(A)}\}_{t\geq 0}$

be the semigroups generated by $-\psi_{m_n}(A)$ and $-\psi_m(A)$ on X, respectively. Then, our first aim is to show that for every $f \in X$,

$$e^{-t\psi_{m_n}(A)}f \to e^{-t\psi_m(A)}f \quad \text{in } X \text{ pointwise for every } t \geq 0. \tag{4.26}$$

According to Theorem 1.13, the operator $\psi_{m_n}(A)$ and $\psi_m(A)$ respectively coincide with their Dirichlet-to-Wenzell operator Λ_{m_n} and Λ_m given by (1.6). In particular, by the integral representation (1.26) of Theorem 1.13, one has that can be rewritten as

$$e^{-t\psi_{m_n}(A)}f = \mathbb{E}\left(e^{-\tilde{L}^{-1}_{m_n,t}A}f\right) \quad \text{and} \quad e^{-t\psi_m(A)}f = \mathbb{E}\left(e^{-\tilde{L}^{-1}_{m,t}A}f\right),$$

where $\{\tilde{L}^{-1}_{m_n,t}\}_{t\geq 0}$ and $\{\tilde{L}^{-1}_{m,t}\}_{t\geq 0}$ are respectively the local inverse times at zero of the generalized diffusions $\{Z_{m_n,t}\}_{t\geq 0}$ associated with m_n and $\{Z_{m,t}\}_{t\geq 0}$ associated with m. From this, one sees that

$$\|e^{-\psi_m(A)t}f - e^{-\psi_{m_n}(A)t}f\|_X \leq \mathbb{E}\left\|e^{-\tilde{L}^{-1}_{m,t}A}f - e^{-\tilde{L}^{-1}_{m_n,t}A}f\right\|_X.$$

Since the semigroup $\{e^{-tA}\}_{t\geq 0}$ is contractive and strongly continuouse^{-tA}, limit (4.26) follows from Lebesgue's dominate convergence theorem and by limit (4.24).

Now, for every $\lambda > 0$, let $R(\lambda, \tilde{\psi}_m(A)) := (\lambda \mathrm{id}_X + \tilde{\psi}_m(A))^{-1}$ be the *resolvent* operator of $\tilde{\psi}_m(A)$ on X and for every $n \geq 1$, let $R(\lambda, \psi_{m_n}(A))$ be the resolvent operator of $\psi_{m_n}(A)$. Then by [14, Theorem 1.10 in Chapter II.], for every $f \in X$, $R(\lambda, \tilde{\psi}_m(A))f$ and $R(\lambda, \psi_{m_n}(A))f$ admit the integral representations

$$R(\lambda, \tilde{\psi}_m(A))f = \int_0^\infty e^{-\lambda t} e^{-t\tilde{\psi}_m(A)}f\,\mathrm{d}t$$

and

$$R(\lambda, \psi_{m_n}(A))f = \int_0^\infty e^{-\lambda t} e^{-t\psi_{m_n}(A)}f\,\mathrm{d}t$$

from where one can conclude that

$$\begin{aligned}
&\|R(\lambda, \tilde{\psi}_m(A))f - R(\lambda, \psi_{m_n}(A))f\|_X \\
&\qquad \leq \int_0^\infty e^{-\lambda t}\left\|e^{-t\tilde{\psi}_m(A)}f - e^{-t\psi_{m_n}(A)}f\right\|_X \mathrm{d}t.
\end{aligned}$$

Thus, by (4.26), and since the semigroups $\{e^{-t\tilde{\psi}_m(A)}\}_{t\geq 0}$ and $\{e^{-t\psi_{m_n}(A)}\}_{t\geq 0}$ are contractive, Lebesgue's dominate convergence theorem yields that

$$\lim_{n\to\infty} R(\lambda, \psi_{m_n}(A))f = R(\lambda, \tilde{\psi}_m(A))f \qquad \text{in } X,$$

for every $\lambda > 0$ and $f \in X$; that is, $\psi_{m_n}(A) \to \tilde{\psi}_m(A)$ strongly in the *resolvent sense*. By Trotter-Kato's first approximation theorem (see Remark 1.18), this type of convergence is equivalent to the type of convergence stated in Theorem 1.17. □

5. Application

In order to demonstrate the usefulness of the main results of this paper, we discuss in this section one classical example.

5.1. **Limits of fractional powers A^σ as $\sigma \to 1-$.** Let A be an m-accretive operator on Banach space X. Then, we intend to give a proof of the limit

$$\lim_{\sigma\to 1-} A^\sigma = A \qquad \text{in the graph sense} \tag{5.1}$$

by using Theorem 1.17. It is worth noting that this limit has been well studied (see, for instance, [7, Lemma 2.3]). Here, we provide an alternative, which from our perspective is much simpler.

For given $0 < \sigma < 1$ and $f \in D(A)$, we begin by realizing the fractional power A^σ by the Dirichlet-to-Neumann operator $\Lambda_{\tilde{m}_\sigma}$ associated with the following incomplete Dirichlet problem

$$\begin{cases} Au(z) - \left(\frac{\sigma}{1-\sigma}\right) z^{\frac{2\sigma-1}{\sigma}} \frac{\mathrm{d}^2 u}{\mathrm{d}z^2}(z) &= 0 \qquad \text{for } z \in (0,\infty), \\ \qquad\qquad\qquad\qquad u(0) &= f, \end{cases} \tag{5.2}$$

for the extension operator

$$\mathcal{A}_{\tilde{m}_\sigma} = A + B_{\tilde{m}_\sigma} \qquad \text{with} \qquad B_{\tilde{m}_\sigma} = -\Big(\frac{\sigma}{1-\sigma}\Big) z^{\frac{2\sigma-1}{\sigma}} \frac{\mathrm{d}^2}{\mathrm{d}z^2}.$$

By comparing Dirichlet problem (5.2) with (1.2) from the introduction, then one realized that the extension equation

$$\mathcal{A}_{\tilde{m}_\sigma} u = 0 \qquad \text{on } \mathcal{X}_+$$

in problem (5.2) is a *scaled version* (by the factor $\sigma/(1-\sigma)$) of the extension equation $\mathcal{A}_{m_\sigma} = 0$ employed in Example 1.2. In fact, this is one possibility to circumvent the fact that for the string m_σ introduced in (1.14) the limit as $\sigma \to 1-$ does not exist. The string $\tilde{m}_\sigma$ corresponding to Dirichlet problem (5.2) is given by

$$\tilde{m}_\sigma(z) = \frac{\sigma}{1-\sigma}\, m_\sigma(z) = \begin{cases} \frac{1}{2} z^{\frac{1-\sigma}{\sigma}} & \text{if } z \geq 0, \\ 0 & \text{if } z < 0, \end{cases} \tag{5.3}$$

for every $z \in \mathbb{R}$, and the associate complete Bernstein function

$$\psi_{\tilde{m}_\sigma}(\lambda) = \frac{\sigma^{\sigma-1}(1-\sigma)^\sigma}{2} \frac{\Gamma(1-\sigma)}{\Gamma(\sigma)}\, \lambda^\sigma \qquad \text{for every } \lambda \geq 0.$$

Then, by our main Theorem 1.13, the Dirichlet-to-Neumann operator $\Lambda_{\tilde{m}_\sigma}$ associated with $\mathcal{A}_{\tilde{m}_\sigma}$ characterizes A^σ up to a multiplicative constant; namely, one has that

$$\Lambda_{\tilde{m}_\sigma} = \frac{\sigma^{\sigma-1}(1-\sigma)^\sigma}{2} \frac{\Gamma(1-\sigma)}{\Gamma(\sigma)} A^\sigma$$

(cf. the results mentioned in Section 2.3). Since for the family $\{\tilde{m}_\sigma\}_{\sigma\in(0,1)}$ of strings,

$$\lim_{\sigma\to 1-} \tilde{m}_\sigma(z) = m(z) := \begin{cases} \frac{1}{2} & \text{if } z \geq 0, \\ 0 & \text{if } z < 0 \end{cases}$$

the *Heaviside step function*, by Theorem 1.17 and Example 1.8, we get that

$$\lim_{\sigma\to 1-} \frac{\sigma^{\sigma-1}(1-\sigma)^\sigma}{2} \frac{\Gamma(1-\sigma)}{\Gamma(\sigma)} A^\sigma = \frac{1}{2} A \qquad \text{in the graph sense.}$$

From this and since

$$\lim_{\sigma\to 1^-} \sigma^{\sigma-1}(1-\sigma)^{\sigma}\frac{\Gamma(1-\sigma)}{\Gamma(\sigma)} = 1,$$

we can conclude that limit (5.1) holds.

Remark 5.1 (Probabilistic justification). Since A^σ coincides up to a scalar multiple with the Dirichlet-to-Neumann operator associated with the extension operator $A + B_{m_\sigma}$ on $\mathcal{X}_+ = X \times (0,\infty)$ and B_{m_σ} is the generator of the $2\sigma^{\text{th}}$-powered process $\{Y_t^{2\sigma}\}_{t\geq 0}$ in $[0,\infty)$ of the Bessel process $\{Y_t\}_{t\geq 0}$ of dimension $2(1-\sigma)$, the limit (5.1) can be justified with probabilistic arguments. In fact, the limit (5.1) means that the process induced by A^σ changes to the process generated by A. This makes sense if one considers the interaction of the Bessel process with the left boundary point $z = 0$. For $\sigma \in (0,1)$, the set $\{0\}$ is reflecting for the Bessel process of dimension $2(1-\sigma)$. For the Bessel process of dimension 0, the case when $\sigma = 1$, the set $\{0\}$ is absorbing.

To conclude this section we note that both strings m_σ from (1.14) and $\tilde{m}_\sigma$ from (5.3) can't be used in combination with Theorem 1.17 for proving that

$$\lim_{\sigma\to 0+} A^\sigma = \mathrm{id}_X \qquad \text{in the graph sense}$$

holds for any given m-accretive operator A on Banach space X. The reason for this is that the pointwise limit

$$\lim_{\sigma\to 0+} m_\sigma(z) = \lim_{\sigma\to 0+} \tilde{m}_\sigma(z) = m(z) := \begin{cases} \infty & \text{if } z > 1,\\ 0 & \text{if } z \leq 1, \end{cases}$$

the *Delta-function*, is not a string.

Appendix A. A Primer on Bessel processes

In this subsection, we give a brief reminder on the theory of Bessel processes. For a detailed review of this subject, we refer the interested reader to [31, Chapter IX] or [39, Chapter 3]. Throughout this paper, we denote by $\{B_t\}_{t\geq 0}$ a standard *Brownian motion* in $\mathbb{R}$ starting at $x = 0$.

We begin by recalling the definition of the *squared Bessel process of dimension* $\delta \geq 0$; this process $\{Y_t^2\}_{t\geq 0}$ is the unique, continuous, non-negative process defined by the stochastic differential equation (SDE)

$$Y_t^2 = y^2 + \delta t + \int_0^t 2\sqrt{Y_s^2}\,\mathrm{d}B_s \qquad \text{for every } t\geq 0,\ y\in\mathbb{R}.$$

Then, for $\delta \geq 0$, the δ-*Bessel process* $\{Y_t\}_{t\geq 0}$ is obtained by taking the square root

$$Y_t := \sqrt{Y_t^2} \qquad \text{for every } t \geq 0. \tag{A.1}$$

In the case $\delta > 1$, Itô's Lemma yields that the δ-Bessel process $\{Y_t\}_{t\geq 0}$ is the unique solution to the SDE

$$Y_t = y + \frac{\delta - 1}{2}\int_0^t \frac{1}{Y_s}\,\mathrm{d}s + B_t \qquad \text{for every } t \geq 0,\ y\geq 0. \tag{A.2}$$

It is well known that equation (A.2) satisfies path-wise uniqueness and has strong solutions due to the drift being monotone decreasing (cf., [39, Theorem

3.2]). For $\delta = 1$, the squared Bessel process $\{Y_t\}_{t\geq 0}$ coincides with the squared Brownian motion and hence, the 1-Bessel process $\{Y_t\}_{t\geq 0}$ coincides with the *reflecting Brownian motion* $\{B_t^+\}_{t\geq 0}$, which is the unique solution of the SDE

$$B_t^+ = y + B_t + L_t \qquad \text{for every } t \geq 0,\ y \geq 0, \tag{A.3}$$

where $\{L_t\}_{t\geq 0}$ is a continuous, monotone, non-decreasing process, with $L_0 = 0$, $\mathrm{d}L_t \geq 0$, and $\int_0^t B_s^+ \,\mathrm{d}L_s = 0$ for all $t \geq 0$. Uniqueness of the solution $\{B_t^+\}_{t\geq 0}$ to (A.3) follows from the Skorokhod lemma (see [39, Lemma 2.1]).

In the case $0 < \delta < 1$, the situation is far more delicate and the δ-Bessel process $\{Y_t\}_{t\geq 0}$ is no longer a solution of equation (A.2). Instead (cf., [39, Proposition 3.8 & 3.12]), for the process $\{Y_t\}_{t\geq 0}$ given by (A.1), there is a continuous family $\{\ell_t^a\}_{a,t\geq 0}$, called *diffusion local times*, satisfying the *occupation times formula*

$$\int_0^t \varphi(Y_r)\,\mathrm{d}r = \int_0^\infty \varphi(a)\,\ell_t^a\,a^{\delta-1}\,\mathrm{d}a$$

for all $t \geq 0$ and bounded and Borel-measurable functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, and $\{Y_t\}_{t\geq 0}$ satisfies

$$Y_t = y + \frac{\delta - 1}{2(2-\delta)} \int_0^\infty \frac{\ell_t^a - \ell_t^0}{a} a^{\delta-1}\,\mathrm{d}a + B_t \qquad \text{for all } t \geq 0,\ y \geq 0.$$

The next result due to Donati-Martin, Roynette, Vallois and Yor [12] is quite important for understanding the probabilistic approach to the Dirichlet-to-Neumann operator in the fractional power case as discussed in Section 2.

Theorem A.1. *For $\sigma \in (0,1)$, let $\delta = 2(1-\sigma)$ and $\{Y_t\}_{t\geq 0}$ be the δ-Bessel process starting at $y = 0$. Then, the $2\sigma^{th}$-powered process $\{Y_t^{2\sigma}\}_{t\geq 0}$ is a sub-martingale with the Doob-Meyer decomposition*

$$Y_t^{2\sigma} = 2\sigma \int_0^t Y_r^{2\sigma-1}\mathrm{d}B_r + L_t$$

where $\{L_t\}_{t\geq 0}$ is a continuous non-decreasing process, carried by the zeros of $\{Y_t\}_{t\geq 0}$; that is, $\{L_t\}_{t\geq 0}$ only increases when $Y_t = 0$. Further, the inverse local time process $\{L_t^{-1}\}_{t\geq 0}$ of $\{L_t\}_{t\geq 0}$ given by

$$L_t^{-1} := \inf\Big\{ r > 0 \,\Big|\, L_r > t \Big\}, \qquad (t \geq 0),$$

is an σ-stable subordinator satisfying

$$\mathbb{E}\Big(e^{-uL_t^{-1}}\Big) = e^{-t\frac{\Gamma(1-\sigma)}{\Gamma(1+\sigma)}\frac{u^\sigma}{2^\sigma}} \qquad \textit{for every } t,\ u > 0$$

and Lévy measure ν given by (2.11).

Appendix B. A distributional definition of $\frac{\mathrm{d}f}{\mathrm{d}m}$

The first section of the appendix deals with some reminder on *distributional definition* of $\frac{\mathrm{d}f}{\mathrm{d}m}$ when $m \in \mathfrak{m}_\infty$ is a string and $f : [0,\infty) \to X$ is a Banach space X-valued measurable function. Throughout this section, let X be a Banach space with norm $\|\cdot\|_X$ and X' its dual space with duality brackets $\langle \cdot, \cdot \rangle_{X',X}$ and $m \in \mathfrak{m}_\infty$ a string with associated measure μ_m.

We begin by considering the *smooth* case.

Definition B.1. We say that a function $f : [0, \infty) \to X$ is *m-differentiable* at $t \in (0, \infty)$ provide the limit

$$\frac{\mathrm{d}f}{\mathrm{d}m}(t) := \lim_{h \to 0} \frac{f(t+h) - f(t)}{m(t+h) - m(t)} \qquad \text{exists in } X.$$

Then, we call $\frac{\mathrm{d}f}{\mathrm{d}m}(t)$ the *(classic) m-derivative* of f at $t \in (0, \infty)$.

Of course, one natural question is to ask, which functions are m-differentiable. One example is given by the following result generalizing the classic theorem of Lebesgue (see, e.g., [28, Theorem 1.18]).

Theorem B.2. *Let $m \in \mathfrak{m}_\infty$ be a string with associated measure μ_m. Then every monotone function $f : [0, \infty) \to \mathbb{R}$ is μ_m-a.e. m-differentiable.*

Proof. By Lebesgue's theorem, the two sets

$$N_f := \Big\{ t \in [0, \infty) \,\Big|\, f'(t) \text{ does not exist } \Big\}$$

and

$$N_m := \Big\{ t \in [0, \infty) \,\Big|\, m'(t) \text{ does not exist } \Big\}$$

are measurable subsets of $[0, \infty)$ with Lebesgue measure $\lambda(N_f) = 0$ and $\lambda(N_m) = 0$. Further, we decompose $[0, r_m)$ into three disjoint Borel-measurable sets

$$A_1 = \Big\{ t \in N_m^c \,\Big|\, m'(t) = 0 \Big\}, \quad A_2 = \Big\{ t \in N_m^c \,|\, m'(t) \neq 0 \Big\}, \quad A_3 = N_m.$$

By the partition $(A_i)_{i=1}^3$ of $[0, \infty)$, one has that

$$[0, \infty) = N_f \,\dot\cup\, N_f^c = \dot{\bigcup}_{i=1}^{3} (N_f^c \cap A_i) \,\dot\cup\, \dot{\bigcup}_{i=1}^{3} (N_f \cap A_i).$$

Since the support $E_m := \mathrm{supp}(\mu_m)$ of the measure μ_m is the set where m increases, one has that $\mu_m(A_1) = 0$ and $\mu_m(A_3) = 0$. Thus, by the monotonicity of μ_m, we can conclude that $\mu_m(N_f \cap A_1) = 0$, $\mu_m(N_f^c \cap A_1) = 0$, $\mu_m(N_f \cap A_3) = 0$, and $\mu_m(N_f^c \cap A_3) = 0$. Therefore, it remains to focus on the two cases

$$N_f \cap A_2 = \Big\{ t \,\Big|\, m'(t) \neq 0 \;\&\; f'(t) \text{ does not exist} \Big\}$$

and

$$N_f^c \cap A_2 = \Big\{ t \,\Big|\, m'(t) \neq 0 \;\&\; f'(t) \text{ does exist} \Big\}.$$

By Lebesgue 's decomposition (cf., [28, Theorem B.67 & Section 6.3]), the Lebesgue-Stieltje measure μ_m has the unique decomposition

$$\mu_m = \mu_{m,ac} + \mu_{m,s},$$

where $\mu_{m,ac}$ is absolutely continuous w.r.t. the Lebesgue measure λ and $\mu_{m,s}$ is singular. Moreover, $\mu_{m,ac}$ is given by

$$\mu_{m,ac}(E) = \int_{E \cap N_m^c} m'(t) \,\mathrm{d}\lambda(t)$$

for every Borel-measurable subset E of $[0, \infty)$. But since N_f has Lebesgue measure $\lambda(N_f) = 0$, we also have that

$$\mu_m(N_f \cap A_2) = \mu_{m,ac}(N_f) = 0.$$

Therefore, we have shown that the set

$$N := [0, \infty) \setminus (N_f \cap A_2)$$

has measure $\mu_m(N) = 0$ and

$$\frac{\mathrm{d}f}{\mathrm{d}m}(t) \text{ exits at all } t \in [0, \infty) \setminus N = \Big\{ t \in [0, \infty) \,\Big|\, m'(t) \neq 0 \;\&\; f'(t) \text{ exists} \Big\}.$$

□

For the moment, suppose the string $m \in \mathfrak{m}_\infty$ is a smooth real-valued function satisfying $m' \geq c_0$ on $\mathbb{R}_+$ for some $c_0 > 0$. Then, the associated measure μ_m to m is absolutely continuous with respect to the Lebesgue measure and if f is differentiable at t, then one easily sees that f is, in particular, m-differentiable and

$$\frac{\mathrm{d}f}{\mathrm{d}m}(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \frac{h}{m(t+h) - m(t)} = f'(t) \frac{1}{m'(t)}.$$

Now, set $g(t) := \frac{\mathrm{d}f}{\mathrm{d}m}(t)$ and recall that Radon-Nikodym derivative $\frac{\mathrm{d}\mu_m}{\mathrm{d}t}(t) = m'(t)$ on $(0, \infty)$. Therefore, an integration by parts shows that

$$\begin{aligned}
\int_0^\infty g(t)\,\xi(t)\mathrm{d}\mu_m(t) &= \int_0^\infty f'(t) \frac{1}{m'(t)}\,\xi(t)\mathrm{d}\mu_m(t) \\
&= \int_0^\infty f'(t)\,\xi(t)\mathrm{d}t \\
&= -\int_0^\infty f(t)\,\xi'(t)\mathrm{d}t
\end{aligned}$$

for every real-valued test function $\xi \in C_c^\infty((0, \infty))$. We emphasize that since the measure μ_m associated to the given string m is a Radon measure on $[0, \infty)$, we have the following result.

Lemma B.3. *Let $m \in \mathfrak{m}_\infty$ be a string with associated measure μ_m. If a function $f \in L^1_{loc,\mu_m}(0, \infty; X)$ satisfies*

$$\int_0^\infty f(t)\,\xi(t)\,\mathrm{d}\mu_m(t) = 0 \tag{B.1}$$

for all $\xi \in C_c^\infty((0, \infty))$, then $f(t) = 0$ in X for a.e. $t \in (0, \infty)$.

Proof. Let $\xi \in C_c^\infty((0, \infty))$ and suppose (B.2) holds. Then multiplying (B.2) by $x' \in X$ gives

$$\int_0^\infty \langle x', f(t) \rangle_{X',X}\, \xi(t)\,\mathrm{d}\mu_m(t) = 0$$

Since the measure μ_m associated to the given string m is a Radon measure on $[0, \infty)$, and the scalar-valued function $t \mapsto \langle x', f(t) \rangle_{X',X}$ belongs to $L^1_{loc,\mu_m}(0, \infty)$, it follows from [32, Theorem 3.14] that $\langle x', f(t) \rangle_{X',X} = 0$ for a.e. $t \in (0, \infty)$. Since $x' \in X'$ was arbitrary, and the dual space X is separating elements of X (Hahn-Banach), we can conclude that $f(t) = 0$ in X for a.e. $t \in (0, \infty)$. □

Because of Lemma B.3, Definition 1.3 of the *weak m-derivative* make sense.

Definition B.4. For a given string $m \in \mathfrak{m}_\infty$ on $\mathbb{R}$, a function $f : [0,\infty) \to \mathbb{R}$ is called to be *m-continuous* at $t_0 \in (0,\infty)$ if for every $\varepsilon > 0$ there is an $\delta > 0$ such that for all $t \in [0,\infty)$ satisfying $|m(t_0)-m(t)| < \delta$, one has $\|f(t)-f(t_0)\|_X < \varepsilon$. Further, f is called *uniformly m-continuous* on $[0,\infty)$ if for every $\varepsilon > 0$ there is an $\delta > 0$ such that for all $t_1, t_2 \in [0,\infty)$ satisfying $|m(t_1)-m(t_2)| < \delta$, one has $\|f(t_1) - f(t_2)\|_X < \varepsilon$. Finally, f is called *absolutely m-continuous* on $[0,\infty)$ if for every $\varepsilon > 0$ there is an $\delta > 0$ such that for every finite family $((a_k, b_k))_{k=1}^n$ of disjoint sub-intervals of $[0,\infty)$ satisfying $\sum_{k=1}^n |m(b_k) - m(a_k)| < \delta$, one has $\sum_{k=1}^n \|f(b_k) - f(a_k)\|_X < \varepsilon$.

Proposition B.5. *Let $m \in \mathfrak{m}_\infty$ be a string with associated measure μ_m. If $g \in L^1_{loc,\mu_m}([0,\infty); X)$ then*

$$f(t) := \int_{t_0}^t g(r)\,\mathrm{d}\mu_m(r), \quad t \in [0,\infty),$$

is locally m-absolutely continuous on $[0,\infty)$ and at μ_m-a.e. $t \in [0,\infty)$, $f(t)$ is m-differentiable with $\frac{\mathrm{d}f}{\mathrm{d}m}(t) = g(t)$.

Proof. Since f satisfies

$$\|f(t) - f(s)\|_X \le \int_s^t \|g(r)\|_X\,\mathrm{d}\mu_m(r)$$

for every t, $s \in [0,\infty)$ with $s < t$, it follows that f is *m-absolutely continuous.* By [1, Corollary 2.23], one has that for every μ_m-Lebesgue point $t \in [0,\infty)$,

$$\lim_{h\to 0+} \tfrac{1}{m(t+h)-m(t-h)} \int_{t-h}^{t+h} \|g(r) - g(t)\|_X\,\mathrm{d}\mu_m(r) = 0.$$

Further, for such $t \in [0,\infty)$ and $h > 0$, one has that

$$\begin{aligned}\left\| g(t) - \frac{f(t+h)-f(t)}{m(t+h)-m(t)} \right\|_X &= \left\| g(t) - \tfrac{1}{m(t+h)-m(t)} \int_t^{t+h} g(r)\,\mathrm{d}\mu_m(r) \right\|_X \\ &\le \tfrac{1}{m(t+h)-m(t)} \int_t^{t+h} \|g(t) - g(r)\|_X\,\mathrm{d}\mu_m(r).\end{aligned}$$

Similarly, for such $t \in [0,\infty)$ and $h < 0$,

$$\left\| g(t) - \frac{f(t+h)-f(t)}{m(t+h)-m(t)} \right\|_X \le \tfrac{1}{m(t)-m(t+h)} \int_{t+h}^{t} \|g(t) - g(r)\|_X\,\mathrm{d}\mu_m(r).$$

Now, by letting $h \to 0+$ in the above estimates yields that f is m-differentiable with $\frac{\mathrm{d}f}{\mathrm{d}m}(t) = g(t)$. □

Further, we have the following important lemma.

Lemma B.6. *If a function $f \in L^1_{loc}(0,\infty; X)$ satisfies*

$$\int_0^\infty f(t)\,\xi'(t)\,\mathrm{d}\mu_m(t) = 0 \tag{B.2}$$

for all $\xi \in C_c^\infty((0,\infty))$, then there is a $C \in X$ such that $f(t) = C$ for a.e. $t \in (0,\infty)$.

Proof. Let ξ and $\eta \in C_c^\infty((0,\infty))$ such that $\eta \neq 0$. Since $\eta - \xi' \in C_c^\infty((0,\infty))$, the function

$$\chi(t) := \frac{\eta(t) - \xi'(t)}{\int_0^\infty \eta(r)\,\mathrm{d}\mu_m(r)}, \qquad t \in (0,\infty),$$

belongs to $C_c^\infty((0,\infty))$ and $\int_0^\infty \chi(r)\,\mathrm{d}\mu_m(r) = 1$. Inserting ξ into (B.2) gives

$$\begin{aligned}
0 &= \int_0^\infty f(t)\,\xi'(t)\,\mathrm{d}\mu_m(t) \\
&= \int_0^\infty f(t)\,\eta(t)\,\mathrm{d}\mu_m(t) - \int_0^\infty \left(f(t)\,\chi(t) \int_0^\infty \eta(r)\,\mathrm{d}\mu_m(r)\right)\,\mathrm{d}\mu_m(t) \\
&= \int_0^\infty \left(f(t) - \int_0^\infty f(r)\,\chi(r)\mathrm{d}\mu_m(r)\right)\,\eta(t)\,\mathrm{d}\mu_m(t)
\end{aligned}$$

Since $\eta \in C_c^\infty((0,\infty))$ was arbitrary, it follows from Lemma B.3 that

$$f(t) = C := \int_0^\infty f(r)\,\chi(r)\mathrm{d}\mu_m(r) \qquad \text{for a.e. } t \in (0,\infty).$$

□

Thanks to the above lemma, we can make the following statements.

Proposition B.7. *Let $m \in \mathfrak{m}_\infty$ be a string with associated measure μ_m. Further, let $f \in L^1_{loc}([0,\infty);X)$ and $g \in L^1_{loc,\mu_m}([0,\infty);X)$. Then, the following statements are equivalent.*

(1) One has that $g = \frac{\mathrm{d}f}{\mathrm{d}m}$ is the weak m-derivative of f;
(2) There is an $x \in X$ such that

$$f(t) = x + \int_0^t g(r)\,\mathrm{d}\mu_m(r), \quad \text{a.e. on } [0,\infty). \tag{B.3}$$

Proof. We show that (1) implies (2). For this, set

$$w(t) = f(t) - \int_0^t g(r)\,\mathrm{d}\mu_m(r), \quad t \in [0,\infty).$$

Then, by Fubini's theorem,

$$\begin{aligned}
\int_0^\infty \left(\int_0^t g(r)\,\mathrm{d}\mu_m(r)\right)\,\xi'(t)\,\mathrm{d}t &= \int_0^\infty \left(\int_r^\infty \xi'(t)\,\mathrm{d}t\right) g(r)\,\mathrm{d}\mu_m(r) \\
&= -\int_0^\infty g(r)\,\xi(r)\mathrm{d}\mu_m(r)
\end{aligned}$$

and so,

$$\int_0^\infty w(t)\,\xi'(t)\,\mathrm{d}t = \int_0^\infty f(t)\,\xi'(t)\,\mathrm{d}t - \int_0^\infty \left(\int_0^t g(r)\,\mathrm{d}\mu_m(r)\right)\,\xi'(t)\,\mathrm{d}t = 0.$$

Therefore, (B.3) follows from Lemma B.6. The implication (2) implies (1) follows from Proposition B.5. □

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